Feedback

On Note 98.27: J. R. Nurcombe writes: On page 494, remark (2), it is stated that 'Quasimonotonic sequences . . . are often the largest class of sequences for which a given proposition is true'.

This is, in general, false. For the proposition in question, if $a_{2n-1} = \frac{1}{2^n}$, $a_{2n} = \frac{1}{n^2}$, $(n = 1, 2, 3, \ldots)$, then $\sum a_n$ converges and $na_n \to 0$, but the sequence (a_n) is not quasimonotonic.

The correct statement, and what was originally intended, is: 'Quasimonotonic sequences . . . often give results which, in a certain sense, are best possible.'

In what sense best possible? It is shown in [3] and [5] in the Note, that no matter how slowly the sequence $(f(n))$ increases to infinity, there are positive sequences (a_n) for which $\frac{a_n}{a_{n+1}} \geqslant 1 - \frac{f(n)}{n}$, $\sum a_n$ converges, but $\limsup_{n\to\infty} na_n = 1$. Thus corollary 2(i) cannot be improved by replacing λ there by an increasing, unbounded sequence of positive numbers.

On 'On the product $\prod_{n=2}^{\infty} \zeta(n)$': Nick Lord writes: In Theorem 5 of the interesting article [1], if we apply Hadamard's inequality to the columns rather than the rows of the determinant D_n of the matrix $\left(k^{-\frac{1}{2}(i+1)}\right)_{1 \leqslant i,k \leqslant n}$ we obtain the bound

$$D_n^2 \leqslant n.\sum_{i=2}^{n+1} 2^{-i}.\sum_{i=2}^{n+1} 3^{-i}.\ldots.\sum_{i=2}^{n+1} n^{-i} < n.\frac{1}{1.2}.\frac{1}{2.3}.\ldots.\frac{1}{(n-1)n} = \frac{1}{(n-1)!^2}$$

on replacing each sum by its sum to infinity. Thus $|D_n| < \frac{1}{(n-1)!}$ which both improves the bound in Theorem 5 and answers the author's query in the ensuing remark because $D_n \to 0$ as $n \to \infty$.

Reference

1. I. Ferit Öktem, On the product $\prod_{n=2}^{\infty} \zeta(n)$, *Math. Gaz.* **99** (November 2015) pp. 422-426.

On 'The 'hitchhiker triangle' and the problem of perimeter = area' Kevin Davis writes: The investigation of triangles in [1] is related in part to Tunnell's Theorem which gives an almost complete answer to an ancient problem studied by the Greeks and later the Arabs. It is to determine whether a given positive integer n is the area of some right triangle all of whose sides are rational numbers. Such numbers are called 'congruent'. This sequence of congruent numbers is A003273 in OEIS.

The statement of Tunnell's theorem is accessible to many students and could form the basis of a investigation:

https://doi.org/10.1017/mag.2016.30 Published online by Cambridge University Press

Let n be an odd, square-free natural number. Consider the two conditions

A) n is congruent;

B) the number of triples of integers (x, y, z) satisfying $2x^2 + y^2 + 8z^2 = n$ is equal to twice the number of triples satisfying $2x^2 + y^2 + 32z^2 = n$.

Then A) implies B); and if a weak form of the Birch-Swinnerton-Dyer conjecture is true, then B) implies A).

However simple and accessible the statement of Tunnell's theorem appears, it is from a 20th century proof encompassing L-functions of elliptic curves, modular forms etc. The proof of Tunnell's theorem forms the skeleton around which Neal Koblitz built his excellent book introducing elliptic curves [1].

Reference

1. Tony Crilly and Colin R. Fletcher, The 'hitchhiker triangle' and the problem of perimeter = area, *Math. Gaz.* **99** (November 2015) pp. 402-415.
2. Neal Koblitz, *Introduction to elliptic curves and modular forms* (2nd edn), Springer (1993).

On 'An octet of circles', Bob Burn writes: It is a delight to be told in [1] of Johnson's theorem, a kind of converse of a familiar triangle theorem, which crops up in reference [4] in the Article.

However, some notes on the paper may help readers.

On Page 468 it is affirmed that Johnson's original paper has no diagram. However in the first line of Johnson's proof, repeated almost verbatim in paragraph 3, the words '(see Figure 1 of the next paper)' have been deleted. In fact Johnson's diagram appears on page 163 of reference [2]. It would have been helpful to the reader to have reproduced this diagram here.

At the bottom of page 468, references [3] and [4] can be found by Googling 'Mathematical Repository IV (1819)'. The website cut-the-knot is huge and I was not able to trace these papers directly. In [3] Bogomolny asserts that Johnson's Theorem had appeared 100 years earlier and gave the reference [4]. On examining [4], I found that the theorem there is the much better-known theorem that if H is the orthocentre of a triangle ABC, then the circumradii of triangles ABC, HBC, HCA and HAB are equal. This theorem is an immediate consequence of the sine rule. Johnson's theorem is in fact the converse of this, where three concurrent equal circles are given and the orthocentre is found. Johnson's theorem does not appear in reference [4], so Bogomolny's assertion in [3] does not hold from the evidence that he provided.

At the bottom of page 472 from Figures 3 and 4, the symmetry referred to is a half-turn. That is a rotation, but not a reflection. So there is no reversal of orientation.

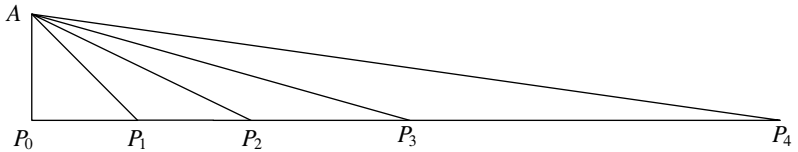
https://doi.org/10.1017/mag.2016.30 Published online by Cambridge University Press

Reference

1. Michael Sewell, An octet of circles, *Math. Gaz.* **99** (November 2015) pp. 468-473.

On 99.39, Tony Robin writes: The diagram below shows Subramaniam's general diagram when $\angle P_0P_1A = 45° = \frac{\pi}{4}$. In general $AP_n = P_nP_{n+1}$. If $AP_0 = 1$, then clearly $P_0P_1 = 1$ and so the other lengths can be calculated using the recurrence relations

$$AP_n^2 = 1 + P_0P_n^2; \qquad P_0P_{n+1} = P_0P_n + AP_n.$$



This means that we can write both the cosecant and cotangent of angles of the form $\pi/2^n$ can be written in terms of expressions involving square roots. For acute angles, θ, we have $\sin\theta < \theta < \tan\theta$, and if the angle is small these values are close together. This also means that $\operatorname{cosec}\theta > \frac{1}{\theta} > \cot\theta$.

We can draw up a table of values as follows. The interested reader might like to draw up their own spreadsheet for this situation.

n	P_0P_n	$P_0P_n^2$	AP_n^2	AP_n	m	Bounds for π	
0	0	0	1	1	2	Infinite	2
1	1	1	2	1.414214	4	4	2.828427125
2	2.414214	5.82842712	6.82842712	2.613126	8	3.313708499	3.061467459
3	5.027339	25.2741424	26.2741424	5.125831	16	3.182597878	3.121445152
4	10.15317	103.086869	104.086869	10.2023	32	3.151724907	3.136548491
5	20.35547	414.345062	415.345062	20.38002	64	3.144118385	3.140331157
6	40.73548	1659.37965	1660.37965	40.74776	128	3.14222363	3.141277251
7	81.48324	6639.51843	6640.51843	81.48938	256	3.141750369	3.141513801
8	162.9726	26560.0737	26561.0737	162.9757	512	3.141632081	3.14157294
9	325.9483	106242.295	106243.295	325.9498	1024	3.14160251	3.141587725
10	651.8981	424971.179	424972.179	651.8989	2048	3.141595118	3.141591422
11	1303.797	1699886.72	1699887.72	1303.797	4096	3.14159327	3.141592346
12	2607.594	6799548.87	6799549.87	2607.595	8192	3.141592808	3.141592577

The first column is a counter, the next four columns give the values of AP_n, P_0P_n and their squares. The sixth column is the denominator of the angle AP_nP_0 in terms of π. The final columns give bounds for the value of π.

doi:10.1017/mag.2016.30

https://doi.org/10.1017/mag.2016.30 Published online by Cambridge University Press