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Hamiltonian Cycles Avoiding Prescribed Arcs in Tournaments

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Thomassen [6] conjectured that if I is a set of $k-1$ arcs in a k-strong tournament T, then $T-I$ has a Hamiltonian cycle. This conjecture was proved by Fraisse and Thomassen [3]. We prove the following stronger result. Let $T=(V,A)$ be a k-strong tournament on n vertices and let $X_1, X_2, \ldots, X_l$ be a partition of the vertex set V of T such that $|X_1| \leqslant |X_2| \leqslant \ldots \leqslant |X_l|$. If $k \geqslant \sum_{i=1}^{l-1} \lfloor |X_i|/2 \rfloor + |X_l|$, then $T - \cup_{i=1}^{l} \{xy \in A : x, y \in X_i\}$ has a Hamiltonian cycle. The bound on k is sharp.

1. Introduction

Thomassen [6] conjectured that if I is a set of $k-1$ arcs in a k-strong tournament T, then $T-I$ has a Hamiltonian cycle. This conjecture was proved by Fraisse and Thomassen [3]. This result is sharp since the deletion of a set I of k arcs from a k-strong tournament may create a vertex of indegree or outdegree 0. However, Fraisse and Thomassen [3] realized that, for some sets I, their bound was far from being the best possible (see, for example, Fraisse and Thomassen [6, Section 5]).

In this paper, we prove the following stronger result. Let $T=(V,A)$ be a k-strong tournament on n vertices and let $X_1, X_2, \ldots, X_l$ be a partition of the vertex set V of T such that $|X_1| \leqslant |X_2| \leqslant \ldots \leqslant |X_l|$. If $k \geqslant \sum_{i=1}^{l-1} \lfloor |X_i|/2 \rfloor + |X_l|$, then $T - \cup_{i=1}^{l} \{xy \in A : x, y \in X_i\}$ has a Hamiltonian cycle. The bound on k is sharp (see Theorem 4.2).

It is easy to see that the above theorem by Fraisse and Thomassen follows from our result. Indeed, let I be a set of arcs in a tournament T, let G be the undirected graph obtained by ignoring all orientations of the arcs of $T\langle I \rangle$, the subgraph of T which has arc set I and no isolated vertices, and let $Y_1, \ldots, Y_m$ be the vertex sets of the connected components of G so that $|Y_1| \leqslant \ldots \leqslant |Y_m|$. By our result, T has a Hamiltonian cycle avoiding the arcs in I if T is k-strong, where $k \geqslant k' = \sum_{i=1}^{m-1} \lfloor |Y_i|/2 \rfloor + |Y_m|$. But

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$k' \leqslant 1 + \sum_{i=1}^{m}(|Y_i| - 1) \leqslant 1 + \sum_{i=1}^{m} e(Y_i) = 1 + |I|$, where $e(Y_i)$ is the number of edges in the component of G induced by Y_i.

A simple analysis of the last calculation shows precisely when the Fraisse–Thomassen and our theorems provide the same value of strong connectivity of T – namely, when I consists of one tree, plus maybe some independent arcs. In all other cases our result gives a better bound. In particular, if $T\langle I\rangle$ is a union of (vertex) disjoint subtournaments of T of order $n_1, ..., n_m$ $(3 \leqslant n_1 \leqslant ... \leqslant n_m)$, then, to guarantee that $T - I$ has a Hamiltonian cycle, we need T to be $(\sum_{i=1}^{m} \binom{n_i}{2} + 1)$-strong by the Fraisse–Thomassen theorem and to be $(n_m + \sum_{i=1}^{m-1} \lfloor n_i/2 \rfloor)$-strong by our result.

Our proof is based on Hoffmann's theorem on circulations in networks [4] and a theorem by the third author on minimal spanning 1-diregular subgraphs in semicomplete multipartite digraphs [7]. The proof of the Fraisse-Thomassen theorem was also based on a non-trivial result from earlier work [5].

2. Terminology and notation

We assume that the reader is familiar with the standard terminology on graphs and digraphs, and refer the reader to Bondy and Murty [1].

By a *cycle* and a *path* in a directed graph we mean a directed simple cycle and path, respectively. Let D be a digraph. $V(D)$ $(A(D))$ denotes the vertex (arc) set of D. Two cycles Q and R (or paths) are *disjoint* if $V(Q) \cap V(R) = \emptyset$. A collection of vertex disjoint cycles of D is called a *cycle subgraph* of D. A cycle subgraph F of D consisting of disjoint cycles $C_1, ..., C_t$ will be denoted $F = C_1 \cup ... \cup C_t$. A digraph D is *strong* if there exists a path from x to y and a path from y to x for every choice of distinct vertices x, y of D. A digraph D is *k-strong* $(k > 0)$ if $D - X$ is strong for any subset X of the vertices of D with $|X| < k$. Let G and H be subgraphs of D and let $P = v_1v_2...v_{k-1}v_k$ be a path in D. P is called a *(G, H)-path* if $v_1 \in V(G)$, $v_k \in V(H)$ and $\{v_2, ..., v_{k-1}\} \cap (V(G) \cup V(H)) = \emptyset$.

Let $C = x_1x_2...x_cx_1$ be a cycle in D. Then we shall usually denote x_{i+1} by x_i^+ and x_{i-1} by x_i^-, where $i = 1, 2, ..., c$ and all subscripts are taken modulo c. For a set X of vertices in C, we denote $X^+ = \{x^+ : x \in X\}$ and $X^- = \{x^- : x \in X\}$. When we use such notation, the digraph D considered has a spanning cycle subgraph F, thus every vertex x of D has a unique predecessor x^- and a unique succesor x^+ on the unique cycle in F containing x. The subpath of a cycle C from a vertex v to a vertex w will be denoted by $C[v, w]$.

If D has an arc $xy \in A(D)$, then we sometimes use the notation $x \rightarrow y$ and say that x *dominates* y and y *is dominated by* x. For disjoint sets X and Y of vertices in D, we say that X *strongly dominates* Y, and use the notation $X \Rightarrow Y$, if there is no arc from Y to X. For arbitrary sets X and Y of vertices in D, $(X, Y)_D = \{xy \in A(D) : x \in X, y \in Y\}$. In particular, if $|X| = 1$, then $(X, X)_D$ is empty.

A digraph H is called *semicomplete* if every two distinct vertices of H are adjacent. A semicomplete digraph without cycles of length two is a *tournament*. Let $X_1, X_2, \ldots, X_p$ $(p \geqslant 2)$ be a partition of the vertex set of a semicomplete digraph H. Then the digraph $D = H - \cup_{i=1}^{p}(X_i, X_i)_H$ is called *semicomplete p-partite* or *semicomplete multipartite* (when the value of p is not important). We call $X_1, X_2, ..., X_p$ the *colour classes* of D.

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3. Preliminaries

In this section we describe some results which will be important tools for the proof of our main result (Theorem 4.1). The following result is a very special case of a theorem proved by the third author [7].

Theorem 3.1. *Let D be a semicomplete p-partite digraph, with colour classes $X_1, X_2, ..., X_p$ and let F be a spanning cycle subgraph of D with the minimum possible number of cycles t. Then either $t = 1$, or the cycles of F can be labelled $C_1, ..., C_t$ such that the following holds: there is a pair of indices $q, q' \in \{1, 2, ..., p\}$ (q and q' may be equal), such that every arc to C_1 from the outside is in $(X_q^-, X_q^+)_D$ and every arc from C_t to outside is in $(X_{q'}^-, X_{q'}^+)_D$.*

We recall the classical theorem by Hoffmann, characterizing the existence of a feasible circulation in a network with upper and lower bounds on the arcs. Below we use the following notation. If X is a subset of the vertex set of a digraph D, then we denote by $\bar{X}$ the set $V(D) \setminus X$. If r is a function on the arc set of D, then $r(X, \bar{X}) = \sum\{r(uv) : uv \in (X, \bar{X})\}$. Let N be a network. A flow f in N is called an *integer* flow if $f(a)$ is integer for every arc a in N. A *circulation* is a flow of value zero in N.

Theorem 3.2. [4] *Let $N = (V, A, \ell, c)$ denote a network with vertex set V, arc set A and lower (upper) bound $\ell(a)$ ($c(a)$) on every arc $a \in A$. Suppose $\ell(a), c(a)$ are non-negative integers for each $a \in A$. There exists an integer circulation f in N such that $\ell(a) \leqslant f(a) \leqslant c(a)$ for all $a \in A$ if and only if $c(\bar{X}, X) \geqslant \ell(X, \bar{X})$ for all proper subsets X of V.*

For a proof of this theorem see, for example, Ford and Fulkerson [2, p. 50]. We finish this section by the following simple but useful lemma. The proof is left for the reader.

Lemma 3.1. *Let T be a tournament, and let $Y_1, Y_2, \ldots, Y_s$ ($s \geqslant 1$) be disjoint sets of vertices in T and let x and y be arbitrary distinct vertices in $V(T) - (Y_1 \cup Y_2 \cup \ldots \cup Y_s)$. If there exist k disjoint (x, y)-paths in T, then there exist at least $k - \sum_{i=1}^{s} \lfloor |Y_i|/2 \rfloor$ disjoint (x, y)-paths in $T - \cup_{i=1}^{s}(Y_i, Y_i)$.*

4. Results

Theorem 4.1. *Let $T = (V, A)$ be a k-strong tournament on n vertices, and also, let $X_1, X_2, \ldots, X_l$ ($l \geqslant 1$) be a partition of V ($V = \cup_{i=1}^{l} X_i, X_i \cap X_j = \emptyset$ for every $i \neq j$). Set $D = T - \cup_{i=1}^{l}(X_i, X_i)_T$, $D_i = D \cup (X_i, X_i)_T$, $D_{i,j} = D \cup (X_i, X_i)_T \cup (X_j, X_j)_T$ ($1 \leqslant i, j \leqslant l$). If $1 \leqslant |X_1| \leqslant |X_2| \leqslant \ldots \leqslant |X_l| \leqslant n/2$ and $k \geqslant |X_l| + \sum_{i=1}^{l-1} \lfloor |X_i|/2 \rfloor$, then the following hold.*

(a) *If $x \in X_i$ and $y \in X_j$ ($1 \leqslant i \neq j \leqslant l$), then there are $\lfloor |X_i|/2 \rfloor + \lfloor |X_j|/2 \rfloor + \lceil |X_l|/2 \rceil$ disjoint (x, y)-paths in $D_{i,j}$.*

(b) *If $x, y \in X_i$ ($x \neq y$), then there are $|X_i|$ disjoint (x, y)-paths in D_i. Furthermore there is an (x, y)-path in D.*

(c) *D is strong.*

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(d) *D contains a spanning cycle subgraph.*
(e) *D is Hamiltonian.*

Proof. (a) This easily follows from Lemma 3.1.

(b) From Lemma 3.1 we easily get that there are $|X_i|$ disjoint (x, y)-paths in D_i. By deleting all arcs in $(X_i, X_i)_T$, we can destroy at most $|X_i| - 1$ of these paths, since at most $|X_i| - 2$ paths can contain vertices from $X_i - \{x, y\}$, and only one path can be the arc xy. Thus there is an (x, y)-path in D.

(c) Let x and y be arbitrary distinct vertices of D. If $\{x, y\} \subseteq X_i$, then there is an (x, y)-path in D, because of (b). Therefore we may now assume that $x \in X_i$ and $y \in X_j$, where $i \neq j$. Let W be the maximal set of vertices such that for all $w \in W$, there is an (x, w)-path in D.

Assume that $W \cap X_j \neq \emptyset$ and let $w \in W \cap X_j$. From (b) there is a (w, y)-path in D. Now we see that there is a (x, y)-path in D.

Assume that $W \cap X_j = \emptyset$. Therefore, there is no (x, y)-path in D_j, and at most $|X_i| - 1$ disjoint (x, y)-paths in $D_{i,j}$, since each (x, y)-path in $D_{i,j}$ must include a vertex from $X_i - x$. This is a contradiction against (a), since $\lfloor |X_i|/2 \rfloor + \lceil |X_l|/2 \rceil \geqslant |X_i| > |X_i| - 1$.

Hence we have proved that there exists a (x, y)-path in D for an arbitrary choice of distinct x and y, which means that D is strong.

(d) Let $D' = (V' \cup V'', A(D'))$ be the digraph obtained from D by replacing each vertex $v \in V$ by two vertices v' and v'' joined by an arc from v' to v''. For each original arc $uv \in A$, D' contains the arc $u''v'$.

Let $N = (V' \cup V'', A(N), \ell, c)$ $(A(N) = A(D'))$ be the network obtained from D' by specifying the following lower and upper bounds for the arcs. Every arc of the kind $u''v'$ (corresponding to an original arc in D) has lower bound zero and an upper bound ∞. Every arc $x'x''$ where $x \in V$ has lower and upper bounds equal to one.

For a subset B of V, B' and B'' will stand for the sets $\{v' : v \in B\}$ and $\{v'' : v \in B\}$, respectively.

It is easy to see that every feasible integer circulation in N corresponds to a spanning cycle subgraph in D and *vice versa*. Hence, by Theorem 3.2, it suffices to prove that for every proper subset U of $V' \cup V''$ we have $c(\bar{U}, U) \geqslant \ell(U, \bar{U})$.

Assume that $c(\bar{U}, U) < \ell(U, \bar{U})$, where U is a proper subset of $V' \cup V''$. The vertex set V of D can be partitioned into the following sets: $Y = \{y \in V : y' \in U,\ y'' \in \bar{U}\}$, $Z = \{z \in V : z'' \in U,\ z' \in \bar{U}\}$, $R_1 = \{v \in V : \{v', v''\} \subseteq U\}$ and $R_2 = \{v \in V : \{v', v''\} \subseteq \bar{U}\}$. We have $|Z| < |Y|$ since $|Z| \leqslant c(\bar{U}, U)$ and $\ell(U, \bar{U}) = |Y|$.

Observe that an arc $uv \in A(D)$ between two vertices in $Y \subseteq V(D)$ will correspond to an arc $u''v'$ in N with $u'' \in \bar{U}$ and $v' \in U$, thus contributing ∞ to $c(\bar{U}, U)$. Therefore there is an $i \in \{1, 2, \ldots, l\}$ such that $Y \subseteq X_i$. If D has an arc yr_1 from Y to R_1, then $y''r_1' \in A(N)$ contributes ∞ to $c(\bar{U}, U)$. Hence, $R_1 \Rightarrow Y$. Analogously, $(R_1 \cup Y) \Rightarrow R_2$. Since $|Z| < |Y| \leqslant |X_i| \leqslant n/2$ and $Y \subseteq X_i$, we obtain that either $W_1 = R_1 - X_i$ or $W_2 = R_2 - X_i$ is non-empty. Since $X_1, \ldots, X_l$ form a partition of $V(D)$, there is j, $j \neq i$, so that $(W_1 \cup W_2) \cap X_j \neq \emptyset$.

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Assume that $|W_1 \cap X_j| \geqslant |W_2 \cap X_j|$ and choose a vertex w_1 in $W_1 \cap X_j$ and vertex $y \in Y$. Set $S = (W_2 \cap X_j) \cup Z \cup (X_i - Y)$. By (a) there exist at least $|X_i| + \lfloor |X_j|/2 \rfloor$ disjoint (y, w_1)-paths in $D_{i,j}$. However, $D_{i,j} - S$ has no (y, w_1)-paths and $|S| \leqslant \lfloor |X_j|/2 \rfloor + |Z| + |X_i| - |Y| < \lfloor |X_j|/2 \rfloor + |X_i|$; a contradiction.

If $|W_1 \cap X_j| < |W_2 \cap X_j|$, then choose a vertex w_2 in $W_2 \cap X_j$ and vertex $y \in Y$. Set $S = (W_1 \cap X_j) \cup Z \cup (X_i - Y)$. Now $D_{i,j} - S$ has no (w_2, y)-path and we obtain a contradiction as above.

(e) Assume that D is not Hamiltonian.

We first observe that D is a semicomplete multipartite digraph with colour classes $X_1, X_2, \ldots, X_l$. By (c) the digraph D is strong. By (d) D contains a spanning cycle subgraph. Let F be a spanning cycle subgraph of D with the minimal possible number of cycles t. Assume that $t \geqslant 2$. Let C_1,..., C_t be a labeling of the cycles of F determined in Theorem 3.1. By Theorem 3.1, there is a pair of indices $q, q' \in \{1, 2, ..., p\}$ such that the conclusion of the theorem holds. Let us fix the labelling and pair of indices above.

If $|V(C_1)| > n/2$, we reverse all the arcs in D and relabel the cycles in the spanning cycle subgraph corresponding to F. Hence, we may assume that $|V(C_1)| \leqslant n/2$. Moreover assume that $|V(C_1)|$ has the minimal possible value. Set $W = V(C_2) \cup \ldots \cup V(C_t)$. For all $i = 1, 2, \ldots, l$ set $z_i = |V(C_1) \cap X_i|$.

Assume that $z_q = |V(C_1)|/2$. There are $|V(C_1)|(|V(C_1)| - 1)/2$ arcs in $(V(C_1), V(C_1))_T$, and, by Theorem 3.1, there are at most $z_q(|X_q| - z_q)$ arcs in $(W, V(C_1))_D$. Let $i \in \{1, ..., l\}$. Since $z_i/|V(C_1)| \leqslant 1/2$, we have $|(X_i \cap W, X_i \cap V(C_1))_T| \leqslant z_i(|X_i| - z_i) \leqslant |V(C_1)|\lfloor |X_i|/2 \rfloor$. Now we obtain the following:

$$\begin{aligned}
|(V, V(C_1))_T| &\leqslant |V(C_1)|(|V(C_1)| - 1)/2 + z_q(|X_q| - z_q) + \textstyle\sum_{i=1}^{l} z_i(|X_i| - z_i) \\
&\leqslant |V(C_1)|(|V(C_1)| - 1)/2 + 2z_q(|X_q| - z_q) + \textstyle\sum_{\substack{i=1 \\ i \neq q}}^{l} |V(C_1)|\lfloor |X_i|/2 \rfloor \\
&= |V(C_1)|\left[|V(C_1)|/2 - 1/2 + |X_q| - |V(C_1)|/2 + \textstyle\sum_{\substack{i=1 \\ i \neq q}}^{l} \lfloor |X_i|/2 \rfloor\right] \\
&< |V(C_1)|\left[|X_q| + \textstyle\sum_{\substack{i=1 \\ i \neq q}}^{l} \lfloor |X_i|/2 \rfloor\right].
\end{aligned}$$

This implies that there is a vertex w in $V(C_1)$ such that

$$|(V, w)_T| < |X_q| + \sum_{i=1,\, i \neq q}^{l} \lfloor |X_i|/2 \rfloor \leqslant k,$$

which is a contradiction against the fact that T is k-strong. Therefore we have shown that $|V(C_1) \cap X_q| < |V(C_1)|/2$. Let

$$S = \{s \in V(C_1) : \text{ there exists an arc from } W \text{ to } s^+ \text{ in } D\}.$$

Note that, by Theorem 3.1, $S \subseteq X_q \cap V(C_1)$. Since $|V(C_1) \cap X_q| < |V(C_1)|/2$, the set $R = V(C_1) - (X_q \cup X_q^+)$ is not empty. We prove that $R \Rightarrow S$ in D. Assume that there exist $s \in S$ and $r \in R$ so that $sr \in A(D)$. There is vertex $w \in V(C_i)$ (for some i, $2 \leqslant i \leqslant l$) so that $ws^+ \in A(D)$. Since $w^+ \in X_q$ and $r^- \notin X_q$, the arc $r^-w^+ \in A(D)$ ($w^+r^- \notin A(D)$ because $w^{++} \notin X_q$). Replace the cycles C_1 and C_i by the cycles $C_1[r, s]r$ and $C_i[w^+, w]C_1[s^+, r^-]w^+$ in F. The new spanning cycle subgraph F' has t cycles as well. However, the first cycle of F' contains less vertices than C_1 does; a contradiction.

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Let X_j be a colour class of D so that $X_j \cap R$ is not empty. Let r be a vertex in $X_j \cap R$ and let w_q be a vertex in $X_q \cap W$. Consider two cases.

Case 1. S is a proper subset of $X_q \cap V(C_1)$
Let $x \in (X_q \cap C_1) - S$. For all colour classes, $X_i \neq X_q$ we have $|X_i \cap V(C_1)| > |X_i \cap W|$ because of the following. Assume that there is a colour class, $X_i \neq X_q$, with $|X_i \cap V(C_1)| \leqslant |X_i \cap W|$. Let $w \in X_i \cap W$ be arbitrary. Claim (a) implies that there exist $\lfloor |X_i|/2 \rfloor + |X_q|$ disjoint (w, x)-paths in $D_{i,q}$. However, $B = [(W \cap X_q) \cup S^+] \cup (X_i \cap V(C_1))$ separates x from w in $D_{i,q}$ and $|B| < |X_q| + \lfloor |X_i|/2 \rfloor$; a contradiction.

Since $|V(C_1)| \leqslant |V(D)|/2$, we obtain $|X_q \cap V(C_1)| \leqslant |X_q \cap W|$. Claim (a) implies that there exist $\lfloor |X_j|/2 \rfloor + |X_q|$ disjoint (w_q, r)-paths in $D_{j,q}$. However, $B = [(V(C_1) \cap X_q) \cup S^+] \cup (X_j \cap W)$ separates r from w_q in $D_{j,q}$ and $|B| < |X_q| + \lfloor |X_j|/2 \rfloor$; a contradiction.

Case 2. $S = X_q \cap V(C_1)$
Subcase 2a. $|X_j \cap V(C_1)| \geqslant |X_j \cap W|$ or $|S| \leqslant |X_q \cap W|$
Set $B = S^+ \cup (X_j \cap W)$. The digraph $D_{j,q} - B$ has no (w_q, r)-paths since $R \Rightarrow S$ in D. If $|X_j \cap V(C_1)| \geqslant |X_j \cap W|$, then $|B| < \lfloor |X_j|/2 \rfloor + |X_q|$ since $X_q \cap W$ is not empty. If $|S| \leqslant |X_q \cap W|$, then $|B| < \lfloor |X_q|/2 \rfloor + |X_j|$ since $X_j \cap V(C_1)$ is not empty. Each of the last two bounds for $|B|$ contradicts (a).

Subcase 2b. $|X_j \cap V(C_1)| < |X_j \cap W|$ and $|S| > |X_q \cap W|$
If $X_j \cap W \subseteq (X_q \cap W)^-$, then set $B = S^+ \cup (X_j \cap W)$. The digraph $D_{j,q} - B$ has no (w_q, r)-path. However, $|B| \leqslant |X_q| < \lfloor |X_j|/2 \rfloor + |X_q|$; a contradiction to (a).

If $(X_j \cap W) - (X_q \cap W)^- \neq \emptyset$, then set $B = (X_q \cap W) \cup (X_q \cap W)^- \cup (X_j \cap V(C_1))$. The digraph $D_{j,q} - B$ has no (w, x)-path, where $x \in V(C_1) \cap X_q$ and $w \in (X_j \cap W) - (X_q \cap W)^-$. However, $|B| < \lfloor |X_j|/2 \rfloor + |X_q|$; a contradiction to (a).

Hence, we have got a contradiction in both cases which implies that D is Hamiltonian. □

The bound for k in Theorem 4.1 is sharp because of the following theorem.

Theorem 4.2. *Let $2 \leqslant r_1 \leqslant r_2 \leqslant \ldots \leqslant r_l$ be arbitrary integers, then there exists a tournament T and a collection $X_1, X_2, \ldots, X_l$ of disjoint sets of vertices in T such that*

(i) *T is $(r_l - 1 + \sum_{i=1}^{l-1} \lfloor r_i/2 \rfloor)$-strong;*
(ii) *$|X_i| = r_i$ for $i = 1, 2, \ldots, l$;*
(iii) *$D = T - \cup_{i=1}^{l} (X_i, X_i)_T$ is not Hamiltonian.*

Proof. Let $k = r_l + \sum_{i=1}^{l-1} \lfloor r_i/2 \rfloor$. We construct a tournament T with the properties (i)-(iii): $V(T) = X_1 \cup X_2 \cup \ldots \cup X_l \cup Y_1' \cup Y_2' \cup Y_1'' \cup Y_2'' \cup Z$, where all the sets in the union are mutually disjoint. Let $|Y_1'| = |Y_2'| = |Y_1''| = |Y_2''| = k$, $|Z| = r_l - 1$, $X_j = \{x_{j1}, \ldots, x_{j,r_j}\}$, $X_j' = \{x_{j1}, \ldots, x_{j,\lfloor r_i/2 \rfloor}\}$ and $X_j'' = \{x_{j,\lfloor r_i/2 \rfloor + 1}, \ldots, x_{j,r_j}\}$ for every $j \in \{1, \ldots, l-1\}$. Set $S' = (\cup_{j=1}^{l-1} X_j') \cup Y_1' \cup Y_2'$, $S'' = (\cup_{j=1}^{l-1} X_j'') \cup Y_1'' \cup Y_2''$. The arc set of T is defined as follows. Let $S' \Rightarrow X_l \Rightarrow S'' \Rightarrow Z \Rightarrow S'$. If $s' \in S'$ and $s'' \in S''$, then $s' \to s''$ unless there exists $j \in \{1, \ldots, l-1\}$ so that both s' and s'' are from X_j in which case $s'' \to s'$. For every $i = \{', ''\}$ and $j \in \{1, \ldots, l-1\}$,

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$X_j^i \Rightarrow Y_1^i \Rightarrow Y_2^i \Rightarrow X_j^i$. The direction of the arcs between the vertices non-adjacent so far can be chosen arbitrarily.

To see that T is $(k-1)$-strong, note that the deletion of any $k-2$ vertices leaves at least one vertex y_i^j in each of the sets Y_i^j, and either (a) a vertex z in Z, or (b) an edge $s'' \to s'$ from some X_j'' to the corresponding X_j'. In case (a), there is a cycle $zy_1'y_2'y_1''y_2''z$ remaining, and in case (b) we have the cycle $s'y_1'y_2'y_1''y_2''s''s'$. In either case, every other vertex sends and receives at least one edge to/from the cycle, so the remaining digraph is strong.

Assume that $D = T - \cup_{i=1}^{l}(X_i, X_i)_T$ is Hamiltonian. In a Hamiltonian cycle of D, after every visit to X_l, the cycle must pass through Z before returning to X_l since $A(D-Z) \subseteq (S', S'')_T \cup (S', X_l)_T \cup (X_l, S'')_T$. However, there are more vertices of X_l than of Z, thus there is no Hamiltonian cycle. □

5. Conclusions and open problems

In fact, this paper is concerned with aspects of the following general question. Which sets B of edges of the complete graph K_n have the property that every k-strong orientation of K_n induces a Hamiltonian digraph on $K_n - B$? The Fraisse–Thomassen theorem says that this is the case whenever A contains at most $k-1$ edges. Here, it has been shown that a union of disjoint cliques of sizes $r_1, \ldots, r_l$ has the property, whenever $\sum_{i=1}^{l} \lfloor r_i/2 \rfloor + \max_{1 \leqslant i \leqslant l} \{\lceil r_i/2 \rceil\} \leqslant k$. This is the best possible result for unions of cliques. Also, it implies the Fraisse–Thomassen theorem.

It seems natural to investigate bounds for k in different cases of the set B. In particular, what are sharp bounds for k when B is a spanning forest of K_n consisting of m disjoint paths containing r_1, ..., r_m vertices, respectively? The same question can be asked if we replace 'paths' by 'stars' or by 'cycles' (in the last case 'spanning forest' should also be replaced by 'spanning cycle subgraph').

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