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METASTABILITY FOR TELEGRAPH PROCESSES IN A DOUBLE-WELL POTENTIAL

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Abstract

We study equations with nonlinearity in the form of a double-well potential, randomised by a velocity-switching (telegraph) stochastic process. If the speed parameters of the randomisation are small, then this dynamics has one metastable uncertainty interval and two invariant attractors. The probabilities of leaving the metastable interval through the upper boundary are determined, as well as characteristics of the first crossing times. Invariant measures are also found. When and if the direction of the telegraph process velocity coincides with the direction of the periodic change in potential, the system can go into a metastable state, having received a time window for the interwell transition. The obtained results can be used as an alternative to stochastic resonance models.

Keywords: Piecewise deterministic process; first-passage time; invariant measure; stochastic resonance; climate models

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Secondary 86A08

1. Introduction

Anomalous relativistic [17] (see also [9]) diffusion processes with finite propagation speed often seem more natural for modelling than the classical diffusion approach. For a detailed discussion, see [32]. There are several different terminological and ideological systems for dealing with such classes of anomalous diffusions. In physics, people prefer to say 'continuous-time random walks' [22], 'flip-flop' [4], or 'zig-zag processes' [9, 35]. In mathematics, the terms 'piecewise-deterministic processes' [8] or 'telegraph processes' [14, 19, 31], are commonly used. The study of classical stochastic differential equations with regime switching (and their applications) is also widely represented; see, for example, [25, 40].

In this paper we study a piecewise deterministic nonlinear system whose dynamics is specified by a double-well potential and a two-state Markov random process. It turns out that such a system can have metastable states as an important feature.

This problem setting continues the author's previous studies on stochastic differential equations based on piecewise deterministic processes. These studies have recently begun with Ornstein–Uhlenbeck-like processes, where the telegraph process replaces the Wiener process in the Langevin equation. Invariant measures and distributions of first-passage times have been obtained for such stochastic dynamics with its various modifications [26, 27, 28, 29, 30]. Here we develop similar ideas in a *nonlinear* setting.

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Dynamic systems subject to random perturbations are actively studied in both physical and mathematical literature. In the simplest case, such systems are usually based on deterministic dynamics, which follows the ordinary differential equation $\mathrm{d}\gamma(t) = -U'(\gamma(t))\,\mathrm{d}t$, $t > 0$, equipped with a random perturbation, where the function U is the potential.

Let x be a stationary point, i.e. $U'(x) = 0$. If $U''(x) > 0$, then the position x is asymptotically stable, i.e. $\gamma(t) \to x$ as $t \to +\infty$ for any starting point near x. On the contrary, if $U''(x) < 0$, then x is unstable and repulsive, i.e. any path that starts near x moves away from x. *The interplay between different kinds of stationary points under different randomisations of this equation is of special interest.* The first and most popular randomisation method is to add a stochastic term, such as white noise, to the deterministic differential equation [18]. Later, other randomisation options have emerged, for example the Lévy process [10], 'coloured' noise (e.g. the Ornstein–Uhlenbeck process) [38, 41] and the fractional Gaussian [15].

When the phases coexist, the phenomenon of hysteresis may occur, see the discussion in [7, 10, 37]. An introduction to metastability can be found in [3].

The simplest case of a one-dimensional setting has recently been generalised in a number of works on metastability effects arising in the case of diffusion processes that are stopped or reflecting on the boundary of a domain [11] and, more generally, on metastability at and near the surfaces that remain invariant under diffusion processes [12].

When using various randomisation methods, stochastic resonance can occur, where a small change in the input signal causes a large output signal in the system. Such dynamics, with a transition from one potential well to another, can be interpreted to qualitatively explain glacial cycles in the history of the Earth (from a warm state to an ice age and back) [18]. Similar phenomena occur in various applied settings, from biology to metallurgy [34]. For example, this approach is effective when modelling a single neuron [5, 16, 24, 26] (see also a recent review of these models in [6]) and artificial neuronal sets [33], as well as dynamics of competing populations [39] and chemical reaction systems [36]. See also detailed overviews of other applications in [13, 20, 23].

The idea of stochastic resonance is successfully used, for example, by adding white noise $\sigma(T)\,\mathrm{d}W(t)$ to a deterministic differential equation, where W is standard Brownian motion and the coefficient $\sigma(T)$ is chosen to *maximise the coefficient of spectral power amplification* [1, 24]. The second approach involves the so-called 'effective dynamics', based on a two-state Markov chain with a time-periodic infinitesimal generator, which artificially ensures a spatial interwell transition [18].

Our randomisation methodology consists in replacing the diffusion term $W(t)$ by a relativistic (finite-velocity) process $\mathbb{T}(t)$. The idea of such replacement has become more and more popular in physics [17, 32]. With this randomisation, the model receives certain time windows for interwell transitions instead of stochastic resonance options. In this setting, we present an infinitesimal generator of the process (Theorem 1), compare the probabilities for which attractor the process will end up on when the velocities are small (Theorem 3), and derive formulae for the average first-passage times when the process has already left the metastable position (Theorem 4). Invariant measures inside the attractors are also found (Theorem 6).

The paper is organised as follows. The next section is devoted to the problem setting and a description of the structure of the underlying dynamics. In Section 3 we analyse the distributions of passage times. Section 4 presents invariant distributions.

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2. Preliminaries and problem statement

Let $\mathbb{T}(t)$ be a telegraph process with alternating velocities c_0, c_1, $c_0 > c_1$: $\mathbb{T}(t) = \int_0^t c_{\xi(s)}\,\mathrm{d}s$, $t > 0$, where $\xi = \xi(t) \in \{0, 1\}$ is a two-state Markov process with alternating switching intensities λ_0, λ_1 [31]. Consider a stochastic process $X(t) = X^x(t)$ defined by the equation

$$\mathrm{d}X(t) = -U'(X(t))\,\mathrm{d}t + \mathrm{d}\mathbb{T}(t), \quad t > 0, \tag{1}$$

and the initial condition $X(0) = x$. The process $X = X(t)$ can be considered as a telegraph process $\mathbb{T}$ in the potential U.

Our previous studies [26, 27, 28, 29, 30] were based on the so-called Kac–Ornstein–Uhlenbeck processes, which are defined by the equation

$$\mathrm{d}\bar{X}(t) = \left(c_{\xi(t)} - \gamma_{\xi(t)}\bar{X}(t)\right)\mathrm{d}t, \quad t > 0,$$

i.e. the case of alternating single-well potentials $U_i = \gamma_i x^2/2 - c_i x$.

For the quasilinear setting (1), suppose that the function $U(x)$, $U \in C^2(-\infty, \infty)$, is a *double-well potential*, i.e. it has a local maximum at the point x_0, two local minima at the points x_-, x_+, $x_- < x_0 < x_+$, and $U''(x) \neq 0$, $x \in \{x_-, x_0, x_+\}$. As a typical example, we can use the symmetric function $U(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$.

The process $X(t)$ is continuous, piecewise continuously differentiable, and is formed by the sequential alternation of two deterministic patterns $\gamma_0(t, x)$ and $\gamma_1(t, x)$, which follow the equations

$$\frac{\partial \gamma_0(t,x)}{\partial t} = c_0 - U'(\gamma_0(t,x)), \quad \frac{\partial \gamma_1(t,x)}{\partial t} = c_1 - U'(\gamma_1(t,x)), \quad t > 0, \ \gamma_0(0,x) = \gamma_1(0,x) = x. \tag{2}$$

One pattern is replaced by another at random times τ_n, when the underlying Markov process $\xi(t)$ switches.

The initial value problems (2) are equivalent to the pair of integral equations

$$\int_x^{\gamma_0(t,x)} \frac{\mathrm{d}z}{c_0 - U'(z)} = t, \quad \int_x^{\gamma_1(t,x)} \frac{\mathrm{d}z}{c_1 - U'(z)} = t, \qquad t \geq 0, \tag{3}$$

which can be rewritten in the equivalent form

$$\Phi_0(\gamma_0(t,x)) = \Phi_0(x) + t, \quad \Phi_1(\gamma_1(t,x)) = \Phi_1(x) + t. \tag{4}$$

Here, Φ_0 and Φ_1 are rectifying diffeomorphisms defined, up to an arbitrary additive constant, by the equations

$$\Phi_0'(y) = \frac{1}{c_0 - U'(y)}, \qquad \Phi_1'(y) = \frac{1}{c_1 - U'(y)}. \tag{5}$$

Notice that the function $\Phi_i = \Phi_i(y)$ is continuous and monotone between the critical points of the potential $U_i = U(x) - c_i x$.

To avoid blow-up, assume that the integral $\int_x \mathrm{d}y/U_i'(y)$ diverges at any critical point of U_i:

$$\int_x \frac{\mathrm{d}y}{U_i'(y)} = \infty, \quad x \in \{x_-, x_0, x_+\}. \tag{6}$$

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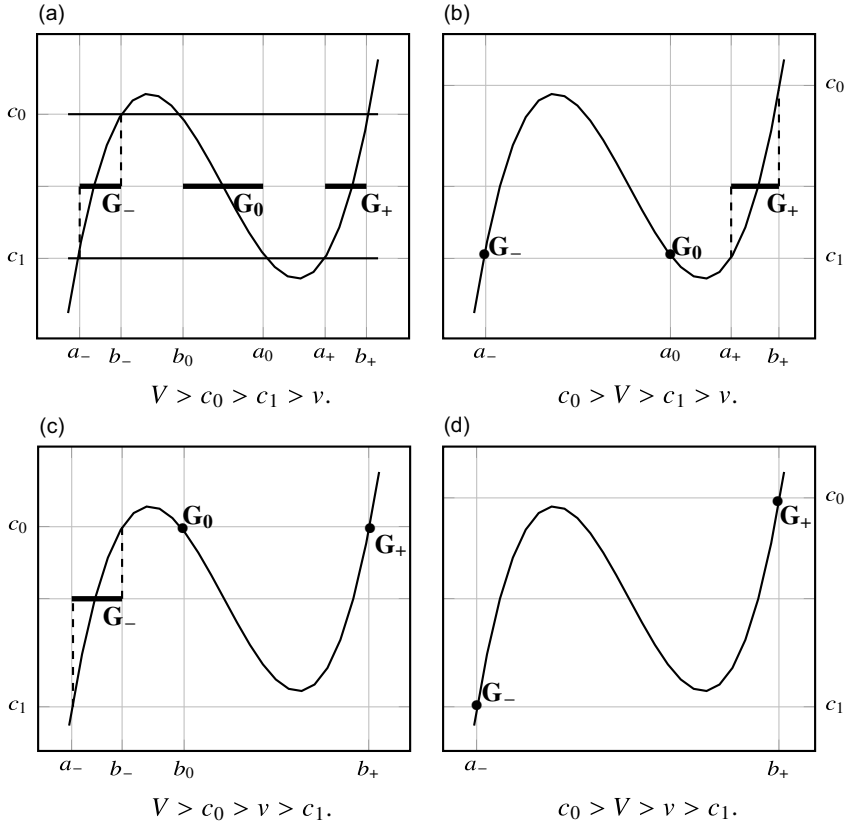


FIGURE 1. $U'(x)$ and G_0, $G_\pm$.

By definition (3), it follows that the pattern $\gamma_i(t, x)$ arising from the state (x, i), $i \in \{0, 1\}$, also satisfies the initial value problem

$$\frac{\partial \gamma_i}{\partial t}(t, x) = L_i^x[\gamma_i(t, x)], \quad t > 0, \ \gamma_i(t, x)\big|_{t\downarrow 0} = x, \tag{7}$$

where $L_i^x = -U_i'(x)(\partial/\partial x) = (c_i - U'(x))(\partial/\partial x)$, $i \in \{0, 1\}$.

The properties of the process X depend significantly on the velocity amplitudes c_0 and c_1.

A: An unstable uncertainty interval and two attractors

Let both parameters c_0 and c_1 be small enough that $V > c_0 > c_1 > v$, where V and v are, respectively, a local maximum and a local minimum of U'. In this case, both $U_0(x) = U(x) - c_0 x$ and $U_1(x) = U(x) - c_1 x$ are still double-well potentials with alternation between them. Let $a_0, a_\pm$ be the critical points of U_1, $a_- < a_0 < a_+$, and $b_0, b_\pm$ be the critical points of U_0, $b_- < b_0 < b_+$ (see Figure 1(a)).

Note that $X(t) \equiv a_-$, $X(t) \equiv a_+$, and $X(t) \equiv a_0$ are stationary solutions of (1) under the initial state $\xi(0) = 1$. The pattern $\gamma_1 = \gamma_1(t, x)$ is repelled by a_0, while being attracted to a_- or a_+. Respectively, $X(t) \equiv b_-$, $X(t) \equiv b_+$, and $X(t) \equiv b_0$ are stationary solutions under $\xi(0) = 0$, and pattern γ_0 is repelled by b_0, being attracted to points $b_\pm$.

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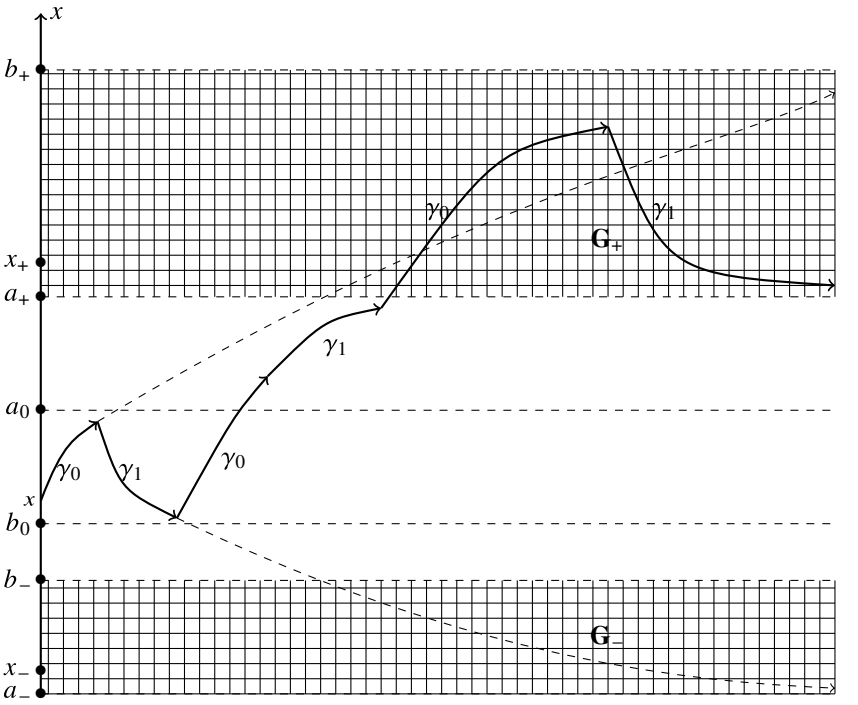


FIGURE 2. Sample path of X, case A.

As a result, the process X has two attractors, $G_- = [a_-, b_-] \subset (-\infty, b_0)$ and $G_+ = [a_+, b_+] \subset (a_0, +\infty)$, and a metastable set $G_0 = (b_0, a_0)$. The process X^x, starting from the metastable set G_0, $x \in G_0$, can temporarily oscillate inside, but once it leaves this interval through the upper or lower bound, it never returns, and then forever remains above a_0 or below b_0, respectively, being attracted to G_+ or G_-. See the sample of a path in Figure 2.

If the process $X = X^x(t)$ starts from $x = X^x(0) > a_0$, then after a finite transition time it ends up in G_+, and when starting from $x < b_0$, the process for almost surely (a.s.) finite time falls into G_-.

Note that both sets G_- and G_+ are invariant under dynamics:

$$\begin{aligned} &\mathbb{P}\{X(t) \in G_- \text{ for all } t > 0 \mid X(0) \in G_-\} = 1, \\ &\mathbb{P}\{X(t) \in G_+ \text{ for all } t > 0 \mid X(0) \in G_+\} = 1. \end{aligned} \tag{8}$$

Figure 3 shows plots of Φ_0 and Φ_1 in this case.

B: Alternating single- and double-well potentials, only one attractor

When c_0 becomes large, so that $c_0 > V$, the potential $U_0(x)$ has only one critical point b_+, becoming single-well, while U_1 is still double-well, and has three critical points. In this case, the process X has only one attractor G_+, see Figure 1(b). After some transition period, with probability 1 X ends up in G_+. A symmetric situation occurs when c_1 becomes less than v, see Fig. 1(c).

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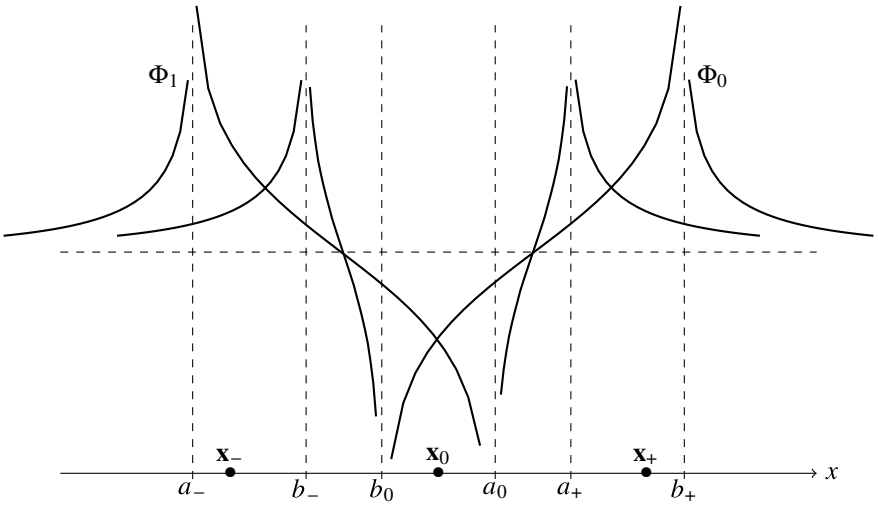


FIGURE 3. Functions $\Phi_0(x)$ and $\Phi_1(x)$, case A.

C: Alternating single-well potentials, one attractor

In the case $c_0 > V > v > c_1$, both potentials U_0 and U_1 become single-well. The dynamics has only one attractor $G = (a_-, b_+)$. When the process X begins in $x = X(0) \in G$, it randomly fluctuates between the attractive levels a_- and b_+. If it starts outside, then it ends at G after some transition.

Two other less interesting single-well situations with a single attractor are not shown in Figure 1. They correspond to $c_0 > c_1 > V > v$ and $c_1 < c_0 < v < V$. See Figure 4 for a diagram of the different regimes.

The process $\Xi = (X^x(t),\ \xi(t))$ is a Markov process with the state space $\mathbb{R} \times \{0, 1\}$. Let τ^0 be the time of first switching from state 0 to state 1; τ^1 corresponds to the reverse switching. Due to the Markov property, the following equalities in distribution hold:

$$\begin{aligned}
[X^x(t) \mid \xi(0) = 0] &\stackrel{\mathrm{D}}{=} \gamma_0(t, x) \cdot 1_{\{\tau^0 > t\}} + 1_{\{\tau^0 < t\}} \cdot \left[X^{\gamma_0(\tau^0, x)}(t - \tau^0) \mid \xi(0) = 1\right], \\
[X^x(t) \mid \xi(0) = 1] &\stackrel{\mathrm{D}}{=} \gamma_1(t, x) \cdot 1_{\{\tau^1 > t\}} + 1_{\{\tau^1 < t\}} \cdot \left[X^{\gamma_1(\tau^1, x)}(t - \tau^1) \mid \xi(0) = 0\right].
\end{aligned} \tag{9}$$

Based on (9), we can obtain an infinitesimal generator of Ξ.

Theorem 1. *The infinitesimal generator of Ξ is given by $\Lambda + \mathcal{L}$, where*

$$\Lambda = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix}$$

is the generator of the two-state Markov process ξ, and

$$\mathcal{L} = \mathcal{L}^x = \begin{pmatrix} L_0^x & 0 \\ 0 & L_1^x \end{pmatrix}. \tag{10}$$

See the proof in the Appendix.

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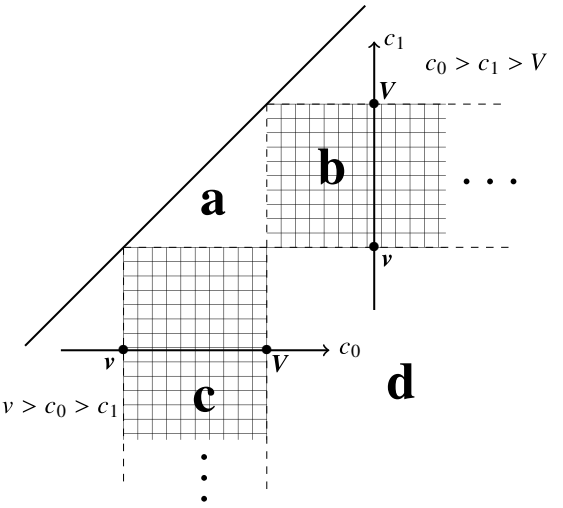


FIGURE 4. Scheme of different modes. Triangle **a**, $V > c_0 > c_1 > v$ (case A); half-bands **b** and **c**, corresponding to $c_0 > V > c_1 > v$ and $V > c_0 > v > c_1$ respectively (case B); quadrant **d**, $c_0 > V > v > c_1$ (case C). (Designations according to Figure 1.)

3. Passage times

Let $\mathcal{T}(x, y)$ be the time when the process $X(t)$ first reaches a given threshold y, starting from point x: $\mathcal{T}(x, y) = \inf\{t > 0 \mid X(0) = x,\ X(t) = y\}$. The distribution of this random variable is determined by the moment-generating function $\boldsymbol{u} = (u_0(q, x, y),\ u_1(q, x, y))^\top$,

$$\begin{aligned} u_0(q, x, y) &= \mathbb{E}[\exp(-q\mathcal{T}(x, y)) \mid \xi(0) = 0], \\ u_1(q, x, y) &= \mathbb{E}[\exp(-q\mathcal{T}(x, y)) \mid \xi(0) = 1], \end{aligned} \qquad q > 0.$$

We denote By $t_0^*(x, y)$ and $t_1^*(x, y)$ the time to reach level y without switching states, starting from $(x, 0)$ and $(x, 1)$, respectively. The value $t_i^*(x, y)$ is finite if the equation $\gamma_i(t, x) = y$ is satisfied for some positive t, $t = t_i^*(x, y)$. In this case we say that level y is reachable from state (x, i). Otherwise we set $t_i^*(x, y) = \infty$.

If y is reachable from the state (x, i), then both x and y belong to the domain of a common continuous piece of Φ_i; moreover, since (4) holds, $\Phi_i(y) > \Phi_i(x)$ and $t_i^*(x, y) = \Phi_i(y) - \Phi_i(x)$. Also, $\lim_{x \to y} t_i^*(x, y) = 0$.

By virtue of (5),

$$L_i^x t_i^*(x, y) = -L_i^x \Phi_i(x) = -1 \quad \text{when } t_i^*(x, y) \text{ is finite},\ i \in \{0, 1\}. \tag{11}$$

Let y be a level reachable from both $(x, 0)$ and $(x, 1)$. Conditioning on the first switching, like in (9), we obtain a system of integral equations:

$$\begin{cases} u_0(q, x, y) = \mathrm{e}^{-(q+\lambda_0)t_0^*(x,y)} + \displaystyle\int_0^{t_0^*(x,y)} \lambda_0 \mathrm{e}^{-(q+\lambda_0)\tau} u_1(q, \gamma_0(\tau, x), y)\, \mathrm{d}\tau, \\ u_1(q, x, y) = \mathrm{e}^{-(q+\lambda_1)t_1^*(x,y)} + \displaystyle\int_0^{t_1^*(x,y)} \lambda_1 \mathrm{e}^{-(q+\lambda_1)\tau} u_0(q, \gamma_1(\tau, x), y)\, \mathrm{d}\tau. \end{cases} \tag{12}$$

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On the contrary, in the case $t_i^*(x, y) = +\infty$, the corresponding integral equation of (12) can be interpreted as

$$u_i(q, x, y) = \int_0^\infty \lambda_i \mathrm{e}^{-(q+\lambda_i)\tau} u_{1-i}(q, \gamma_i(\tau, x), y)\, \mathrm{d}\tau, \quad i \in \{0, 1\}. \tag{13}$$

Theorem 2. *The moment-generating function* $\boldsymbol{u} = (u_0(q, x, y),\ u_1(q, x, y))^\top$ *of the first-passage time* $\mathcal{T}(x, y)$ *satisfies*

$$\big(\Lambda + \mathcal{L}\big)\boldsymbol{u}(q, x, y) = q\boldsymbol{u}(q, x, y), \quad q > 0, \tag{14}$$

where $\Lambda + \mathcal{L}$ *is the generator of the Markov process* Ξ.

Both entries $u_0(q, x, y)$ *and* $u_1(q, x, y)$ *of* $\boldsymbol{u}$ *separately obey telegraph [2] equations of the form*

$$\begin{aligned} &L_1^x L_0^x u_0(q, x, y) - \big[(q + \lambda_1)L_0^x + (q + \lambda_0)L_1^x\big]u_0(q, x, y) + q(q + \lambda_0 + \lambda_1)u_0(q, x, y) = 0, \\ &L_0^x L_1^x u_1(q, x, y) - \big[(q + \lambda_1)L_0^x + (q + \lambda_0)L_1^x\big]u_1(q, x, y) + q(q + \lambda_0 + \lambda_1)u_1(q, x, y) = 0. \end{aligned} \tag{15}$$

Equations (14) and (15) are provided with boundary conditions that depend on whether the level y is reachable from $(x, 0)$ and $(x, 1)$.

Proof. Apply L_0^x and L_1^x to the first and second equations of the system (12), respectively.

First, let the threshold y be reachable from $(x, 0)$. By virtue of (7) and (11), the first equation of (12) yields

$$\begin{aligned} L_0^x u_0(q, x, y) = (q + \lambda_0)\mathrm{e}^{-(q+\lambda_0)t_0^*(x,y)} - \lambda_0 \mathrm{e}^{-(q+\lambda_0)t_0^*(x,y)} u_1(q, \gamma_0(t_0^*(x, y), x), y) \\ + \int_0^{t_0^*(x,y)} \lambda_0 \mathrm{e}^{-(q+\lambda_0)\tau} \frac{\partial}{\partial \tau}[u_1(q, \gamma_0(\tau, x), y)]\, \mathrm{d}\tau. \end{aligned}$$

Integrating by parts, we can see that

$$\begin{aligned} L_0^x u_0(q, x, y) = (q + \lambda_0)\mathrm{e}^{-(q+\lambda_0)t_0^*(x,y)} - \lambda_0 u_1(q, x, y) \\ + (q + \lambda_0) \int_0^{t_0^*(x,y)} \lambda_0 \mathrm{e}^{-(q+\lambda_0)\tau} u_1(q, \gamma_0(\tau, x), y)\, \mathrm{d}\tau. \end{aligned}$$

Proceeding similarly with the second equation of (12), both in the case of finite and infinite $t_1^*(x, y)$, we obtain the system

$$\begin{cases} L_0^x u_0(q, x, y) = (q + \lambda_0)u_0(q, x, y) - \lambda_0 u_1(q, x, y), \\ L_1^x u_1(q, x, y) = -\lambda_1 u_0(q, x, y) + (q + \lambda_1)u_1(q, x, y), \end{cases} \tag{16}$$

which in matrix form coincides with (14).

Equations (15) follow from (16) by eliminating variables. Indeed, applying L_1^x to the first equation of system (16), we get $L_1^x L_0^x u_0(q, x, y) = (q + \lambda_0)L_1^x u_0(q, x, y) - \lambda_0 L_1^x u_1(q, x, y)$. From the second equation of (16) we obtain

$$L_1^x L_0^x u_0(q, x, y) = (q + \lambda_0)L_1^x u_0(q, x, y) + \lambda_0\lambda_1 u_0(q, x, y) - \lambda_0(q + \lambda_1)u_1(q, x, y).$$

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The first equation of (16) can be rewritten as $\lambda_0 u_1(q,x,y) = (q+\lambda_0)u_0(q,x,y) - L_0^x u_0(q,x,y)$, which gives the first equation of (15); the second equation is obtained similarly. □

If $\mathbb{P}\{\mathcal{T}(x,y) < \infty\} = 1$, then the mean value $\boldsymbol{m}(x,y) = (m_0(x,y),\ m_1(x,y))^\top$,

$$m_0(x,y) = \mathbb{E}[\mathcal{T}(x,y) \mid \xi(0) = 0], \quad m_1(x,y) = \mathbb{E}[\mathcal{T}(x,y) \mid \xi(0) = 1],$$

can be found by differentiation:

$$m_0(x,y) = -\frac{\partial u_0(q,x,y)}{\partial q}\bigg|_{q=0}, \quad m_1(x,y) = -\frac{\partial u_1(q,x,y)}{\partial q}\bigg|_{q=0}.$$

Theorem 2 implies the following result, general for Markov processes.

Corollary 1. *Let the mean value of $\mathcal{T}(x,y)$ be finite. Then the mean vector $\boldsymbol{m}(x,y)$ satisfies the equation*

$$(\Lambda + \mathcal{L}^x)\boldsymbol{m}(x,y) = -\mathbf{1} = (1,1)^\top. \tag{17}$$

Let us first consider the case of small c_0 and c_1, i.e. $V > c_0 > c_1 > v$, see Figure 1(a).

3.1. Small c_0 and c_1: Probabilities of leaving the uncertainty interval

In the case $V > c_0 > c_1 > v$, let the initial point x be inside the interval G_0, i.e. $X(0) = x \in G_0 = (b_0, a_0)$ – case A, Figure 1(a).

Since both a_0 and b_0 are repulsive, a trajectory that has crossed one of these thresholds once never returns to the interval (b_0, a_0). We are interested in the conditional probabilities of exiting the interval $G_0 = (b_0, a_0)$ through the upper bound a_0,

$$p_0(x) = \mathbb{P}\{\mathcal{T}(x,a_0) < \infty \mid \xi(0) = 0\}, \quad p_1(x) = \mathbb{P}\{\mathcal{T}(x,a_0) < \infty \mid \xi(0) = 1\}. \tag{18}$$

The moment-generating functions $u_0(q,x,a_0)$ and $u_1(q,x,a_0)$, $x \in G_0$, are increasing. Further, $0 < u_0(q,x,a_0) < 1$ and $0 < u_1(q,x,a_0) < 1$ for $b_0 < x < a_0$. Moreover,

$$\lim_{q \downarrow 0}[u_0(q,x,a_0)] = \mathbb{P}\{\mathcal{T}(x,a_0) < \infty \mid \varepsilon(0) = 0\} = p_0(x),$$

$$\lim_{q \downarrow 0}[u_1(q,x,a_0)] = \mathbb{P}\{\mathcal{T}(x,a_0) < \infty \mid \varepsilon(0) = 1\} = p_1(x)$$

are the probabilities of leaving the interval (b_0, a_0) through the upper bound a_0, starting from the position $X(0) = x$ with the initial state $\xi(0) = 0$ and $\xi(0) = 1$, respectively.

The probabilities $p_0(x)$ and $p_1(x)$ depend on the constants

$$B_0 = \int_{b_0}^{a_0} \frac{\Psi(y)}{c_0 - U'(y)}\,\mathrm{d}y, \qquad B_1 = \int_{b_0}^{a_0} \frac{\Psi(y)}{c_1 - U'(y)}\,\mathrm{d}y. \tag{19}$$

Here, $\Psi(y) = \exp(\lambda_0\Phi_0(y) + \lambda_1\Phi_1(y))$.

The integrals in (19) converge, since

$$\left|\frac{\Psi(y)}{c_i - U'(y)}\right| \le \text{const} \cdot \exp(\lambda_i\Phi_i(y)) \cdot |\Phi_i'(y)|, \quad y \in G_0,$$

and

$$\int_{b_0}^{a_0} \mathrm{e}^{\lambda_i\Phi_i(y)}|\Phi_i'(y)|\,\mathrm{d}y = \left|\exp(\lambda_i\Phi_i(y))\Big|_{y=b_0+}^{y=a_0-}\right| < \infty.$$

See Figure 3.

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Theorem 3. *Let $X(0)=x$, $x\in(b_0,\ a_0)$. The probabilities $p_0(x)$ and $p_1(x)$ defined by (18) have the form*

$$p_0(x)=\mathbb{P}\{\mathcal{T}(x,a_0)<\infty\mid\xi(0)=0\}=1-B_0^{-1}\int_x^{a_0}\frac{\exp\left(\lambda_0\Phi_0(y)+\lambda_1\Phi_1(y)\right)}{c_0-U'(y)}\,\mathrm{d}y, \tag{20}$$

$$p_1(x)=\mathbb{P}\{\mathcal{T}(x,a_0)<\infty\mid\xi(0)=1\}=B_1^{-1}\int_{b_0}^{x}\frac{\exp\left(\lambda_0\Phi_0(y)+\lambda_1\Phi_1(y)\right)}{c_1-U'(y)}\,\mathrm{d}y, \tag{21}$$

where B_0, B_1 are defined by (19).

Proof. By definition, it follows that

$$p_0(x)\big|_{x\uparrow a_0}=1,\quad p_1(x)\big|_{x\downarrow b_0}=0,\quad q>0. \tag{22}$$

Conditioning on the first switching, like in (9), similarly to (12), we obtain

$$u_0(0,x,a_0)=p_0(x)=\mathrm{e}^{-\lambda_0 t_0^*(x,a_0)}+\int_0^{t_0^*(x,a_0)}\lambda_0\mathrm{e}^{-\lambda_0\tau}p_1(\gamma_0(\tau,x))\,\mathrm{d}\tau. \tag{23}$$

Since the process X leaving the interval G_0 through the lower bound forever remains below b_0, instead of (23) and (13) we obtain

$$u_1(0,x,a_0)=p_1(x)=\int_0^{t_1^*(x,b_0)}\lambda_1\mathrm{e}^{-\lambda_1\tau}p_0(\gamma_1(\tau,x))\,\mathrm{d}\tau. \tag{24}$$

Here, $t_0^*(x,a_0)=\Phi_0(a_0)-\Phi_0(x)$ and $t_1^*(x,b_0)=\Phi_1(b_0)-\Phi_1(x)$; function Φ_0 is increasing, and function Φ_1 is decreasing on (b_0,a_0).

From (23) and (24), in the usual way by applying L_0^x and L_1^x followed by integration by parts, differential equations similar to (15) and (16) are derived: first $(\Lambda+\mathcal{L})\mathbf{p}(x)=\mathbf{0}$ and $\mathbf{p}(x)=(p_0(x),p_1(x))^\top$, and then

$$\begin{aligned}&L_1^xL_0^xp_0(x)-(\lambda_1L_0^x+\lambda_0L_1^x)p_0(x)=0,\\&L_0^xL_1^xp_1(x)-(\lambda_1L_0^x+\lambda_0L_1^x)p_1(x)=0.\end{aligned} \tag{25}$$

See Theorem 2.

Let

$$v_0(x)=L_0^x[p_0](x),\qquad v_1(x)=L_1^x[p_1](x). \tag{26}$$

Note that the functions $p_0(x)$ and $p_1(x)$ increase, $x\in G_0$.

Since $c_1<U'(x)<c_0$, $v_0(x)>0$ and $v_1(x)<0$ for all $x\in G_0$. Using this notation, equations (25) take the form

$$(c_1-U'(x))\frac{\partial v_0}{\partial x}(x)-\lambda_1v_0(x)-\lambda_0\frac{c_1-U'(x)}{c_0-U'(x)}v_0(x)=0,$$

$$(c_0-U'(x))\frac{\partial v_1}{\partial x}(x)-\lambda_1\frac{c_0-U'(x)}{c_1-U'(x)}v_1(x)-\lambda_0v_1(x)=0,$$

which can be rewritten as

$$\frac{\mathrm{d}v_0}{\mathrm{d}x}(x)=\psi(x)v_0(x),\qquad\frac{\mathrm{d}v_1}{\mathrm{d}x}(x)=\psi(x)v_1(x),\qquad b_0<x<a_0, \tag{27}$$

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where

$$\psi(x) = \frac{\lambda_0}{c_0 - U'(x)} + \frac{\lambda_1}{c_1 - U'(x)}. \tag{28}$$

From (27) and (28), by definition (5) we obtain

$$v_0(x) = A_0 \exp\left(\lambda_0 \Phi_0(x) + \lambda_1 \Phi_1(x)\right), \qquad v_1(x) = A_1 \exp\left(\lambda_0 \Phi_0(x) + \lambda_1 \Phi_1(x)\right)$$

with indefinite constants A_0 and A_1.

Taking into account boundary conditions (22), we obtain

$$p_0(x) = -\int_x^{a_0} \frac{v_0(y)}{c_0 - U'(y)}\,\mathrm{d}y + 1, \qquad p_1(x) = \int_{b_0}^{x} \frac{v_1(y)}{c_1 - U'(y)}\,\mathrm{d}y.$$

By the no-blow-up condition (6), we can see that $A_0 = B_0^{-1}$ and $A_1 = B_1^{-1}$, which gives the solution to equations (26) in the form (20) and (21). □

3.2. Starting near the attractor

Let the initial point x and the threshold y be located near (or inside) the attractor. We study only the case of the upper half-line, $x, y > a_0$, and the attractor G_+, see Figure 1(a) and (b). This is sufficient because the case of $x, y < b_0$ and the attractor G_-, Figure 1(a) and (c), is symmetric.

Since both levels a_+ and b_+ are attractive, and a_0, b_0 are repulsive, the process $X = X(t)$ increases a.s. moving along the set (a_0, a_+) and decreases moving along the set $[b_+, +\infty)$. Therefore, $\mathcal{T}(x, y)\big|_{x>y\geq a_0} = \infty$, $\mathcal{T}(x, y)\big|_{b_+\leq x<y} = \infty$ a.s. Inside the attractor G_+, the process $X = X(t)$ randomly oscillates between the attractive levels a_+ and b_+.

The mean value $\boldsymbol{m}(x, y) = (m_0(x, y),\ m_1(x, y))^\top$ can be expressed through the integrals

$$I_0(x, y) = \int_x^y \frac{\beta(z_0, y)\,\mathrm{d}z_0}{c_0 - U'(z_0)}, \qquad I_1(x, y) = \int_x^y \frac{\beta(z_1, y)\,\mathrm{d}z_1}{c_1 - U'(z_1)}, \tag{29}$$

$$J_0(x, y) = \int_{\Delta_0(x,y)} \frac{\beta(z_0, z_1)}{(c_0 - U'(z_0))(c_1 - U'(z_1))}\,\mathrm{d}z_0\,\mathrm{d}z_1, \tag{30}$$

$$J_1(x, y) = \int_{\Delta_1(x,y)} \frac{\beta(z_1, z_0)}{(c_0 - U'(z_0))(c_1 - U'(z_1))}\,\mathrm{d}z_0\,\mathrm{d}z_1. \tag{31}$$

Here, $\beta(x, y) = \Psi(x)/\Psi(y) = \exp\left(\lambda_0(\Phi_0(x) - \Phi_0(y)) + \lambda_1(\Phi_1(x) - \Phi_1(y))\right)$, and $\Delta_0(x, y)$, $\Delta_1(x, y) \subset \mathbb{R}^2$ are two triangles defined by

$$\Delta_0(x, y) = \{(z_0, z_1) \mid x < z_0 < y,\ z_0 < z_1 < y\},$$

$$\Delta_1(x, y) = \{(z_0, z_1) \mid x < z_1 < y,\ z_1 < z_0 < y\}.$$

According to the above definitions, in both cases $a_0 < x < y \leq a_+$ (with $U'(x)$, $U'(y) < c_1 < c_0$) and $x > y \geq b_+$ (with $U'(x)$, $U'(y) > c_0 > c_1$), the integrals $I_0(x, y)$, $I_1(x, y)$ and $J_0(x, y)$, $J_1(x, y)$ converge.

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Theorem 4.

(i) Let $x, y > a_0$, see Figure 1(a) and (b). The mean values $m_0(x, y)$ and $m_1(x, y)$ are given explicitly as follows.

- *In both cases $x < y < a_+$ and $x > y > b_+$,*

$$m_0(x, y) = I_0(x, y) + (\lambda_0 + \lambda_1)J_0(x, y), \tag{32}$$

$$m_1(x, y) = I_1(x, y) + (\lambda_0 + \lambda_1)J_1(x, y). \tag{33}$$

- *If $x < y$ and $y \in [a_+, b_+)$,*

$$m_0(x, y) = (1 + \lambda_0/\lambda_1)I_0(x, y) + (\lambda_0 + \lambda_1)J_0(x, y), \tag{34}$$

$$m_1(x, y) = 1/\lambda_1 + (\lambda_0 + \lambda_1)J_1(x, y). \tag{35}$$

- *If $x > y$ and $y \in (a_+, b_+]$,*

$$m_0(x, y) = 1/\lambda_0 + (\lambda_0 + \lambda_1)J_0(x, b_+), \tag{36}$$

$$m_1(x, y) = (1 + \lambda_0/\lambda_1)I_1(x, b_+) + (\lambda_0 + \lambda_1)J_1(x, b_+). \tag{37}$$

Here, I_0, I_1 and J_0, J_1 are defined by (29)–(31).

(ii) Let $x, y < b_0$, see Figure 1(a) and (c). The mean values $m_0(x, y)$ and $m_1(x, y)$ are determined by the following formulae, symmetrical to those given above.

- *In both cases $x > y > b_-$ and $x < y < a_-$,*

$$m_0(x, y) = I_0(x, y) + (\lambda_0 + \lambda_1)J_0(x, y), \tag{38}$$

$$m_1(x, y) = I_1(x, y) + (\lambda_0 + \lambda_1)J_1(x, y). \tag{39}$$

- *If $x > y$ and $y \in [a_-, b_-)$,*

$$m_0(x, y) = 1/\lambda_0 + (\lambda_0 + \lambda_1)J_0(x, y), \tag{40}$$

$$m_1(x, y) = (1 + \lambda_1/\lambda_0)I_1(x, y) + (\lambda_0 + \lambda_1)J_1(x, y). \tag{41}$$

- *If $x < y$ and $y \in (a_-, b_-]$,*

$$m_0(x, y) = (1 + \lambda_1/\lambda_0)I_0(x, b_+) + (\lambda_0 + \lambda_1)J_0(x, b_+), \tag{42}$$

$$m_1(x, y) = 1/\lambda_1 + (\lambda_0 + \lambda_1)J_1(x, b_+). \tag{43}$$

Notice that X is structured in such a way that if this process begins with $x \in (a_0, a_+]$ or with $x \in [b_+, +\infty)$, (symmetrically, $x \in [b_-, b_0)$ or $x \in (-\infty, a_-]$) then with probability 1 it ends up inside the set G_+ (G_-) and remains there forever, randomly fluctuating between two attracting thresholds a_+ and b_+.

In the rest of the paper, all proofs for the lower half-line ($x < b_0$) are symmetric to the proofs for $x > a_0$, so we present only the latter.

To prove Theorem 4, we need an auxiliary result that is important in itself.

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Lemma 1. *Under the conditions of Theorem 4, the mean value of $\mathcal{T}(x, y)$ is finite and the entries $m_0(x, y)$ and $m_1(x, y)$ of the mean vector $\boldsymbol{m}(x, y)$ are positive solutions of (17). Furthermore:*

- *If $a_0 < x < y < a_+$ ($b_0 > x > y > b_-$) or $x > y > b_+$ ($x < y < a_-$), then (17) is supplied with the zero boundary condition $\boldsymbol{m}(x, y)|_{x\to y} = \mathbf{0}$.*
- *If $x < y$ and $y \in [a_+, b_+)$,*

$$m_0(x, y)|_{x\to y} = 0, \qquad m_1(x, y)|_{x\to y} = 1/\lambda_1; \tag{44}$$

if $x > y$ and $y \in (a_-, b_-]$,

$$m_0(x, y)|_{x\to y} = 1/\lambda_0, \qquad m_1(x, y)|_{x\to y} = 0. \tag{45}$$

- *If $x > y$ and $y \in (a_+, b_+]$,*

$$m_0(x, y)|_{x\to y} = 1/\lambda_0, \qquad m_1(x, y)|_{x\to y} = 0; \tag{46}$$

if $x < y$ and $y \in [a_-, b_-)$,

$$m_0(x, y)|_{x\to y} = 0, \qquad m_1(x, y)|_{x\to y} = 1/\lambda_1. \tag{47}$$

Moreover, $m_0(x, y)$ and $m_1(x, y)$ satisfy separate second-order equations:

$$\begin{cases} L_1^x L_0^x m_0(x, y) - \big(\lambda_1 L_0^x + \lambda_0 L_1^x\big) m_0(x, y) = \lambda_0 + \lambda_1, \\ L_0^x L_1^x m_1(x, y) - \big(\lambda_1 L_0^x + \lambda_0 L_1^x\big) m_1(x, y) = \lambda_0 + \lambda_1. \end{cases} \tag{48}$$

Proof. First, note that if $a_0 < x < y < a_+$ or $x > y > b_+$, then $t_0^*(x, y)$ and $t_1^*(x, y)$ are both finite, $t_0^*(x, y)|_{x\to y} = t_1^*(x, y)|_{x\to y} = 0$, functions u_0 and u_1 follow the coupled integral equations (12), $\mathcal{T}(x, y) < \infty$ a.s., and $u_0(0, x, y) = u_1(0, x, y) = 1$. Differentiating in (12) with respect to q and then setting $q = 0$, we obtain the integral equations

$$\begin{cases} m_0(x, y) = t_0^*(x, y)\mathrm{e}^{-\lambda_0 t_0^*(x,y)} + \displaystyle\int_0^{t_0^*(x,y)} \lambda_0 \tau \mathrm{e}^{-\lambda_0 \tau}\, \mathrm{d}\tau + \int_0^{t_0^*(x,y)} \lambda_0 \mathrm{e}^{-\lambda_0 \tau} m_1(\gamma_0(\tau, x), y)\, \mathrm{d}\tau, \\ m_1(x, y) = t_1^*(x, y)\mathrm{e}^{-\lambda_1 t_1^*(x,y)} + \displaystyle\int_0^{t_1^*(x,y)} \lambda_1 \tau \mathrm{e}^{-\lambda_1 \tau}\, \mathrm{d}\tau + \int_0^{t_1^*(x,y)} \lambda_1 \mathrm{e}^{-\lambda_1 \tau} m_0(\gamma_1(\tau, x), y)\, \mathrm{d}\tau. \end{cases} \tag{49}$$

After simplification, the system (49) becomes

$$\begin{cases} m_0(x, y) = \dfrac{1 - \mathrm{e}^{-\lambda_0 t_0^*(x,y)}}{\lambda_0} + \displaystyle\int_0^{t_0^*(x,y)} \lambda_0 \mathrm{e}^{-\lambda_0 \tau} m_1(\gamma_0(\tau, x), y)\, \mathrm{d}\tau, \\ m_1(x, y) = \dfrac{1 - \mathrm{e}^{-\lambda_1 t_1^*(x,y)}}{\lambda_1} + \displaystyle\int_0^{t_1^*(x,y)} \lambda_1 \mathrm{e}^{-\lambda_1 \tau} m_0(\gamma_1(\tau, x), y)\, \mathrm{d}\tau. \end{cases} \tag{50}$$

Note that in cases A and B, the interval G_+ attracts both patterns, $\gamma_0(t, x)$ and $\gamma_1(t, x)$. Therefore, $t_i^*(x, y)|_{x\to y} = 0$ if $x < y < a_+$ or $x > y > b_+$. By (50), this yields the zero boundary condition for (17): $m_i(x, y)|_{x\to y} = 0$, $i \in \{0, 1\}$.

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If $x < y = a_+$, then $t_0^*(x, y)$ is still finite, $t_0^*(x, y)|_{x\uparrow y} = 0$, and $t_1^*(x, y) = \infty$. In this case, the first equations in (49) and (50) remain the same. Hence, $m_0(x, y)|_{x\uparrow y} = 0$. The second equation in (50) takes the form

$$m_1(x, y) = \frac{1}{\lambda_1} + \int_0^\infty \lambda_1 \mathrm{e}^{-\lambda_1 \tau} m_0(\gamma_1(\tau, x), y)\, \mathrm{d}\tau,$$

which gives the same differential equation as before and the boundary condition (44).

If $x > y = b_+$, then similar reasoning again leads to the boundary condition (46).

By eliminating variables (as in Theorem 2), we can see that $m_0(x, y)$ and $m_1(x, y)$ satisfy the two separate second-order equations (48). □

Proof of Theorem 4. This is similar to the proof of Theorem 3. It consists of two steps. First, we transform the system (48) of second-order equations into a system of first-order equations for the functions

$$v_0(x, y) := L_0^x m_0(x, y), \qquad v_1(x, y) := L_1^x m_1(x, y). \tag{51}$$

Note that (51) can be solved in the form

$$\begin{aligned} m_0(x, y) &= m_0(x, y)|_{x=y} - \int_x^y \frac{v_0(z_0, y)}{c_0 - U'(z_0)}\, \mathrm{d}z_0, \\ m_1(x, y) &= m_1(x, y)|_{x=y} - \int_x^y \frac{v_1(z_1, y)}{c_1 - U'(z_1)}\, \mathrm{d}z_1. \end{aligned} \tag{52}$$

Second, the formulae in (52) will give the answer if we find v_0 and v_1.

For the first step, by the definition of L_0^x and L_1^x,

$$L_0^x m_1(x, y) = \frac{c_0 - U'(x)}{c_1 - U'(x)} \cdot v_1(x, y), \qquad L_1^x m_0(x, y) = \frac{c_1 - U'(x)}{c_0 - U'(x)} \cdot v_0(x, y).$$

Applying the change of variables (51) to (48), we get

$$\begin{aligned} (c_1 - U'(x))\frac{\partial v_0(x, y)}{\partial x} - \lambda_1 v_0(x, y) - \lambda_0 \frac{c_1 - U'(x)}{c_0 - U'(x)} v_0(x, y) &= \lambda_0 + \lambda_1, \\ (c_0 - U'(x))\frac{\partial v_1(x, y)}{\partial x} - \lambda_1 \frac{c_0 - U'(x)}{c_1 - U'(x)} v_1(x, y) - \lambda_0 v_1(x, y) &= \lambda_0 + \lambda_1. \end{aligned}$$

Thus, system (48) is equivalent to

$$\begin{cases} \dfrac{\partial v_0}{\partial x}(x, y) = \psi(x) v_0(x, y) + \dfrac{\lambda_0 + \lambda_1}{c_1 - U'(x)}, \\ \dfrac{\partial v_1}{\partial x}(x, y) = \psi(x) v_1(x, y) + \dfrac{\lambda_0 + \lambda_1}{c_0 - U'(x)}, \end{cases} \tag{53}$$

where $\psi(x)$ is defined by (28):

$$\psi(x) = \frac{\lambda_0}{c_0 - U'(x)} + \frac{\lambda_1}{c_1 - U'(x)} = \frac{\Psi'(x)}{\Psi(x)}.$$

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The solution of (53) is determined by the boundary conditions that follow from Lemma 1.

For the second step, consider the case $a_0 < x < y < a_+$ with $U'(x), U'(y) < c_1 < c_0$. The case $x > y > b_+$ is symmetric with $U'(x), U'(y) > c_0 > c_1$.

Since, in this case, $\boldsymbol{m}(x, y)|_{x\to y} = \mathbf{0}$ (Lemma 1), by virtue of (17), the system (53) is supplied with the boundary conditions $v_0(x, y)|_{x\uparrow y} = v_1(x, y)|_{x\uparrow y} = -1$. Therefore,

$$v_0(z_0, y) = \Psi(z_0)\Psi(y)^{-1}\left[-1 - (\lambda_0 + \lambda_1)\int_{z_0}^{y} \frac{\Psi(y)\Psi(z_1)^{-1}}{c_1 - U'(z_1)}\,\mathrm{d}z_1\right]$$

$$= -\beta(z_0, y) - (\lambda_0 + \lambda_1)\int_{z_0}^{y} \frac{\beta(z_0, z_1)}{c_1 - U'(z_1)}\,\mathrm{d}z_1, \tag{54}$$

$$v_1(z_1, y) = \Psi(z_1)\Psi(y)^{-1}\left[-1 - (\lambda_0 + \lambda_1)\int_{z_1}^{y} \frac{\Psi(y)\Psi(z_0)^{-1}}{c_0 - U'(z_0)}\,\mathrm{d}z_0\right]$$

$$= -\beta(z_1, y) - (\lambda_0 + \lambda_1)\int_{z_1}^{y} \frac{\beta(z_1, z_0)}{c_0 - U'(z_0)}\,\mathrm{d}z_0, \qquad z_0, z_1 < y. \tag{55}$$

The formulae in (32) and (33) follow from (52), (54), and (55).

In the case $a_0 < x < y = a_+$, by (44) and (17), we obtain the boundary conditions for (53):

$$v_0(x, y)|_{x\uparrow y} = -1 - \lambda_0/\lambda_1, \qquad v_1(x, y)|_{x\uparrow y} = 0.$$

Similar to (54), we obtain

$$v_0(z_0, y) = -\beta(z_0, y)(1 + \lambda_0/\lambda_1) - (\lambda_0 + \lambda_1)\int_{z_0}^{y} \frac{\beta(z_0, z_1)}{c_1 - U'(z_1)}\,\mathrm{d}z_1, \tag{56}$$

$$v_1(z_1, y) = -(\lambda_0 + \lambda_1)\int_{z_1}^{y} \frac{\beta(z_1, z_0)}{c_0 - U'(z_0)}\,\mathrm{d}z_0. \tag{57}$$

The formulae in (34) and (35) follow from (52), (56), and (57).

The rest of the proof follows by symmetry. □

3.3. Large parameters

If both parameters are large, i.e. $c_0 > V$ and $c_1 < v$, see Figure 1(d), then both potentials $U(x) - c_0x$ and $U(x) - c_1x$ are single-well, repulsion points disappear, and the attractors G_- and G_+ merge into one, $G = [a_-, b_+]$. In this case, process $X^x(t)$, $x \in G$, randomly oscillates between two attracting stationary states $x = a = a_-$ and $x = b = b_+$.

If both parameters are large, the mean first-passage time is derived in closed form using the technique of Theorem 4.

Theorem 5. *Let the parameters c_0, c_1 be large, as described above. The average values $m_0(x, y) = \mathbb{E}[\mathcal{T}(x, y) \mid \xi(0) = 0]$ and $m_1(x, y) = \mathbb{E}[\mathcal{T}(x, y) \mid \xi(0) = 1]$ are determined as follows:*

- *if $x < y < b$ or $x > y > a$, then m_0 and m_1 are determined by (32) and (33);*
- *if $a < x < y < b$, then (34) and (35) are valid;*
- *if $a < y < x < b$, then (36) and (37) are valid.*

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4. Stationary measure

Let the interval $G = (a, b)$ be invariant under the dynamics X, see (8), so that $c_1 \le U'(x) \le c_0$, $x \in G$, and $U'(b) = c_0$, $U'(a) = c_1$, and thus $G = G_-$ or $G = G_+$ – see Figure 1.

The explicit form of the operator $\mathcal{L}^*$ adjoint to $\mathcal{L}$, (10), on the interval G is obtained by integrating by parts the integral

$$\langle \mathcal{L}[\boldsymbol{f}](x),\ \boldsymbol{g}(x)\rangle_{L_2(G)} = \int_G \big[(c_0 - U'(x))f_0'(x)g_0(x) + (c_1 - U'(x))f_1'(x)g_1(x)\big]\,\mathrm{d}x$$

for any test functions $\boldsymbol{f} = (f_0, f_1)$ and $\boldsymbol{g} = (g_0, g_1)$. We have

$$\begin{aligned}\langle \mathcal{L}[\boldsymbol{f}](x),\ \boldsymbol{g}(x)\rangle_{L_2(G)} &= [(c_0 - U'(x))f_0(x)g_0(x)]\big|_{\partial G} + [(c_1 - U'(x))f_1(x)g_1(x)]\big|_{\partial G}\\ &\quad - \int_G \left\{f_0(x)\frac{\mathrm{d}}{\mathrm{d}x}[(c_0 - U'(x))g_0(x)] + f_1(x)\frac{\mathrm{d}}{\mathrm{d}x}[(c_1 - U'(x))g_1(x)]\right\}\mathrm{d}x.\end{aligned}$$

Therefore, the formal adjoint to the infinitesimal generator $\mathcal{L}$ is given by

$$\mathcal{L}^*\boldsymbol{g}(x) = \begin{pmatrix} -\dfrac{\mathrm{d}}{\mathrm{d}x}[(c_0 - U'(x))g_0(x)] & 0\\ 0 & -\dfrac{\mathrm{d}}{\mathrm{d}x}[(c_1 - U'(x))g_1(x)]\end{pmatrix} \tag{58}$$

with a singular part corresponding to the boundary condition

$$g_0(a) = 0, \qquad g_1(b) = 0. \tag{59}$$

As we will show, there exist invariant measures μ^- and μ^+ supported on the attractors $G_- = (a_-, b_-)$ and $G_+ = (a_+, b_+)$, respectively.

Theorem 6. *Let $G = (a, b)$ be an invariant attractor (G_- or G_+). The invariant measure with support G exists and is determined by the probability density functions $\pi_0(x)$ and $\pi_1(x)$:*

$$\pi_0(x) = C_0\frac{\Psi(x)^{-1}}{c_0 - U'(x)}, \qquad \pi_1(x) = C_1\frac{\Psi(x)^{-1}}{U'(x) - c_1}, \quad x \in G.$$

Here, $\Psi(x)^{-1} = \exp\left(-\lambda_0\Phi_0(x) - \lambda_1\Phi_1(x)\right)$, and C_0, C_1 are normalising constants,

$$C_0 = \frac{\lambda_1}{\lambda_0 + \lambda_1}\left(\int_a^b \frac{\Psi(x)^{-1}\,\mathrm{d}x}{c_0 - U'(x)}\right)^{-1}, \qquad C_1 = \frac{\lambda_0}{\lambda_0 + \lambda_1}\left(\int_a^b \frac{\Psi(x)^{-1}\,\mathrm{d}x}{U'(x) - c_1}\right)^{-1}. \tag{60}$$

Proof. The invariant density $\boldsymbol{\pi} = (\pi_0(x), \pi_1(x))$, $x \in G$, which is defined by

$$\mathbb{P}\{X(t) \in \mathrm{d}x,\ \xi(t) = 0\} \equiv \pi_0(x)\,\mathrm{d}x, \qquad \mathbb{P}\{X(t) \in \mathrm{d}x,\ \xi(t) = 1\} \equiv \pi_1(x)\,\mathrm{d}x,$$

follows the Fokker–Planck equation $(\Lambda^* + \mathcal{L}^*)\boldsymbol{\pi} = \boldsymbol{0}$, $x \in G$, $\Lambda^* = \Lambda^\top$, with boundary conditions $\pi_0(a) = 0$ and $\pi_1(b) = 0$, see (58) and (59). That is, the entries π_0 and π_1 are positive

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solutions to the coupled equations

$$-\frac{\mathrm{d}}{\mathrm{d}x}[(c_0 - U'(x))\pi_0(x)] - \lambda_0\pi_0(x) + \lambda_1\pi_1(x) = 0, \tag{61}$$

$$-\frac{\mathrm{d}}{\mathrm{d}x}[(c_1 - U'(x))\pi_1(x)] + \lambda_0\pi_0(x) - \lambda_1\pi_1(x) = 0, \tag{62}$$

$a < x < b$.

Summing (61) and (62), we obtain $(c_0 - U'(x))\pi_0(x) + (c_1 - U'(x))\pi_1(x) \equiv \text{const}$. The constant is zero. This follows, for example, from $U'(a) = c_1$ and due to the boundary conditions, (59), $\pi_0(a) = 0$. Therefore,

$$\frac{\pi_0(x)}{\pi_1(x)} = \frac{U'(x) - c_1}{c_0 - U'(x)}. \tag{63}$$

Equation (61) can be rewritten as

$$-\frac{[(c_0 - U'(x))\pi_0(x)]'}{(c_0 - U'(x))\pi_0(x)} - \frac{\lambda_0}{c_0 - U'(x)} + \frac{\lambda_1}{c_0 - U'(x)} \cdot \frac{\pi_1(x)}{\pi_0(x)} = 0.$$

By (63), we have

$$\frac{\mathrm{d}}{\mathrm{d}x} \log [(c_0 - U'(x))\pi_0(x)] = -\psi(x), \qquad x \in G = (a, b),$$

where $\psi(x)$ is defined before, (28), and $\pi_0(a) = 0$. Integrating, we obtain

$$\pi_0(x) = C_0 \frac{\exp(-\lambda_0\Phi_0(x) - \lambda_1\Phi_1(x))}{c_0 - U'(x)}, \qquad a < x < b,$$

where Φ_0 and Φ_1 are defined in (5). The boundary condition holds, since $\Phi_0|_{x\to a} < \infty$, $\Phi_1(x)|_{x\to a} = +\infty$. Similarly,

$$\pi_1(x) = C_1 \frac{\exp(-\lambda_0\Phi_0(x) - \lambda_1\Phi_1(x))}{U'(x) - c_1}, \qquad a < x < b.$$

Therefore, C_0 and C_1 are given by (60), since the invariant distribution of $\boldsymbol{\xi}(t)$ is determined by

$$\int_a^b \pi_0(x)\,\mathrm{d}x = \mathbb{P}_0\{\xi(t) = 0\} = \frac{\lambda_1}{\lambda_0 + \lambda_1}, \qquad \int_a^b \pi_1(x)\,\mathrm{d}x = \mathbb{P}_1\{\xi(t) = 0\} = \frac{\lambda_0}{\lambda_0 + \lambda_1}. \quad \square$$

Discussion

The mathematical interpretation of physical phenomena often uses very sophisticated and sometimes very artificial tools. For example, a very popular climate change model, widely studied in recent decades, is based on a stochastic differential equation with periodic potential,

$$\mathrm{d}X_\varepsilon(t) = -U'(X_\varepsilon(t), t/T_\varepsilon)\,\mathrm{d}t + \sigma(T_\varepsilon)\,\mathrm{d}W(t), \quad t > 0. \tag{64}$$

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Here, the period T_ε is usually assumed to be exponentially large in ε, and the periodic fluctuations of the potential are due to periodic changes in the parameters of the Earth's orbit (the so-called Milankovitch astronomical cycles, [21]).

The key property postulated for this model is stochastic resonance. This methodology involves choosing the diffusion coefficient $\sigma(T_\varepsilon)$ taking into account the spectral properties of the mean solution of (64). In my opinion, this method looks artificial, since the use of white noise for such modelling (especially for climate models) is not confirmed. Moreover, it is worth noting that the global and long-term behaviour of the atmosphere does not undergo abrupt and rapid changes, and the climate usually changes very slowly. This makes the classical diffusion process with infinite variation and infinitely fast propagation problematic for describing climate change.

We propose to add to the (deterministic) continuous periodic changes of potential randomly alternating trends corresponding to internal processes in the atmosphere and at the surface, see (1). In other words, we can then modify the model (64), replacing the white-noise term $\sigma(T_\varepsilon)\,\mathrm{d}W(t)$ by a telegraph process $\mathbb{T}(t)$.

When and if this trend coincides with the direction of the periodic change in potential, the system can go into a metastable state, receiving a time window for the interwell transition. Otherwise, this internal trend has only a retarding effect on the global changes that occur from the laws of celestial mechanics.

Appendix: Proof of Theorem 1

Proof. Let $\tau^0 \sim \mathrm{Exp}(\lambda_0)$ be the time of the first switching of the process $\xi(t)$ from state 0 to state 1, and $\tau^1 \sim \mathrm{Exp}(\lambda_1)$ be the time of the first switching from state 1 to state 0.

Let $\mathcal{P}(t,\mathrm{d}y \mid x) = (p_{ij}(t,\mathrm{d}y \mid x))_{i,j\in\{0,1\}}$ be the transition probability matrix, where

$$p_{ij}(t,\mathrm{d}y \mid x) = \mathbb{P}\{X(t)\in \mathrm{d}y,\ \xi(t)=j \mid X(0)=x,\ \xi(0)=i\}.$$

The corresponding Markov semigroup P_t is defined on the test function $\boldsymbol{f}=(f_0,f_1)$ as follows:

$$(P_t\boldsymbol{f})_i(x) = \mathbb{E}[\boldsymbol{f}(\Xi(t)) \mid \Xi(0)=(x,i)] = \int_{-\infty}^{\infty} f_0(y)p_{i0}(t,\mathrm{d}y \mid x) + \int_{-\infty}^{\infty} f_1(y)p_{i1}(t,\mathrm{d}y \mid x),$$

$i\in\{0,1\}$.

Conditioning on the first switching, see (9), we obtain the following system of integral equations for $j\in\{0,1\}$:

$$\begin{cases} p_{0j}(t,\mathrm{d}y \mid x) = \mathrm{e}^{-\lambda_0 t}\,\delta_{\gamma_0(t,x)}(\mathrm{d}y) + \displaystyle\int_0^t \lambda_0 \mathrm{e}^{-\lambda_0\tau} p_{1j}(t-\tau,\mathrm{d}y \mid \gamma_0(\tau,x))\,\mathrm{d}\tau, \\ p_{1j}(t,\mathrm{d}y \mid x) = \mathrm{e}^{-\lambda_1 t}\,\delta_{\gamma_1(t,x)}(\mathrm{d}y) + \displaystyle\int_0^t \lambda_0 \mathrm{e}^{-\lambda_0\tau} p_{0j}(t-\tau,\mathrm{d}y \mid \gamma_1(\tau,x))\,\mathrm{d}\tau. \end{cases} \tag{65}$$

Here, $\delta_a(\mathrm{d}y)$ is Dirac's δ-measure at a.

By applying the operator $(\partial/\partial t) - L_0^x$ to the first equation of (65), and the operator $(\partial/\partial t) - L_1^x$ to the second, we can find that the system of integral equations in (65) is equivalent to some initial-value problem for partial differential equations. Indeed, by virtue of (7),

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$$\begin{cases} \dfrac{\partial p_{0j}(t,\mathrm{d}y\mid x)}{\partial t} - L_0^x[p_{0j}(t,\mathrm{d}y\mid x)] = -\lambda_0 \mathrm{e}^{-\lambda_0 t}\,\delta_{\gamma_0(t,x)}(\mathrm{d}y) + \lambda_0 \mathrm{e}^{-\lambda_0 t} p_{1j}(0,\mathrm{d}y\mid \gamma_0(t,x)) \\ \qquad\qquad - \displaystyle\int_0^t \lambda_0 \mathrm{e}^{-\lambda_0 \tau}\,\frac{\partial}{\partial \tau}[p_{1j}(t-\tau,\mathrm{d}y\mid \gamma_0(\tau,x))]\,\mathrm{d}\tau, \\ \dfrac{\partial p_{1j}(t,\mathrm{d}y\mid x)}{\partial t} - L_1^x[p_{0j}(t,\mathrm{d}y\mid x)] = -\lambda_1 \mathrm{e}^{-\lambda_1 t}\,\delta_{\gamma_1(t,x)}(\mathrm{d}y) + \lambda_1 \mathrm{e}^{-\lambda_1 t} p_{0j}(0,\mathrm{d}y\mid \gamma_1(t,x)) \\ \qquad\qquad - \displaystyle\int_0^t \lambda_1 \mathrm{e}^{-\lambda_1 \tau}\,\frac{\partial}{\partial \tau}[p_{0j}(t-\tau,\mathrm{d}y\mid \gamma_1(\tau,x))]\,\mathrm{d}\tau. \end{cases}$$

Integrating by parts, we obtain, for $j \in \{0,1\}$,

$$\begin{cases} \dfrac{\partial p_{0j}(t,\mathrm{d}y\mid x)}{\partial t} - L_0^x[p_{0j}(t,\mathrm{d}y\mid x)] = -\lambda_0 p_{0j}(t,\mathrm{d}y\mid x) + \lambda_0 p_{1j}(t,\mathrm{d}y\mid x), \\ \dfrac{\partial p_{1j}(t,\mathrm{d}y\mid x)}{\partial t} - L_1^x[p_{1j}(t,\mathrm{d}y\mid x)] = \lambda_1 p_{0j}(t,\mathrm{d}y\mid x) - \lambda_1 p_{1j}(t,\mathrm{d}y\mid x), \qquad j \in \{0,1\}, \end{cases}$$

which can be rewritten in matrix form as

$$\frac{\partial \mathcal{P}(t,\mathrm{d}y\mid x)}{\partial t} = (\Lambda + \mathcal{L}^x)[\mathcal{P}(t,\mathrm{d}y\mid x)], \quad t>0,$$

with initial condition

$$\mathcal{P}(t,\mathrm{d}y\mid x)\big|_{t\downarrow 0} = \begin{pmatrix} \delta_x(\mathrm{d}y) & 0 \\ 0 & \delta_x(\mathrm{d}y) \end{pmatrix}. \qquad \square$$

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