Student Problems

Students up to the age of 19 are invited to send solutions to either or both of the following problems to Beth Woollacott, 20 Rectory Close, Sutton Bonington, Loughborough, Nottinghamshire, LE12 5PJ.

Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is not necessary to submit solutions to both. Entries should arrive by 20th January 2022 and solutions will be published in the March 2022 edition.

The Mathematical Association and the *Gazette* comply fully with the provisions of the 2018 GDPR legislation. Submissions **must** be accompanied by the SPC permission form which is available on the MA website

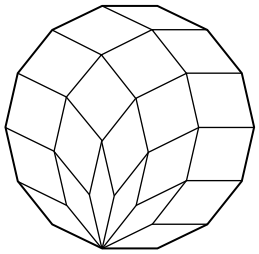
https://www.m-a.org.uk/the-mathematical-gazette

Note that if permission is not given, a pupil ***may still participate and will be eligible for a prize in the same way as others***.

Problem 2021.5 **(Paul Stephenson)**

The figure shows that a regular polygon with an even number of sides, $2n$, can be dissected into equilateral rhombuses.

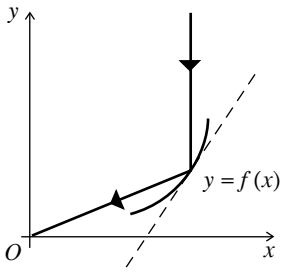
Find a simple expression for the total number of rhombus sides, which holds regardless of the arrangement of the rhombuses within the polygon.



Problem 2021.6 **(Mark Emson)**

A light ray parallel to the y-axis hits a curve $y = f(x)$ and subsequently passes through the origin O, as shown.

Given that the incident angle is equal to the reflection angle at the tangent, show that $y = f(x)$ is a parabola ($x \neq 0$).



Note from the Editor

See *www.m-a.org.uk/the-mathematical-gazette* for the SPC questions.

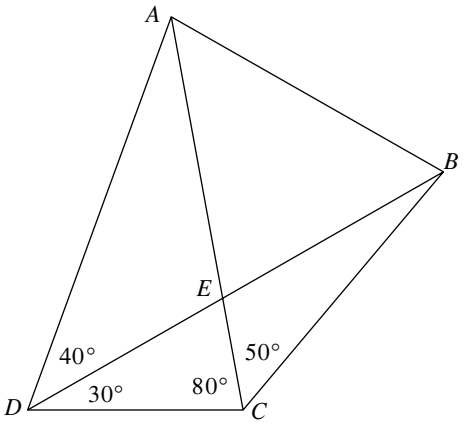
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Solutions to 2021.3 and 2021.4

Problem 2021.3 was solved by Aparup Roy of St Peter's School, Durgabur, India, Nathan Burn of Bishop Wordsworth's School, Salisbury, UK, and another student from Kolkata, India. Problem 2021.4 was solved by Bura Charan of Raman Bhavan-1, Gudavalli, India and James Keating of Kings College London Mathematics School, London, UK. Thank you for your brilliant solutions, I enjoyed reading them all.

Problem 2021.3 **(Geoffrey Strickland)**

The diagram shows a quadrilateral $ABCD$ with diagonals AC, BD intersecting at E. $\angle ADB = 40°$, $\angle ACB = 50°$, $\angle ACD = 80°$ and $\angle BDC = 30°$.



Find the angles of triangle AEB.

Solution

This was Nathan's solution, which I thoroughly enjoyed reading.

Consider ΔDAC, $\angle DAC = 180° - 70° - 80° = 30°$ so

$$\frac{DC}{\sin 30°} = \frac{AC}{\sin 70°}. \tag{1}$$

Now consider ΔDBC, $\angle DBC = 180° - 30° - 130° = 20°$ so

$$\frac{DC}{\sin 20°} = \frac{BD}{\sin 130°}. \tag{2}$$

Dividing (1) by (2) yields

$$\frac{\sin 20°}{\sin 30°} = \frac{AC \sin 130°}{BD \sin 70°} = \frac{AC \sin 50°}{BD \cos 20°}.$$

Rearranging:

$$BD \sin 20° \cos 20° = AC \sin 30° \sin 50°$$

$$\tfrac{1}{2} BD \sin 40° = \tfrac{1}{2} AC \sin 50°$$

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$$BD \sin 40° = AC \sin 50°$$

$$BD \sin 40° = AC \cos 40°$$

$$AC = BD \tan 40° \qquad (*)$$

Consider $\triangle ABD$ and let $\angle CAB = x°$. So

$$\frac{BD}{\sin (x + 30)°} = \frac{AB}{\sin 40°}. \qquad (3)$$

Consider $\triangle CAB$ $$\frac{AC}{\sin (130 - x)°} = \frac{AB}{\sin 50°}. \qquad (4)$$

(4) ÷ (3) $$\frac{\sin 40°}{\sin 50°} = \frac{AC \sin (x + 30)°}{BD \sin (130 - x)°} = \frac{\sin 40°}{\cos 40°}.$$

Using (∗) $$\tan 40° = \frac{BD \tan 40° \sin (x + 30)°}{BD \sin (130 - x)°}.$$

Hence $x + 30 = 130 - x$ $\Rightarrow 2x = 100 \Rightarrow x = 50$,
or $x + 30 = 180 - (130 - x)$, which is impossible.

Therefore, $\angle EAB = 50°$ and we found that $\angle AEB = 70°$, so $\angle EBA = 60°$.

Problem 2021.4 (Paul Stephenson)

Let n and k be integers greater than 1 with k a power of 2. Prove that $n^{2k} + n^k + 1$ has at least $2k$ different factors.

Solution

Bura and James both gave excellent proofs; however, I give Paul's proof here for brevity.

$$n^{4m} + n^{2m} + 1 = \left(n^{2m} - n^m + 1\right)\left(n^{2m} + n^m + 1\right).$$

A common factor of $n^{2m} - n^m + 1$ and $n^{2m} + n^m + 1$ would also divide their difference, 2^{nm}, and so they are coprime. Furthermore, $n^{2m} - n^m + 1 > 1$ and so $n^{2m} - n^m + 1$ has a prime factor not in $n^{2m} + n^m + 1$. Then

$$n^{2k} + n^k + 1 = \left(n^{2k} - n^{k/2} + 1\right) \ldots \left(n^4 - n^2 + 1\right)\left(n^2 - n + 1\right)\left(n^2 + n + 1\right)$$

has at least $\log_2 2k$ distinct prime factors. In a product of some of these primes, each can either be chosen or not, to make at least 2k distinct factors.

Prizewinners

I have awarded Bura the first prize of £25 and Nathan the second prize of £20.

Beth Woollacott

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