Integral inequalities in probability theory revisited

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1. *Introduction*

In [1], the following conjecture was proposed concerning the distribution of ages in a closed interval $[0, A]$:

If $f(x)$ is decreasing for $0 \leqslant x \leqslant A$, $f(A) = 0$ and $\int_0^A f(x)\,dx = 1$ (and continuous? differentiable?) then $\int_0^A f(x)\,dx > \dfrac{1}{2f(0)}$.

To the best of our knowledge the first person to gave an answer to the conjecture was A. F. Beardon (see [2]), who proved the following inequality for a continuous probability density function f in $[0, A]$ with $0 \leqslant f \leqslant M$ and n a positive integer:

$$\frac{1}{(n+1)M^n} \leqslant \int_0^A x^n f(x)\,dx \leqslant \frac{M}{n+1}\left(A^{n+1} - B^{n+1}\right), \tag{1}$$

where $B = A - \frac{1}{M}$.

Later the lower bound of (1) for $n = 1$ was proved by a different method in [3].

That proof, as it was noted in [3], was also valid for $A = \infty$, i.e. for probability density functions on $[0, \infty)$ and for $n \geqslant 1$.

Another article which deals with the same problem is [4], where (1) is proved differently for $n = 1$.

In this Article, we present new simple proofs for the bounds of (1) and we also present and prove other integral inequalities for continuous probability density functions. Moreover, we find a better upper and lower bound than the bounds given in (1), when f is a decreasing or an increasing function respectively. We end this Article by giving some examples of our results.

2. *First results*

In this section we first present a new simple proof for the lower bound of (1), when a continuous probability density function f is defined either on $[0, A]$ or on $[0, \infty)$. Let us note here that the case $[0, \infty)$ is not always a trivial generalisation of the case $[0, A]$. For instance, in [2], the fact that $\int_0^A nx^{n-1}F(x)\,dx + \int_0^A x^n f(x)\,dx = A^n$ (where $F(x) = \int_0^x f(t)\,dt$) is used in the proof and this equality would not be useful if $A = \infty$. On the other hand, in [4], the proof for the lower bound of (1) would be valid even if $A = \infty$. However, as we said in the Introduction, only the case $n = 1$ was encountered in [4]. Lastly, in [3], the case $A = \infty$ is presented and an example using the chi-squared distribution is also given.

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So, let $F(x) = \int_0^x f(t)\,dt$, where f is a continuous probability density function on $[0, \infty)$ with $0 \leqslant f(x) \leqslant M$. Then

$$F(x) = \int_0^x f(t)\,dt \leqslant Mx.$$

Hence for a given integer $n \geqslant 1$ we have

$$[F(x)]^n \leqslant M^n x^n.$$

Multiplying both sides by $(n+1)f(x)$ we get

$$(n+1)f(x)[F(x)]^n \leqslant (n+1)f(x)M^n x^n.$$

Integrating both sides of the inequality from 0 to $y > 0$, and taking into account that $\frac{d}{dx}[F(x)]^{n+1} = (n+1)f(x)[F(x)]^n$ and $F(0) = 0$, we obtain

$$\frac{1}{(n+1)M^n}[F(y)]^{n+1} \leqslant \int_0^y x^n f(x)\,dx.$$

Since f is a probability density function on $[0, \infty)$, $\lim_{y\to\infty} F(y) = 1$, and for a given $n \geqslant 1$ we have $\lim_{y\to\infty}[F(y)]^{n+1} = 1$. So by letting $y \to \infty$ we get the requested lower bound

$$\frac{1}{(n+1)M^n} \leqslant \int_0^\infty x^n f(x)\,dx. \tag{2}$$

Let us observe that (2) is always true since the improper integral $\int_0^\infty x^n f(x)\,dx$ will either converge to a real number $L > 0$ or diverge to $+\infty$. Moreover, the same proof can be used to find a lower bound for a continuous probability density function f on $[0, A]$. In this case we get

$$\frac{1}{(n+1)M^n} \leqslant \int_0^A x^n f(x)\,dx,$$

which is of course the same lower bound as in (1).

Next we will find an upper bound for the case of a continuous probability density function f which is also decreasing on $[0, \infty)$ or on $[0, A]$. Let us note here that an upper bound for the improper integral $\int_0^\infty x^n f(x)\,dx$ is not always possible since the integral may diverge to $+\infty$. Having this in mind, we find instead an upper bound for the integral $\int_0^\infty x^n [f(x)]^{n+1}\,dx$ and prove that it converges. So we further assume that the continuous probability density function f is decreasing on $[0, \infty)$, and we get

$$xf(x) \leqslant F(x) = \int_0^x f(t)\,dt.$$

For an integer $n \geqslant 1$

$$(n+1)x^n[f(x)]^{n+1} \leqslant (n+1)f(x)[F(x)]^n.$$

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Integrating both sides from 0 to $y > 0$ and by using

$$\frac{d}{dx}[F(x)]^{n+1} = (n+1)f(x)[F(x)]^n$$

and $F(0) = 0$, we get

$$I(y) = \int_0^y x^n [f(x)]^{n+1}\,dx \leqslant \frac{[F(y)]^{n+1}}{n+1} \leqslant \frac{1}{n+1},$$

where the last inequality holds since $F(y) \leqslant 1$ for $y \in [0, \infty)$. Now since $I(y)$ is an increasing function on $[0, \infty)$ and bounded from above, it is also convergent as $y \to \infty$. So by letting $y \to \infty$ we have the following upper bound

$$\int_0^\infty x^n [f(x)]^{n+1}\,dx \leqslant \frac{1}{n+1} \tag{3}$$

for $n \geqslant 1$, and for this case the improper integral $\int_0^\infty x^n [f(x)]^{n+1}\,dx$ converges. Moreover, we can use the same proof for a continuous probability density function f on $[0, A]$, which is also decreasing, to get the following upper bound

$$\int_0^A x^n [f(x)]^{n+1}\,dx \leqslant \frac{1}{n+1} \tag{4}$$

for $n \geqslant 1$.

3. *Finding other bounds*

In this section we will prove three integral inequalities for a continuous probability density function f on $[0, A]$ with $0 \leqslant f(x) \leqslant M$, $x \in [0, A]$. Firstly, we will find a better upper bound than that of (1) when f is decreasing on $[0, A]$. Secondly, we will find a better lower bound than that of (1) when f is increasing on $[0, A]$. Lastly we will present a new simple proof for the upper bound of (1). So, for a continuous probability density function f which is decreasing on $[0, A]$ and for which $0 \leqslant f(x) \leqslant M$, we will prove that

$$\int_0^A x^n f(x)\,dx \leqslant \frac{A^n}{n+1} \tag{5}$$

for an integer $n \geqslant 1$. To prove this upper bound, we use Hölder's inequality for the functions $x^{n^2/(n+1)}$ and $x^{n/(n+1)} f(x)$. Hence for $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\int_0^A \left[x^{\frac{n^2}{n+1}}\right]\left[x^{\frac{n}{n+1}} f(x)\right] dx \leqslant \left[\int_0^A \left[x^{\frac{n^2}{n+1}}\right]^p dx\right]^{1/p} \left[\int_0^A \left[x^{\frac{n}{n+1}} f(x)\right]^q dx\right]^{1/q},$$

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where $n \geqslant 1$. Thus for $p = \dfrac{n+1}{n}$ and $q = n+1$, we obtain

$$\int_0^A x^n f(x)\,dx \leqslant \left[\int_0^A x^n dx\right]^{n/(n+1)} \left[\int_0^A x^n [f(x)]^{n+1}\,dx\right]^{1/(n+1)}$$

for $n \geqslant 1$. Applying (4) to the right-hand side of the above, we easily get (5). We can further show that the upper bound (5) is better than the upper bound given in (1). Let us first note that $M > 1/A$, because if $M < 1/A$ then $F(A) < 1$ and f would not be a probability density function on $[0, A]$. Moreover if $M = 1/A$ we get the trivial case $f(x) = 1/A$ and for this case the upper bounds of (1) and (5) are identical.

Let us now prove that the upper bound $\dfrac{A^n}{n+1}$ of (5) is better than the upper bound $\dfrac{M}{n+1}\left(A^{n+1} - B^{n+1}\right)$ of (1). So we want to prove

$$\frac{A^n}{n+1} < \frac{M}{n+1}\left(A^{n+1} - B^{n+1}\right)$$

for $AM > 1$. A simple calculation leads to

$$(AM)^n < (AM)^{n+1} - (AM-1)^{n+1}.$$

By dividing both sides by $(AM)^{n+1}$ and letting $v = \dfrac{1}{AM} < 1$ we have

$$v < 1 - (1-v)^{n+1} \Leftrightarrow (1-v)^n < 1,$$

which holds for $n \geqslant 1$ since $0 < 1 - v < 1$. This means that $\dfrac{A^n}{n+1}$ is a sharper upper bound for decreasing functions f.

Next we will prove that a better lower bound when f is increasing on $[0, A]$ can be found. Of course, as before, it is true that $AM > 1$ and we can define

$$g(x) = \frac{M - f(x)}{AM - 1} \geqslant 0,$$

which is decreasing on $[0, A]$ and also $\int_0^A g(x)\,dx = 1$. Let us note that this form of $g(x)$ was also used in [4]. Now we can use (5) to get

$$\int_0^A x^n g(x)\,dx \leqslant \frac{A^n}{n+1},$$

or

$$\int_0^A x^n \frac{M}{AM-1}dx - \int_0^A x^n \frac{f(x)}{AM-1}dx \leqslant \frac{A^n}{n+1},$$

and after the calculations we get

$$\frac{A^n}{n+1} \leqslant \int_0^A x^n f(x)\,dx, \tag{6}$$

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and a lower bound has been found for $n \geqslant 1$. Note also that for $AM = 1$ we get the trivial case $f(x) = \dfrac{1}{A}$ and the above bound is still valid as equality. To see that it is a better bound than the lower bound of (1), we want

$$\frac{A^n}{n+1} > \frac{1}{(n+1)M^n},$$

or equivalently $(AM)^n > 1$, which is true since $AM > 1$ and $n \geqslant 1$. Observe also that for $AM = 1$ the bound of (6) is identical to the lower bound of (1).

Finally we present a new simple proof for the upper bound of (1). Let $G(x) = \int_0^x g(t)\,dt, x \in [0, A]$, where

$$g(x) = \frac{M - f(x)}{AM - 1} \geqslant 0,$$

with $AM > 1$. Note that g is a probability density function on $[0, A]$ since $G(A) = 1$. Also $g(x) \leqslant \dfrac{M}{AM-1}$, and we have

$$G(x) = \int_0^x g(t)\,dt \leqslant \frac{M}{AM-1}\,x,$$

or

$$[G(x)]^n \leqslant \left(\frac{M}{AM-1}\right)^n x^n,$$

for a given $n \geqslant 1$. By multiplying both sides by $(n+1)\,g(x)$ we get

$$(n+1)\,g(x)\,[G(x)]^n \leqslant (n+1)\,g(x)\left(\frac{M}{AM-1}\right)^n x^n.$$

Integrating both sides from 0 to A and taking into account that

$$\frac{d}{dx}[G(x)]^{n+1} = (n+1)g(x)[G(x)]^n, G(0) = 0 \text{ and } G(A) = 1,$$

we obtain

$$\frac{1}{n+1}\left(\frac{AM-1}{M}\right)^n \leqslant \int_0^A x^n g(x)\,dx.$$

By substituting $g(x)$, we get the following upper bound

$$\int_0^A x^n f(x)\,dx \leqslant \frac{M}{n+1}\left[A^{n+1} - \left(A - \frac{1}{M}\right)^{n+1}\right], \tag{7}$$

for $n \geqslant 1$, and this is the same upper bound as in (1).

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4. *Related inequalities*

Our aim in this section is to prove a few more simple integral inequalities for probability density functions on $[0, A]$. Using Hölder's inequality for a continuous function h on $[0, A]$, we can easily get

$$\int_0^A h(x)\,dx \leqslant A^{(\alpha-1)/\alpha}\left(\int_0^A [h(x)]^\alpha\,dx\right)^{1/\alpha},$$

where $\alpha > 1$ is a real number. Let f be a continuous probability density function on $[0, A]$ where $f(x) \leqslant M$, $x \in [0, A]$ and let $\mu = \int_0^A xf(x)\,dx$ be the mean of f. Using the above inequality with $h(x) = xf(x)$ we have the following lower bound

$$A\left(\frac{\mu}{A}\right)^\alpha \leqslant \int_0^A x^\alpha [f(x)]^\alpha\,dx,$$

where $\alpha > 1$. We can now use (1) for $n = 1$ to get

$$\frac{A}{(2AM)^\alpha} \leqslant A\left(\frac{\mu}{A}\right)^\alpha \leqslant \int_0^A x^\alpha [f(x)]^\alpha\,dx,$$

for $\alpha > 1$. Moreover, if we assume that f is increasing on $[0, A]$ we can use (6) to get

$$\frac{A}{2^\alpha} \leqslant A\left(\frac{\mu}{A}\right)^\alpha \leqslant \int_0^A x^\alpha [f(x)]^\alpha\,dx,$$

for $\alpha > 1$. Lastly, if we set $h(x) = f(x)$, we get

$$\int_0^A [f(x)]^\alpha\,dx \geqslant \frac{1}{A^{\alpha-1}}, \tag{8}$$

for $\alpha > 1$.

5. *Examples*

We end this Article by giving examples to demonstrate some applications of our results. Let us first use (4) with $f(x) = 2(1 - x)$, which is a decreasing function on [0, 1] and for which $\int_0^1 f(x)\,dx = 1$, to get

$$\int_0^1 x^{n-1}(1 - x)^n\,dx = 2^n B(n, n + 1) \leqslant \frac{1}{n},$$

where $B(n, n + 1)$ is the beta function. So we have found the upper bound $B(n, n + 1) \leqslant \frac{1}{n2^n}$. Using the formula $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}$ for x, y positive integers, we have $B(n, n + 1) = \frac{B(n, n)}{2}$ and so

$$B(n, n) \leqslant \frac{1}{n2^{n-1}},$$

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which is an upper bound for $B(n, n)$. In order to check this upper bound, we use the following inequality that was proved in [5] for real numbers $x, y \geqslant 1$:

$$\alpha \leqslant \frac{1}{xy} - B(x, y) \leqslant \beta,$$

where $\alpha = 0$ and $\beta = 0.08731\ldots$. So for $x = y = n$ the above inequality gives

$$B(n, n) \leqslant \frac{1}{n^2}.$$

It is easy to see that our upper bound is better for the special case of $B(n, n)$, since

$$\frac{1}{n2^{n-1}} < \frac{1}{n^2},$$

for $n \geqslant 3$.

Next, let the continuous probability density function $f(x) = \dfrac{2(1 + x)}{A(A + 2)}$ on $[0, A]$. Applying (8) we have

$$\left(\frac{2}{A(A + 2)}\right)^{\alpha} \int_0^A (1 + x)^{\alpha} dx \geqslant \frac{1}{A^{\alpha - 1}},$$

for $\alpha > 1$. Equivalently,

$$\left(\frac{2}{A + 2}\right)^{\alpha} \left[\frac{(1 + A)^{\alpha + 1} - 1}{\alpha + 1}\right] \geqslant A.$$

Since $A > 0$ and $\alpha > 1$ it holds that $\left(\dfrac{2}{A + 2}\right)^{\alpha} < 1$, hence

$$(1 + A)^{\alpha + 1} - 1 > (\alpha + 1)A.$$

Letting $\beta = \alpha + 1$ we have

$$(1 + A)^{\beta} > 1 + \beta A,$$

for real numbers $A > 0$ and $\beta > 2$. This is a strict version of Bernoulli's inequality.

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1933 Cumulative Index

In amongst the hundreds of the *Gazette*, this booklet is rarely noticed. Paper copies must be next to impossible to find but, thanks to the Internet, it is readily available. It runs to some 170 pages and covers Volumes I to XV. Naturally, it contains all that is to be expected of an Index but has some amplification as well. For example, in Volume XIV, the obituary of Alfred George Greenhill is shown to be by J.W.N. but, in this Index, we are told the author was J.W. Nicholson. This list appears.

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