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SOME RESULTS ON INDEFINITE SUBLINEAR ELLIPTIC RADIAL PROBLEMS WITH SIMULTANEOUS DIRICHLET AND NEUMANN BOUNDARY CONDITIONS

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Abstract

Let B be a ball in $\mathbb{R}^N$, $N \geq 1$, let $m \in C(\overline{B})$ be a radial function that changes sign in B and let $0 < q < 1$. We consider the existence and nonexistence of (nontrivial) nonnegative solutions and positive solutions for semilinear elliptic problems of the form $-\Delta u = m(x)u^q$ in B, $u = \partial_\nu u = 0$ on ∂B, where ∂_ν is the exterior normal derivative.

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1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$, let $q \in (0, 1)$ and let $m \in C(\overline{\Omega})$ be a function that changes sign in Ω. Let us consider the problem

$$(\mathcal{P}) \quad \begin{cases} -\Delta u = m(x)u^q & \text{in } \Omega, \\ 0 \not\equiv u \geq 0 & \text{in } \Omega, \\ u = \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

where ∂_ν is the exterior normal derivative. To simplify the presentation of Theorem 1.1, we assume that the set

$$\Omega^+ := \{x \in \Omega : m(x) > 0\}$$

has a smooth boundary and a finite number of connected components.

Let us also introduce the problems

$$(\mathcal{P}_D) \quad \begin{cases} -\Delta u = m(x)u^q & \text{in } \Omega, \\ 0 \not\equiv u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\mathcal{P}_N) \quad \begin{cases} -\Delta u = m(x)u^q & \text{in } \Omega, \\ 0 \not\equiv u \geq 0 & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

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An overview of the main results and the existing literature concerning $(\mathcal{P}_D)$ and $(\mathcal{P}_N)$ can be found in the review paper [15] and its references. Of course, $(\mathcal{P})$ is closely related to these problems. In fact, to the best of our knowledge, the only known results for $(\mathcal{P})$ come from $(\mathcal{P}_D)$ and $(\mathcal{P}_N)$. We compile these results in the next theorem. As usual, we set $m^{\pm} := \max(\pm m, 0)$.

THEOREM 1.1. *The following assertions hold.*

(i) *If $(\mathcal{P})$ has a positive solution, then $\int_\Omega m < 0$; and if Ω^+ is connected and $(\mathcal{P})$ has a solution, then $\int_\Omega m < 0$* [2, *Corollaries 2.1 and 2.2*].

(ii) *The problem $(\mathcal{P})$ has at most one solution u such that $u > 0$ in Ω^+; and if Ω^+ is connected, then $(\mathcal{P})$ has at most one solution* [1, *Theorem 2.1*].

(iii) *There exists some $q_0 \in [0, 1)$ such that $(\mathcal{P})$ does not have a solution for $q \in (q_0, 1)$* [13, *Theorem 1.9*]

(iv) *Under further conditions on Ω^+ (which, in particular, imply $m \le 0$ in the set $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}$ for some $\delta > 0$), there exists $\mu_0 > 0$ such that for all $\mu \ge \mu_0$, $(\mathcal{P})$ has a solution with μm^- in place of m^-* [3, *Proposition 4.2*].

Let us also mention that, for any fixed $q \in (0, 1)$, one can construct a weight $m = m_q$ such that $(\mathcal{P})$ has a *positive* solution (see [13, Proposition 2.1]). In particular, q_0 in Theorem 1.1(iii) can be arbitrarily close to 1.

Let us add that the existence and qualitative properties of nonnegative (and nontrivial) solutions of the problem

$$\begin{cases} -\Delta u = \lambda u + m(x)|u|^{q-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where λ is a real parameter, have been recently studied in [10]. When the solution is strictly positive in the interior while simultaneously satisfying both Dirichlet and Neumann boundary conditions, it is referred to as a flat solution, while if the solution is nonnegative in the interior, it is called a compact support solution, since its support does not necessarily coincide with $\overline{\Omega}$. For further articles considering flat and compact support solutions for similar problems, we refer to [5–9] and the references therein.

Our aim in this paper is to study the existence and nonexistence of radial solutions for $(\mathcal{P})$ when Ω is a ball in $\mathbb{R}^N$, $N \ge 1$. Concerning this problem, employing energy methods, in Theorem 2.1, we provide a necessary condition for the existence of solutions for $(\mathcal{P})$, while in Theorem 2.3, we present a necessary condition for the existence of positive solutions for $(\mathcal{P})$. Also, in Theorem 2.5 (see also Remark 2.6 and Corollary 2.7), we give sufficient conditions for the existence of solutions for $(\mathcal{P})$, relying on a sub- and supersolution approach.

2. Main results

By a solution of $(\mathcal{P})$, we mean a strong solution, that is, a function $u \in W^{2,r}(\Omega)$ with $r > N$ (and, hence, $u \in C^1(\overline{\Omega})$) that satisfies the equation for the weak derivatives and

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the boundary conditions in the usual sense. If $u > 0$ in Ω, we call it a *positive* solution. Given $0 < r_0 < R$, we write

$$B_R := \{x \in \mathbb{R}^N : |x| < R\},$$
$$A_{r_0,R} := \{x \in \mathbb{R}^N : r_0 < |x| < R\}.$$

If f is a radial function, we write (with a slight abuse of notation) $f(x) = f(|x|) := f(r)$.

The first result is a necessary condition for the existence of solutions for $(\mathcal{P})$, which strengthens Theorem 1.1(i) and (iii). The idea for the proof is inspired by some of the computations in the proofs of [12, Theorems 2.7 and 3.4].

THEOREM 2.1. *Let $\Omega := B_R$ and let $m \in C(\overline{\Omega})$ be a radial function. Assume that there exists $r_0 > 0$ such that:*

- *$m \geq 0$ in B_{r_0};*
- *$m \leq 0$ in $A_{r_0,R}$;*
- *$r \to m(r)$ is differentiable and nonincreasing in (r_0, R).*

If $(\mathcal{P})$ has a solution, then

$$\int_{B_{r_0}} m^+ < \frac{1-q}{1+q} \int_{A_{r_0,R}} m^-. \tag{2.1}$$

REMARK 2.2.

(i) Note that (2.1) says that if $(\mathcal{P})$ has a solution, then

$$q < \frac{-\int_{B_R} m}{\int_{B_R} |m|}. \tag{2.2}$$

(ii) If $m \leq 0$ in B_{r_0} and $m \geq 0$ in $A_{r_0,R}$, then $(\mathcal{P})$ does not have a solution. Indeed, if u is solution, from Hopf's lemma for strong solutions [16, Theorem 7.11] applied in $A_{r_0,R}$, we deduce that $u \equiv 0$ in $A_{r_0,R}$. Hence, by the strong maximum principle for strong solutions [16, Theorem 7.10], $u \equiv 0$ in B_R.

(iii) In [14, Theorem 1.6], it is proved that if the first three conditions of Theorem 2.1 hold but the inequality in (2.1) is the opposite, then there exists a solution u of $(\mathcal{P}_N)$ with $u > 0$ in Ω. Theorem 2.1 says that u *cannot* be a solution of $(\mathcal{P})$.

PROOF. Suppose $(\mathcal{P})$ has a solution u. From the strong maximum principle, $u > 0$ in B_{r_0} and so, in particular, $u > 0$ in Ω^+. Since $u(Sx)$ is also a solution if S is an isometry of $\mathbb{R}^N$, we infer from the first assertion in Theorem 1.1(ii) that u is radial. Let us first assume that $u > 0$ in B_R. Now, in $(0, r_0)$, the equation reads as $-(r^{N-1}u')' = r^{N-1}m^+(r)u^q$. Thus, $(r^{N-1}u')' \leq 0$ for all such r. Since $u'(0) = 0$, integrating in $(0, r)$ gives $r^{N-1}u'(r) \leq 0$ and so $u'(r) \leq 0$ (and u' is not identically 0). In other words, $u(r)$ is nonincreasing. Thus,

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$$-r_0^{N-1}u'(r_0) = -\int_0^{r_0}(s^{N-1}u'(s))'\,ds = \int_0^{r_0} s^{N-1}m^+(s)u(s)^q\,ds$$
$$> u(r_0)^q\int_0^{r_0} s^{N-1}m^+(s)\,ds$$

and so

$$\int_0^{r_0} s^{N-1}m^+(s)\,ds < \frac{-r_0^{N-1}u'(r_0)}{u(r_0)^q}. \tag{2.3}$$

On the other hand, in (r_0, R), the equation can be written as $r^{N-1}m^-(r) = (r^{N-1}u'(r))'/u(r)^q$. Integrating by parts in (r_0, R) gives

$$\begin{aligned}\int_{r_0}^R s^{N-1}m^-(s)\,ds &= \int_{r_0}^R \frac{(s^{N-1}u'(s))'}{u(s)^q}\,ds\\ &= \left(\frac{s^{N-1}u'(s)}{u(s)^q}\right)\Big|_{r_0}^R - \int_{r_0}^R s^{N-1}u'(s)(-qu(s)^{-q-1}u'(s))\,ds\\ &= \lim_{r\to R}\frac{r^{N-1}u'(r)}{u(r)^q} - \frac{r_0^{N-1}u'(r_0)}{u(r_0)^q} + q\int_{r_0}^R s^{N-1}\frac{(u'(s))^2}{u(s)^{1+q}}\,ds.\end{aligned} \tag{2.4}$$

We claim that $\lim_{r\to R} r^{N-1}u'(r)/u(r)^q = 0$. Indeed, integrating in (r, R) with $r > r_0$, since $u'(R) = 0$,

$$-r^{N-1}u'(r) = \int_r^R (s^{N-1}u'(s))'\,ds = \int_r^R s^{N-1}m^-(s)u(s)^q\,ds.$$

Hence, $u'(r) \le 0$ and $u(r)$ is also nonincreasing in (r_0, R). Therefore, $u(s) \le u(r)$ for $s \in (r, R)$, and then

$$0 \le -r^{N-1}u'(r) \le u(r)^q\int_r^R s^{N-1}m^-(s)\,ds.$$

Dividing by $u(r)^q$, the claim follows. Moreover, from (2.4),

$$-\frac{r_0^{N-1}u'(r_0)}{u(r_0)^q} = \int_{r_0}^R s^{N-1}m^-(s)\,ds - q\int_{r_0}^R s^{N-1}\frac{(u'(s))^2}{u(s)^{1+q}}\,ds. \tag{2.5}$$

For $r \in [r_0, R]$, we now define the energy functional

$$E(r) := \frac{1}{2}(u'(r))^2 + \frac{m(r)}{1+q}u(r)^{1+q}.$$

We have $E(R) = 0$ (because $u(R) = u'(R) = 0$) and

$$E'(r) = u'(r)u''(r) + \frac{m'(r)}{1+q}u(r)^{1+q} + m(r)u(r)^q u'(r).$$

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Since $u'' = m^-(r)u^q - ((N-1)/r)u'$ in (r_0, R) and $r \to m(r)$ is nonincreasing in (r_0, R),

$$E'(r) = u'(r)\left(m^-(r)u(r)^q - \frac{N-1}{r}u'(r)\right) + \frac{m'(r)}{1+q}u(r)^{1+q} - m^-(r)u(r)^q u'(r)$$
$$= -\frac{N-1}{r}(u'(r))^2 + \frac{m'(r)}{1+q}u(r)^{1+q} \le 0. \tag{2.6}$$

Hence, $E(r) \ge 0$ for all $r \in [r_0, R]$. Thus,

$$\frac{(u'(r))^2}{u(r)^{1+q}} \ge \frac{2}{1+q}m^-(r),$$

and so multiplying by r^{N-1} and integrating in (r_0, R),

$$\int_{r_0}^{R} s^{N-1}\frac{(u'(s))^2}{u(s)^{1+q}}\,ds \ge \frac{2}{1+q}\int_{r_0}^{R} s^{N-1}m^-(s)\,ds.$$

Now, applying this bound to (2.5),

$$-\frac{r_0^{N-1}u'(r_0)}{u(r_0)^q} \le \int_{r_0}^{R} s^{N-1}m^-(s)\,ds - \frac{2q}{1+q}\int_{r_0}^{R} s^{N-1}m^-(s)\,ds = \frac{1-q}{1+q}\int_{r_0}^{R} s^{N-1}m^-(s)\,ds.$$

Finally, recalling (2.3), we see that

$$\int_0^{r_0} s^{N-1}m^+(s)\,ds < \frac{1-q}{1+q}\int_{r_0}^{R} s^{N-1}m^-(s)\,ds$$

and this part of the theorem is proved.

To conclude the proof, suppose there exists a solution u of $(\mathcal{P})$ that vanishes for some r. Let r^* be the smallest r such that $u(r) = 0$. From the strong maximum principle, $r^* > r_0$. In addition, u is a positive solution of $(\mathcal{P})$ in B_{r^*}. Therefore, by the first part of the proof,

$$\int_{B_{r_0}} m^+ < \frac{1-q}{1+q}\int_{A_{r_0,r^*}} m^- \le \frac{1-q}{1+q}\int_{A_{r_0,R}} m^-. \qquad \square$$

When $N = 1$, we can also give a lower bound on q and $\int_{B_{r_0}} m^+$ for the existence of *positive* solutions for $(\mathcal{P})$.

THEOREM 2.3. *Let $\Omega := (-R, R)$ and $m \in C^1(\overline{\Omega})$ be an even function. Assume that there exists $r_0 > 0$ such that:*

- *$m \ge 0$ in B_{r_0};*
- *$m \le 0$ in $A_{r_0,R}$;*
- *$x \to m(x)$ is nonincreasing in $(0, R)$.*

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Let

$$C_{m^-} := \int_0^R M^-(x)\,dx \quad \text{where } M^-(x) := \begin{cases} \dfrac{1}{R-x}\displaystyle\int_x^R m^-(s)\,ds & \text{if } x \geq r_0, \\ M^-(r_0) & \text{if } x < r_0. \end{cases} \tag{2.7}$$

If $(\mathcal{P})$ *has a positive solution, then*

$$q > \frac{-\int_0^R m}{\int_0^R m + 2C_{m^-}}. \tag{2.8}$$

REMARK 2.4.

(i) Let us note that $C_{m^-} < R\|m^-\|_\infty$ and that also $\int_0^R m + 2C_{m^-} > 0$ by the final paragraph of the following proof. In addition, note that (2.8) can be written as

$$\int_0^{r_0} m^+ > \int_{r_0}^R m^- - \frac{2q}{1-q} C_{m^-}. \tag{2.9}$$

(ii) In view of Theorem 2.5, we *cannot* expect a similar result for solutions of $(\mathcal{P})$ that may vanish in Ω.

(iii) Under the assumptions of the above theorem, recalling (2.2), we see that if $(\mathcal{P})$ has a positive solution, then

$$\frac{-\int_0^R m}{\int_0^R m + 2C_{m^-}} < q < \frac{-\int_0^R m}{\int_0^R |m|},$$

or equivalently,

$$\int_{r_0}^R m^- - \frac{2q}{1-q} C_{m^-} < \int_0^{r_0} m^+ < \frac{1-q}{1+q}\int_{r_0}^R m^-.$$

PROOF. Let u be a positive solution of $(\mathcal{P})$. Then, u is even, and multiplying by u^{-q} and integrating by parts,

$$-\int_0^R m = \int_0^R u'' u^{-q} = u' u^{-q}|_0^R + q\int_0^R \frac{(u')^2}{u^{1+q}} = q\int_0^R \frac{(u')^2}{u^{1+q}}, \tag{2.10}$$

where we have used $u'(0) = 0$ (because u is even) and $\lim_{x\to R} u'(x)/u(x)^q = 0$ (this follows exactly as in the proof of Theorem 2.1). We now define $E(x)$ also as in the proof of Theorem 2.1, but for all $x \in (0, R)$, that is,

$$E(x) := \frac{1}{2}(u'(x))^2 + \frac{m(x)}{1+q} u(x)^{1+q}.$$

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As in (2.6) (with $N = 1$), we have $E'(x) = (m'(x)/(1+q))u(x)^{1+q}$ for all x and so, since $E(R) = 0$, integrating gives $E(x) = -\int_x^R (m'(s)/(1+q))u(s)^{1+q}\,ds$ and then

$$\frac{1}{2}(u'(x))^2 = \int_x^R \frac{-m'(s)}{1+q}u(s)^{1+q}\,ds - \frac{m(x)}{1+q}u(x)^{1+q}.$$

Therefore, since u and m are nonincreasing,

$$\begin{aligned}\frac{(u'(x))^2}{u(x)^{1+q}} &= 2\int_x^R \frac{-m'(s)}{1+q}\Big(\frac{u(s)}{u(x)}\Big)^{1+q}\,ds - \frac{2}{1+q}m(x)\\ &\le \frac{2}{1+q}\Big(\int_x^R -m'(s)\frac{u(s)}{u(x)}\,ds - m(x)\Big).\end{aligned} \tag{2.11}$$

We claim that

$$\frac{u(s)}{u(x)} < \frac{R-s}{R-x} \quad \text{for } r_0 \le x < s < R. \tag{2.12}$$

Indeed, note first that since $u'(R) = 0$, $u'(z) = -\int_z^R u''(t)\,dt$; and since $u(R) = 0$,

$$u(x) = \int_x^R \Big(\int_z^R u''(t)\,dt\Big)dz = \int_x^R (t-x)u''(t)\,dt \quad \text{for all } x \in (0,R). \tag{2.13}$$

In addition, for $s \le t \le R$, let $f(t) = (t-s)/(t-x)$. Then, f is increasing and so $f(t) < f(R)$ if $t \in [s,R)$, that is,

$$t - s < \frac{R-s}{R-x}(t-x). \tag{2.14}$$

Now, using (2.13) and (2.14), since $u'' \ge 0$ in (r_0, R), for $r_0 \le x < s < R$,

$$\frac{u(s)}{u(x)} = \frac{\int_s^R (t-s)u''(t)\,dt}{\int_x^R (t-x)u''(t)\,dt} \le \frac{\int_s^R (t-s)u''(t)\,dt}{\int_s^R (t-x)u''(t)\,dt} < \frac{R-s}{R-x},$$

and the claim is proved. We now divide our argument into two cases.

Case 1: $x \in [r_0, R)$. Recalling that m is nonincreasing, from (2.11) and (2.12), and integrating by parts,

$$\begin{aligned}\frac{(u'(x))^2}{u(x)^{1+q}} &< \frac{2}{1+q}\Big(\int_x^R -m'(s)\frac{R-s}{R-x}\,ds - m(x)\Big)\\ &= \frac{2}{1+q}\Big(\frac{-1}{R-x}\Big(m(s)(R-s)\big|_x^R + \int_x^R m(s)\,ds\Big) - m(x)\Big)\\ &= \frac{2}{1+q}\frac{1}{R-x}\int_x^R m^-(s)\,ds = \frac{2}{1+q}M^-(x).\end{aligned} \tag{2.15}$$

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Case 2: $x \in (0, r_0)$. Using (2.11), from the computation made above and arguing as in Case 1,

$$\begin{aligned}\frac{(u'(x))^2}{u(x)^{1+q}} &< \frac{2}{1+q}\Big(\int_x^{r_0} -m'(s)\frac{u(s)}{u(x)}\,ds + \int_{r_0}^{R} -m'(s)\frac{u(s)}{u(x)}\,ds - m(x)\Big)\\ &\le \frac{2}{1+q}\Big(\int_x^{r_0} -m'(s)\,ds + m(r_0) - \frac{1}{R-r_0}\int_{r_0}^{R} m(s)\,ds - m(x)\Big)\\ &= \frac{2}{1+q}M^-(r_0). \end{aligned} \tag{2.16}$$

Finally, from (2.10), (2.15) and (2.16),

$$-\int_0^R m < \frac{2q}{1+q}\Big(r_0 M^-(r_0) + \int_{r_0}^R M^-\Big) = \frac{2q}{1+q}C_{m^-}$$

and so $q(2C_{m^-} + \int_0^R m) > -\int_0^R m > 0$. Thus, (2.8) is proved. □

We now give a sufficient condition for the existence of a solution for $(\mathcal{P})$.

THEOREM 2.5. *Let $\Omega := B_R$ and let $m \in C(\overline{\Omega})$ be a radial function that changes sign in Ω. Assume that there exists $r_0 > 0$ such that $0 \not\equiv m \le 0$ in $A_{r_0,R}$. For $r \in (0, R)$ and $\rho \in (r_0, R)$, let*

$$I_1(r) = \int_0^r s^{N-1}m^+(s)\,ds, \quad I_2(r) = \int_0^r t^{1-N}I_1(t)\,dt, \quad m_-(\rho) := \min_{s\in[\rho,R]} m^-(s).$$

If there exists some $\overline{r} \in (r_0, R)$ such that

$$I_1(\overline{r})^{1-q}\Big[I_1(\overline{r})(R-\overline{r}) + \frac{2}{1-q}\overline{r}^{N-1}I_2(\overline{r})\Big]^q \le m_-(\overline{r})\frac{1-q}{1+q}\overline{r}^{N-1}(R-\overline{r})^{1+q}, \tag{2.17}$$

then $(\mathcal{P})$ has a solution.

REMARK 2.6. Since $I_1(\overline{r}) \le \|m^+\|_\infty \overline{r}^N/N$ and $I_2(\overline{r}) \le \|m^+\|_\infty \overline{r}^2/(2N)$, one can see that if

$$\|m^+\|_\infty \le \frac{1-q}{1+q}N \sup_{r\in(r_0,R)}\Big\{m_-(r)\frac{(R-r)^{1+q}}{r(r/(1-q)+R-r)^q}\Big\},$$

then (2.17) holds. Note that clearly the supremum is attained at some $\overline{r} \in (r_0, R)$.

PROOF. Let $\overline{r}$ be given by (2.17), and define

$$\gamma := \frac{2}{1-q} > 2, \quad k := \Big(\frac{m_-(\overline{r})}{\gamma(\gamma-1)}\Big)^{1/(1-q)}, \quad C := \frac{k\gamma(R-\overline{r})^{\gamma-1}\overline{r}^{N-1}}{I_1(\overline{r})} \tag{2.18}$$

and

$$\overline{u}(r) := \begin{cases} w(r) := k(R-r)^\gamma & \text{if } \overline{r} \le r \le R,\\ v(r) := w(\overline{r}) + C\int_r^{\overline{r}} t^{1-N}I_1(t)\,dt & \text{if } 0 \le r < \overline{r}. \end{cases}$$

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We note that $\overline{u}(R) = \overline{u}'(R) = \overline{u}'(0) = 0$. In addition, by the choice of C,

$$w'(\overline{r}) = -k\gamma(R-\overline{r})^{\gamma-1} = -C\overline{r}^{1-N}I_1(\overline{r}) = v'(\overline{r})$$

and hence $\overline{u} \in C^1(\overline{B}_R)$. Furthermore, since $\gamma - 2 = \gamma q$ and recalling the definition of k, for $r \in (\overline{r}, R)$,

$$\begin{aligned}-\Delta\overline{u}(r) &= -\overline{u}''(r) - \frac{N-1}{r}\overline{u}'(r) = -k\gamma(\gamma-1)(R-r)^{\gamma-2} + \frac{N-1}{r}k\gamma(R-r)^{\gamma-1}\\ &\geq -k\gamma(\gamma-1)(R-r)^{\gamma-2} = -k\gamma(\gamma-1)(R-r)^{\gamma q}\\ &= -m_-(\overline{r})k^q(R-r)^{\gamma q} \geq -m^-(r)k^q(R-r)^{\gamma q} = m(r)\overline{u}(r)^q.\end{aligned}$$

On the other hand, for $r \in (0, \overline{r})$, we claim that

$$-\Delta\overline{u}(r) = Cm^+(r) \geq m^+(r)v(0)^q \geq m^+(r)v(r)^q \geq m(r)\overline{u}^q.$$

Indeed, we only have to check that $C \geq v(0)^q$, since the other inequalities are clearly true. Moreover, recalling (2.18),

$$C \geq v(0)^q \Leftrightarrow C^{1-q} \geq (k(R-\overline{r})^\gamma C^{-1} + I_2(\overline{r}))^q. \tag{2.19}$$

Now, since $\gamma = 2/(1-q)$ and $\gamma - 1 = (1+q)/(1-q)$,

$$(k\gamma)^{1-q} = \frac{m_-(\overline{r})}{\gamma^q(\gamma-1)} = m_-(\overline{r})\left(\frac{1-q}{2}\right)^q\frac{1-q}{1+q}$$

and so (again using $(\gamma-1)(1-q) = 1+q$),

$$C^{1-q} = \frac{m_-(\overline{r})}{I_1(\overline{r})^{1-q}}\left(\frac{1-q}{2}\right)^q\frac{1-q}{1+q}(R-\overline{r})^{1+q}\overline{r}^{(N-1)(1-q)}. \tag{2.20}$$

Next, observe that

$$k(R-\overline{r})^\gamma C^{-1} = k(R-\overline{r})^\gamma\frac{I_1(\overline{r})}{k\gamma(R-\overline{r})^{\gamma-1}\overline{r}^{N-1}} = \frac{1-q}{2}\frac{R-\overline{r}}{\overline{r}^{N-1}}I_1(\overline{r}) \tag{2.21}$$

and thus, from (2.19), (2.20) and (2.21), it follows that $C \geq v(0)^q$ if

$$\frac{m_-(\overline{r})}{I_1(\overline{r})^{1-q}}\left(\frac{1-q}{2}\right)^q\frac{1-q}{1+q}(R-\overline{r})^{1+q}\overline{r}^{(N-1)(1-q)} \geq \left(\frac{1-q}{2}\frac{R-\overline{r}}{\overline{r}^{N-1}}I_1(\overline{r}) + I_2(\overline{r})\right)^q, \tag{2.22}$$

and a quick look at (2.17) shows that (2.22) holds. So, $\overline{u}$ is a positive weak supersolution of $(\mathcal{P})$.

To provide an arbitrarily small weak subsolution of $(\mathcal{P})$, we proceed as follows. Let $A_{r_1,r_2} \Subset \Omega^+$ and $0 < \underline{m} := \min_{\overline{A}_{r_1,r_2}} m^+$, and take $\underline{u} := \varepsilon\phi$, with $\varepsilon > 0$ small and $\phi > 0$ in A_{r_1,r_2}, where ϕ is defined such that $-\Delta\phi = \lambda_1(\underline{m})\underline{m}\phi = \lambda_1\phi$ in A_{r_1,r_2} and $\phi = 0$ on $\partial A_{r_1,r_2}$, and extend $\underline{u}$ by 0 to $\overline{\Omega}$. It is easy to check that if ε is small enough, then $-\Delta\underline{u} \leq m(r)\underline{u}^q$

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in A_{r_1,r_2} (and $-\Delta \underline{u} = 0 = m(r)\underline{u}^q$ in $\Omega \setminus \overline{A}_{r_1,r_2}$), and choosing ε smaller if necessary, we also have $\varepsilon\phi \le \overline{u}$ in $\overline{\Omega}$. Moreover, $\underline{u}|_{A_{r_1,r_2}} \in W^{2,2}(A_{r_1,r_2})$, $\underline{u}|_{\Omega\setminus\overline{A}_{r_1,r_2}} \in W^{2,2}(\Omega \setminus \overline{A}_{r_1,r_2})$ and $\underline{u} \in C(\overline{\Omega}) \cap W_0^{1,2}(\Omega)$. Taking these facts into account, using the definition of $\underline{u}$ and integrating by parts, we deduce that $\underline{u}$ is a weak subsolution of $(\mathcal{P})$, that is, $\int_\Omega \langle \nabla \underline{u}, \nabla w\rangle \le \int_\Omega m\underline{u}^q w$ for $0 \le w \in W^{1,2}(\Omega)$. Indeed, let ν_A denote the exterior unit normal to A_{r_1,r_2}. Then, $\partial_{\nu_A}\underline{u} < 0$ on $\partial A_{r_1,r_2}$. In addition, by the divergence theorem (see [4, Lemma A.1]), we see that the usual Green's formulae hold here. So, recalling that $w \ge 0$,

$$\begin{aligned}\int_\Omega \langle \nabla \underline{u}, \nabla w\rangle &= \int_{A_{r_1,r_2}} \langle \nabla \underline{u}, \nabla w\rangle = -\int_{A_{r_1,r_2}} w\Delta \underline{u} + \int_{\partial A_{r_1,r_2}} w\partial_{\nu_A}\underline{u}\\ &\le -\int_{A_{r_1,r_2}} w\Delta \underline{u} \le \int_{A_{r_1,r_2}} m\underline{u}^q w = \int_\Omega m\underline{u}^q w,\end{aligned}$$

as desired. Hence, the weak sub- and supersolutions method (see [11, Theorem 4.9]) gives a weak solution u of $(\mathcal{P})$, and by elliptic regularity, $u \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ for some $r > N$. □

Extending the solution given by the above theorem by 0 gives the following corollary.

COROLLARY 2.7. *Suppose $0 \in \Omega$ and let $m \in C(\overline{\Omega})$. If there exists some $B_R \subset \Omega$ such that m fulfils the hypothesis of Theorem 2.5 in B_R, then $(\mathcal{P})$ has a solution.*

In addition, as a direct consequence of Theorems 2.3 and 2.5, we get the existence of a solution of $(\mathcal{P})$ that must vanish somewhere in Ω.

COROLLARY 2.8. *Let $\Omega := (-R, R)$ and $m \in C^1(\overline{\Omega})$ be an even function.Assume that there exists $r_0 > 0$ such that:*

- *$m \ge 0$ in B_{r_0};*
- *$m \le 0$ in $A_{r_0,R}$;*
- *$x \to m(x)$ is nonincreasing in $(0, R)$.*

Let C_{m^-} be given by (2.7) and let

$$0 < q < \frac{\int_{r_0}^R m^-}{\int_{r_0}^R m^- + 2C_{m^-}}. \tag{2.23}$$

If m satisfies (2.17) and

$$\int_0^{r_0} m^+ \le \int_{r_0}^R m^- - \frac{2q}{1-q}C_{m^-}, \tag{2.24}$$

then there exists an even solution u of $(\mathcal{P})$ such that $u > 0$ in $(-r^*, r^*)$ and $u \equiv 0$ in $[r^*, R] \cup [-R, -r^*]$ for some $r^* \in (r_0, R)$.

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PROOF. The existence of u follows directly from Theorem 2.5. Also, note that (2.23) implies that the right member of (2.24) is positive. Now, recalling Theorem 2.3 (and 2.9), from (2.24), u must vanish at some $r^* \in (r_0, R)$, and it is easy to see that then it vanishes in $[r^*, R] \cup [-R, -r^*]$ using a convexity argument. □

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