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Dual modules

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Introduction and statement of results. It is well known that if V is a vector space over a field k then its dual vector space $\mathrm{Hom}_k(V, k)$ has dimension not less than that of V, and that the dimensions are equal precisely when V is finite-dimensional. In this note we introduce a 'dual module' for a module over a principal ideal domain, and show that some though not all of the properties of the field case carry over.

In what follows L will be a principal ideal domain, assumed to be countable for simplicity (see remarks at the end). We write HM for $\mathrm{Hom}_L(M, L)$ and EM for $\mathrm{Ext}_L(M, L)$, where M is an L-module. $|M|$ denotes the cardinality of M, and tM the torsion submodule of M. Q is the field of fractions of L. If $p \in L$ is prime, $L(p^\infty)$ is defined as $\varinjlim (L/p^kL)$ with obvious onto bonding maps.

Definition. The *dual module* DM of an L-module M is defined by $DM = HM \oplus EM$.

Example. If M is the singular homology of a space with coefficients in L, then DM is the corresponding cohomology module (5).

Theorem. *Let L be a countable principal ideal domain.*

(1) *$DM \cong M$ if and only if M is finitely generated.*
(2) *M is not finitely generated if and only if $|DM| = 2^{|M|}$.*

Corollary 1. *If M is not finitely generated, one of EM, HM has cardinality strictly greater than that of M.*

Corollary 2 (cf.(1)). *$EM \cong M$ if and only if M is finitely generated and torsion.*

Proofs. Proof of theorem. Clearly $|HM| \leqslant 2^{|M|}$, while by the long exact cohomology sequence for Hom and Ext, EM is a quotient of $\mathrm{Hom}_L(M, Q/L)$, and so

$$|EM| \leqslant |Q/L|^{|M|} = 2^{|M|}. \quad \text{Thus} \quad |DM| = |HM| . |EM| \leqslant 2^{|M|}.$$

If M is finitely generated, by the standard structure theorems ((3), 15·5) $M \cong F \oplus T$, where F is free and T is torsion, and both are finitely generated. Then $HM \cong HF \cong F$ and $EM \cong ET \cong T$, whence $DM \cong M$. It remains to show that $|DM| = 2^{|M|}$ for infinitely generated M.

In general we have the exact sequence

$$0 \to tM \to M \to M/tM \to 0.$$

Since $H(tM) = 0$, the long exact sequence gives a short exact sequence

$$0 \to D(M/tM) \to DM \to D(tM) \to 0.$$

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Now $D(tM) = E(tM) \cong \mathrm{Hom}_L(tM, Q/L)$. Hence by (3), 47·5, $|D(tM)| = 2^{|tM|}$ if tM is infinitely generated. We need:

LEMMA. *If N is infinitely generated and torsion-free, then $|DN| = 2^{|N|}$.*

Proof of lemma. If L is a field the result is well known. Assume L is not a field. By a well-known argument L is infinite. Let $p \in L$ be prime and let B be p-basic in N ((3), §32). Write $\mathrm{rank}\,(B) = b$, $\mathrm{rank}_0(N/B) = c$. For N to be infinitely generated, either b is infinite or c is non-zero. (By (2) we may assume that N has infinite rank.)

N/B contains a submodule isomorphic to $\bigoplus_c Q^{(p)}$, where $Q^{(p)}$ is the submodule of Q consisting of fractions with p-power denominators. Using the long exact cohomology sequence we find

$$|E(N/B)| \geqslant |E(\bigoplus_c Q^{(p)})| = |\prod_c E(Q^{(p)})| = (2^{\aleph_0})^c.$$

(By the exact sequence $0 \to L \to Q^{(p)} \to L(p^\infty) \to 0$, one sees that $E(Q^{(p)})$ is a quotient of $E(L(p^\infty))$ by L; but $E(L(p^\infty))$ is isomorphic to $\mathrm{Hom}_L(L(p^\infty), Q/L)$ and this is uncountable as above (cf. lemma 3·5 of (2)).)

We have the exact sequences

$$0 \to HN \to HB \to E(N/B) \to EN \to 0$$

and $$0 \to HN \xrightarrow{p} HN \to \mathrm{Hom}_L(N, L/pL) \to EN \xrightarrow{p} EN \to \mathrm{Ext}_L(N, L/pL) \to 0.$$

From the first sequence we find $|HB| \,.\, |EN| = |HN| \,.\, |E(N/B)|$, from which we deduce $|HB| \,.\, |DN| = |HN|^2 \,.\, |E(N/B)| \geqslant |E(N/B)| \geqslant 2^{\aleph_0 c}$. (*).

From the second sequence we find that:

$$\begin{aligned} |\mathrm{Hom}_L(B/pB, L/pL)| &= |\mathrm{Hom}_L(N/pN, L/pL)| = |\mathrm{Hom}_L(N, L/pL)| \\ &= |EN[p]| \,.\, |HN/pHN| \leqslant |EN| \,.\, |HN| = |DN|. \qquad (**) \end{aligned}$$

Case 1: *b is finite.* Then $|HB| = |B| = \aleph_0$ as above, and so by (*)

$$\max(|HB|, |DN|) = |HB| \,.\, |DN| \geqslant 2^{\aleph_0 c} \geqslant 2^{\aleph_0}, \quad \text{as} \quad c > 0.$$

Thus $$|DN| \,.\, |HB| = |DN| \quad \text{and} \quad |DN| \geqslant 2^{\aleph_0 c} = 2^{\aleph_0 (c+b)} = 2^{|N|}.$$

Case 2: *b infinite.* Since B is free, we have

$$|\mathrm{Hom}_L(B/pB, L/pL)| = |(L/pL)|^b = \aleph_0^b = |HB| \,(\text{as } 2^b = \aleph_0^b).$$

Thus by (**) $|HB| = 2^b \leqslant |DN|$, while by (*) $|HB| \,.\, |DN| = |DN| \geqslant 2^{\aleph_0 c}$. Hence $|DN| \geqslant \max(2^{\aleph_0 c}, 2^b) = 2^{\aleph_0 (b+c)} = 2^{|N|}$.

As we also have $|DN| \leqslant 2^{|N|}$ always, this completes the proof.

The Lemma proves the result when L is a field, so we may assume L is infinite. Then $|M| = \max(|tM|, |M/tM|)$ and $|DM| = \max(|D(tM)|, |D(M/tM)|)$ since M/tM is infinite and $|D(M/tM)| \geqslant |M/tM|$. If M is infinitely generated, so is at least one of $tM, M/tM$. As a finitely generated module is countable and an infinitely generated one is infinite, and $2^{\aleph_0} > \aleph_0$, the result is immediate. This ends the proof of the theorem.

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Proof of corollaries. Corollary 1 is trivial. To prove Corollary 2, note that if $EM \neq 0$, L is not a field. $M \cong EM$ implies M is algebraically compact ((3), 52 (N)). Now L is not algebraically compact, so as algebraic compactness is inherited by direct summands, $HM = 0$. Thus $EM \cong DM$ and the result is clear from the theorem. (The converse assertion is trivial.)

Remarks. (1) If L is not countable, a similar result holds, but it is complicated by the fact that if rank (M) is infinite the equality rank (M) = $|M|$ need not hold.

(2) We sketch an amusing alternative proof of (1) of the theorem.

By (4) if $M \cong DM$, then $M \cong A \oplus D \oplus K$, where A is reduced algebraically compact, D is divisible and K is $\aleph_1$-free. Since $HA = 0$, $DDM \cong DM$ gives

$$EA \oplus ED \oplus EK \oplus HK \cong A \oplus D \oplus K.$$

Since $EQ \cong Q^2 \not\supset D$ is torsion. If $\bar{L}_p$, the completion of the valuation ring L_p, is a summand of A, EA then contains $E\bar{L}_p$, which has non-zero torsion-free divisible part. Hence by (3), 40·4, A is torsion. If $D \neq 0$, ED contains some $\bar{L}_p$, so by the above $D = 0$. Since EK is divisible ((3), 52 I), $EK = 0$. Hence K is a W-module and so unless rank (K) $< +\infty$, $|HK| > |K|$ ((3), 99·4). Thus since $HK \cong K$, K has finite rank and is free. Finally we have $EA \cong A$, so A is finite, and the result is proved.

(3) If L is not a field, the functor $D-$ has no good exactness properties. Indeed on applying D to the exact sequences

$$0 \to L \to Q \quad \text{and} \quad \bigoplus_1^\infty L \to Q \to 0$$

one finds that all the induced maps are zero.

(4) By (3) there is no *natural* embedding of M in DDM. However, I conjecture that there is always *some* embedding of M in DDM; it is easy to verify this when M is countable, or torsion, or torsionless (cf. (2)).

(5) If an infinitely generated M is 'Hom-reproduced' (i.e. $HM \cong M$) we easily deduce from the theorem that $|EM| = 2^{|M|}$. Moreover (2) M is locally free and separable, and so $\aleph_1$-free. However, examples of such modules other than finitely generated ones exist. Indeed if I is a non-measurable cardinal, by (3), 94·6,

$$M = (\bigoplus_I L) \oplus (\prod_I L)$$

is such a module. It would be interesting to know of a characterization of 'Hom-reproduced' modules.

(6) In appendix I of (4) Lacher showed that for a finitely generated L-module M there is a natural isomorphism $EEM \cong tM$. Our Corollary 2 (or (1)) also shows this, and moreover shows that in general this isomorphism fails.

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