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AGGREGATION AND IDEMPOTENCE

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Abstract. A 1-ary sentential context is aggregative (according to a consequence relation) if the result of putting the conjunction of two formulas into the context is a consequence (by that relation) of the results of putting first the one formula and then the other into that context. All 1-ary contexts are aggregative according to the consequence relation of classical propositional logic (though not, for example, according to the consequence relation of intuitionistic propositional logic), and here we explore the extent of this phenomenon, generalized to having arbitrary connectives playing the role of conjunction; among intermediate logics, LC, shows itself to occupy a crucial position in this regard, and to suggest a characterization, applicable to a broader range of consequence relations, in terms of a variant of the notion of idempotence we shall call componentiality. This is an analogue, for the consequence relations of propositional logic, of the notion of a conservative operation in universal algebra.

§1. Introduction. We begin by introducing some basic terminology and notation; for a summary of the paper, see the Abstract as well as the discussion after Proposition 1.3 below. The notions of consequence relation and generalized (alias 'multiple conclusion') consequence relation will be taken to be familiar. Though not part of the definition, all such relations encountered below will be finitary, substitution-invariant, and will relate (sets of) formulas from languages of the kind familiar in propositional logic, freely generated from a countably infinite supply of propositional variables ('sentence letters') by application of primitive connectives. A *valuation* for such a language is a mapping, v, from the formulas to the set {T, F} of truth-values, which is *consistent with* a consequence relation $\vdash$ (generalized consequence relation $\Vdash$) just in case we never have $\Gamma \vdash B$ while $v(A) = \mathrm{T}$ for all $A \in \Gamma$ and $v(B) = \mathrm{F}$ (resp., $\Gamma \Vdash \Delta$ while $v(A) = \mathrm{T}$ for all $A \in \Gamma$ and $v(B) = \mathrm{F}$ for all $B \in \Delta$). $\vdash$ (resp. $\Vdash$) is *determined by* a set V of valuations for the language of $\vdash$ (of $\Vdash$) when for all Γ, B (resp. all Γ, Δ) $\Gamma \vdash B$ ($\Gamma \Vdash \Delta$) if and only if for all $v \in V$, if $v(B) = \mathrm{T}$ whenever $v(A) = \mathrm{T}$ for all $A \in \Gamma$ (resp. $v(D) = \mathrm{T}$ for some $D \in \Delta$ whenever $v(A) = \mathrm{T}$ for all $A \in \Gamma$).

For convenience of exposition, we think of the propositional variables as

$$p_1, q_1, p_2, q_2, \ldots, p_n, q_n, \ldots,$$

with the p_i playing a special place-holder role in the specification of contexts, which are formally just formulas, but will be denoted using the schematic letter C thus: $C(p_1, \ldots, p_n)$, this being an n-ary sentential context, the result of uniformly substituting formulas $A_1, \ldots, A_n$ for $p_1, \ldots, p_n$, respectively, in which we denote by $C(A_1, \ldots, A_n)$. Here the focus is on the special case of $n = 1$ and we will say "context" for 1-ary sentential context, though the 2-ary case does come up more than once in §3. Note that in general

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$C(p_1, \ldots, p_n)$ will also contain additional variables (from the range $q_1, \ldots, q_n, \ldots$); when it does not, it may be regarded as defining—indeed on some views of the matter, as *being*—a derived (or 'defined') n-ary connective of the language in question. For brevity, p_1 will be written as p, and q_1, q_2, q_3 as q, r, s, respectively. Sometimes a 1-ary context $C(p)$ will be indicated by $C(\cdot)$ to emphasize that we are thinking of C as a formula with a gap in it rather than as a complete formula in its own right. With the $\vdash$ and $\Vdash$ notation, we take the usual notational liberties: thus $\Gamma, A \vdash B$ means $\Gamma \cup \{A\} \vdash B$, and so on. When there is only one formula on the left in the case of the $\vdash$ notation we use the following further abbreviations: "$A \dashv\vdash B$" for "$A \vdash B$ and $B \vdash A$" (marking any sub- or superscripting only on the "$\vdash$" part of the composite "$\dashv\vdash$" notation[1]); further, "$A_1 \vdash A_2 \vdash \ldots A_{k-1} \vdash A_k$" abbreviates "$A_1 \vdash A_2$ and $A_2 \vdash A_3$ and $\ldots$ and $A_{k-1} \vdash A_k$". Note that we use A, B, D, sometimes decorated, as schematic letters for formulas, reserving C for the case in which, as above, we have a context-specifying formula in mind; later "#" will be used as a temporary symbol for this or that connective, often (though not always) binary.

DEFINITION 1.1. *Given a consequence relation* $\vdash$ *on a language with* L *as its set of formulas, the* replacement core, $\vdash^{\mathsf{rep}}$, *of* $\vdash$, *is defined as follows. For all* $B \in L$, $\Gamma \subseteq L$: $\Gamma \vdash^{\mathsf{rep}} B \Leftrightarrow$ *for every context* $C(p) \in L$: $\{C(A) \mid A \in \Gamma\} \vdash C(B)$. ◀

Note that because we can always take $C(p)$ as the null or empty context p, $\vdash^{\mathsf{rep}} \subseteq \vdash$. Aspects of the behaviour of contexts and replacement in various logics were studied in Humberstone (2013) by raising questions about the replacement core of this or that consequence relation: how much of the original consequence relation $\vdash$ is inherited by $\vdash^{\mathsf{rep}}$? When do we have $A \vdash^{\mathsf{rep}} B$ implying $B \vdash^{\mathsf{rep}} A$ (for all A, B)? And so on. Abbreviating "classical (propositional) logic" to "CL", examples given in Humberstone (2013) include the fact that $A, B \vdash^{\mathsf{rep}}_{\mathsf{CL}} A \wedge B$—in words: all contexts are aggregative according to $\vdash_{\mathsf{CL}}$—to which, along with some variations, we return in Proposition 1.3 below. and also the following:

More explicitly in respect of notation, we take $\vdash_{\mathsf{CL}}$ to be the consequence relation over a language with a functionally complete set of primitive connectives which for convenience we take to include the nullary $\bot$ and $\top$, determined by the class of all boolean valuations (i.e., valuations on which the connectives are associated with the conventional truth-functions). Similarly, we take $\Vdash_{\mathsf{CL}}$ to be the generalized consequence relation determined by the class of all boolean valuations (for a language as before). Later when we want to talk about intuitionistic (propositional) logic, we use the generic abbreviation IL, and for suitable consequence and generalized consequence relations, $\vdash_{\mathsf{IL}}$ and $\Vdash_{\mathsf{IL}}$,[2] and with

[1] Thus "$A \dashv\vdash_{\mathsf{KT}_c} B$", to cite a notation appearing in Example 2.5 below, means that both $A \vdash_{\mathsf{KT}_c} B$ and $B \vdash_{\mathsf{KT}_c} A$.

[2] Since there is no unique historically sanctioned candidate for being *the* generalized consequence relation of intuitionistic logic (see Gabbay, 1981, p. 28, Theorem 6), a word of clarification is in order here: by $\Vdash_{\mathsf{IL}}$ is meant the largest generalized consequence relation which agrees with the consequence relation $\vdash_{\mathsf{IL}}$ for the case in which there is exactly one formula on the right of the "$\Vdash$". This amounts to saying that $\Gamma \Vdash_{\mathsf{IL}} \Delta$ when some disjunction of formulas from Δ is a consequence by $\vdash_{\mathsf{IL}}$ of Γ. Because it is the generalized consequence relation most immediately suggested by the Kripke semantics for IL, Gabbay calls it $\Vdash_K$ in Gabbay (1981). The contrast is with the Beth semantics, which allows a disjunction to be true at a point in a model without either disjunct's being true at that point, giving the *smallest* generalized consequence relation—$\Vdash_B$ as Gabbay accordingly calls it—agreeing with $\vdash_{\mathsf{IL}}$. The background idea of defining the smallest and largest generalized consequence relations agreeing with a given (finitary) consequence relation is from Scott (1974). (The word *finitary* has been inserted here because, according to

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similar notation in the case of some other logics). We adapt Definition 1.1 to the case of generalized consequence relations in the obvious manner:

DEFINITION 1.2. *Given a generalized consequence relation* $\Vdash$ *on a language with* L *as its set of formulas, the* replacement core, $\Vdash^{\mathsf{rep}}$, *of* $\Vdash$, *is defined by:* $\Gamma \Vdash^{\mathsf{rep}} \Delta \Leftrightarrow$ *for every context* $C(p) \in L$: $\{C(A) \mid A \in \Gamma\} \Vdash \{C(B) \mid B \in \Delta\}$ *(all* $\Gamma, \Delta \subseteq L$*).* ◀

Part (i) of Proposition 1.3 below repeats in the $\Vdash$ notation the first of the two examples alluded to above, as it was put in Humberstone (2013), that for classical propositional logic every context $C(\cdot)$ is *aggregative* in the sense that for all formulas A, B, the formula $C(A \wedge B)$ is a classical consequence of $C(A)$ taken together with $C(B)$. Here we write B_1 and B_2 rather than A and B for the sake of a similar formulation in Corollary 1.8. Most of the content of Proposition 1.3 was familiar—in a different formulation—to Łukasiewicz and can be extracted from his Łukasiewicz (1953) where derivations are given from an axiomatic basis, as is explained in Humberstone (2013), which instead gives semantic arguments like that offered for the proof of Proposition 1.3 above. In Proposition 2.7, we give a more general formulation covering a wider range of cases; alternatively, one may derive Proposition 1.3 as a corollary to Proposition 1.7 below; however, the greater generality of the former treatment does not subsume the case of Proposition 1.7, so we devote some effort to furnishing a single explanation covering all cases, with the discussion of conservativity in §3. A quick verification of the claims made is possible because although the definitions of $\vdash^{\mathsf{rep}}$ and $\Vdash^{\mathsf{rep}}$ quantify over all contexts, for the case $\vdash^{\mathsf{rep}}_{\mathsf{CL}}$ and $\Vdash^{\mathsf{rep}}_{\mathsf{CL}}$, it suffices to consider only two: $C(p)$ as p and $C(p)$ as $\neg p$. (See Proposition 3.1 of Humberstone, 2013.)

PROPOSITION 1.3. *For any formulas* B_1, B_2*, we have the following:*

(i) $B_1, B_2 \Vdash^{\mathsf{rep}}_{\mathsf{CL}} B_1 \wedge B_2$; (ii) $B_1 \wedge B_2 \Vdash^{\mathsf{rep}}_{\mathsf{CL}} B_1, B_2$;

(iii) $B_1, B_2 \Vdash^{\mathsf{rep}}_{\mathsf{CL}} B_1 \vee B_2$; (iv) $B_1 \vee B_2 \Vdash^{\mathsf{rep}}_{\mathsf{CL}} B_1, B_2$.

Proof. We address explicitly only part (i), and for notational convenience, re-letter B_1, B_2 to A, B. (A similar argument works for (ii)–(iv).) We must show that $C(A), C(B) \Vdash_{\mathsf{CL}} C(A \wedge B)$ for all formulas A, B and all contexts $C(p)$. (We could just as well write "$\vdash_{\mathsf{CL}}$" for "$\Vdash_{\mathsf{CL}}$" in the current case.) Take any boolean valuation v and write a for $v(A)$, b for $V(B)$, and $a \wedge b$ for $v(A \wedge B)$. Since any q_i in $C(p)$ remain fixed as we consider the fate of $C(A)$, $C(B)$ and $C(A \wedge B)$, $C(p)$ can be regarded as inducing on v a 1-ary truth-function g. Thus suppose $g(a) = g(b) = \mathrm{T}$, with a view to showing that $g(a \wedge b) = \mathrm{T}$. There are two cases to consider: (1) $a \neq b$ and (2) $a = b$. In case (1), since there are only the two truth-values T and F to consider and $g(a) = g(b) = \mathrm{T}$, $a \wedge b$ must be a or else b, so in either case $g(a \wedge b) = \mathrm{T}$. Case (2): if $a = b$ then $a \wedge b = a$ (and also $= b$), by the idempotence of (the truth-function) $\wedge$, so since $g(a) = \mathrm{T}$, $g(a \wedge b) = \mathrm{T}$. □

The principal concern of the discussion to follow is with Proposition 1.3, the range of connectives which can replace $\wedge$ and $\vee$ in (i)–(ii) and (iii)–(iv), respectively, as well as the range of alternatives to CL for which such results hold, though we are mainly interested in consequence relations (*cf.* the reformulation in Corollary 1.8 below), with generalized consequence relations playing an auxiliary role. In §2, several extensionality-like

Shoesmith & Smiley (1978), Theorems 5.7 and 5.11, p. 75*f.*, in the nonfinitary case—not arising for Scott or Gabbay or in the present paper—there may be no maximum agreeing generalized consequence relation.)

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conditions (on contexts) and in §3 several idempotence-like conditions (on connectives) will come in for attention as we try to get at the bottom of what it going on in Propositions 1.3 and 1.7 in the present section. For the sake of a naturally flowing discussion some material is postponed to two appendices (§5 and §6). Part (i) of Proposition 1.3 is the aggregation property already alluded to or more precisely the claim that for classical logic all contexts are aggregative. It will be helpful to introduce a more refined terminology.

DEFINITION 1.4.

(i) *A context $C(p)$ is* #-aggregative *according to the generalized consequence relation $\Vdash$, for some (not necessarily primitive) n-ary connective # of the language of $\Vdash$, just in case for all formulas $A_1, \ldots, A_n$ of that language, we have $C(A_1), \ldots, C(A_n) \Vdash C(\#(A_1, \ldots, A_n))$. We define what it is for a context to be* #-aggregative *according to a consequence relation $\vdash$ in exactly the same way, with "$\vdash$" in pace of $\Vdash$.*

(ii) *Under the same conditions as in (i), $C(p)$ is* #-disaggregative *according to a generalized consequence relation $\Vdash$ when for all formulas $A_1, \ldots, A_n$, we have $C(\#(A_1, \ldots, A_n)) \Vdash C(A_1), \ldots, C(A_n)$, and to be* #-disaggregative *according to a consequence relation $\vdash$ when for all formulas $A_1, \ldots, A_n, D$, and sets of formulas Γ, we have: if $\Gamma, C(A_i) \vdash D$ for each i $(1 \leq i \leq n)$ then $\Gamma, C(\#(A_1, \ldots, A_n)) \vdash D$.*

(iii) *With #, $\Vdash$, $\vdash$, as in (i), (ii), we say that # is aggregative (resp. disaggregative) according to the generalized consequence relation $\Vdash$ when every context is #-aggregative (resp. disaggregative) according to $\Vdash$, and that # is aggregative (resp. disaggregative) according to the consequence relation $\vdash$ when every context is #-aggregative (resp. disaggregative) according to $\vdash$.* ◀

Thus the aggregativity of a context in the traditional sense is a matter of its $\wedge$-aggregativity in the present terminology, and n-ary # is aggregative according to $\vdash$ just when for all $A_1, \ldots, A_n$ we have $A_1, \ldots, A_n \vdash^{\mathsf{rep}} \#(A_1, \ldots, A_n)$. Parts (i), (iii) of Proposition 1.3 record the aggregativity of $\wedge$ and $\vee$ according to $\Vdash_{\mathsf{CL}}$, while parts (ii) and (iv) the disaggregativity of these connectives, respectively, according to to $\Vdash_{\mathsf{CL}}$. The definition given for #-disaggregativity of a context according to a consequence relation given under Definition 1.4(ii) inherits a complexity arising from the desire to keep it neutral over what connectives the language of $\vdash$ provides and how they behave according to $\vdash$. It is often more less cumbersome to sacrifice this neutrality and restrict attention to what have been called 'disjunctive' (e.g., Czelakowski, 2001, p. 73) or '$\vee$-classical' (Humberstone, 2011, p. 62) consequence relations:

DEFINITION 1.5. *A consequence relation $\vdash$ is* disjunctive *if the language of $\vdash$ provides a (not necessarily primitive) binary connective (we shall write as $\vee$) such that for all sets Γ of formulas of that language and all formulas A, B, D, we have: $\Gamma, A \vee B \vdash D$ if and only if $\Gamma, A \vdash D$ and $\Gamma, B \vdash D$.* ◀

For a disjunctive consequence relation we can offer as a more manageable rendering of the second half of Definition 1.4(ii): $C(p)$ is #-disaggregative just in case for all $A_1, \ldots, A_n$, we have $C(\#(A_1, \ldots, A_n)) \vdash C(A_1) \vee \ldots \vee C(A_n)$.[3]

[3] The associativity of $\vee$, and thus the legitimacy of suppressing any particular bracketing of the disjuncts here, follows from the properties in virtue of which $\vdash$ is a disjunctive consequence relation.

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None of parts (i)–(iv) of Proposition 1.3 would be correct in the case of intuitionistic propositional logic, which is to say, with $\Vdash^{\mathrm{rep}}_{\mathsf{IL}}$ (see note 2) in place of $\Vdash^{\mathrm{rep}}_{\mathsf{CL}}$, as Examples 1.6 below show.

EXAMPLE 1.6.

(i) $C(p) = (p \to q) \to r$ *yields a counterexample to the analogue of Proposition 1.3*(i) *with "*IL*" for "*CL*", as is mentioned in Proposition 1.2 of Humberstone (2013).*

(ii) $C(p) = p \to q$ *provides a counterexample to the analogue of Proposition 1.3*(ii)*, as does* $C(p) = \neg p$.

(iii) $C(p) = (q \to p) \to r$ *(or* $(q \to p) \to q$*) serves a counterexample in the case of the analogue of Proposition 1.3*(iii)*. (On the other hand, the occurrence of* r *in the* $C(p)$ *of* (i) *above cannot similarly be traded in for* q*, since for all* B_1*,* B_2*, we* do *have* $(B_1 \to q) \to q, (B_2 \to q) \to q \vdash_{\mathsf{IL}} ((B_1 \wedge B_2) \to q) \to q$*.)*

(iv) Take $C(p) = q \to p$ *for part* (iv) *of Proposition 1.3.* ◀

On the other hand, for the case of Dummett's intermediate logic LC, the situation remains, in respect of aggregativity and disaggregativity, as reported in Proposition 1.3 for CL; a formulation of this appears as Proposition 1.7 below. Understand by $\Vdash_{\mathsf{LC}}$ the relation holding between sets Γ and Δ when some conjunction of formulas from Γ provably implies some disjunction of formulas from Δ in the Hilbert system for LC given by Dummett in Dummett (1959). We denote by $\vdash_{\mathsf{LC}}$ the consequence relation associated with LC, in play later, which may be defined as for $\Vdash_{\mathsf{LC}}$ except that we restrict attention to the case of Δ a singleton (in which case the reference to disjunctions drops out).

PROPOSITION 1.7. *Claims* (i)–(iv) *of Proposition 1.3 hold with* LC *in place of* CL.

Proof. Use the fact that $A \leftrightarrow B \Vdash_{\mathsf{LC}} C(A) \leftrightarrow C(B)$ for all contexts $C(p)$ and all formulas A, B (something holding already for $\Vdash_{\mathsf{IL}}$) and the facts (this time specific to the LC case) that $\Vdash_{\mathsf{LC}} ((B_1 \wedge B_2) \leftrightarrow B_1) \vee ((B_1 \wedge B_2) \leftrightarrow B_2)$, for (i) and (ii), and that $\Vdash_{\mathsf{LC}} ((B_1 \vee B_2) \leftrightarrow B_1) \vee ((B_1 \vee B_2) \leftrightarrow B_2)$, for (iii) and (iv). Thus, if # is $\wedge$ or $\vee$, $C(A \# B)$ is equivalent to one or other of $C(A)$, $C(B)$, which suffices for each of (i)–(iv). □

The reference to one formula being equivalent to another in the above proof is not to provable LC-equivalence, but rather just a way of alluding to the truth of the $\leftrightarrow$-formula involved on the hypothesis that the disjunct with it as main connective is true.

The fact that $A \leftrightarrow B \Vdash_{\mathsf{IL}} C(A) \leftrightarrow C(B)$ for all contexts $C(p)$, mentioned in the proof of Proposition 1.7, is sometimes put using the terminology of extensionality. (It is also part of what is recalled in §3 as the fact that $\{p_1 \leftrightarrow p_2\}$ is a 'set of equivalence formulas' for $\vdash_{\mathsf{LC}}$.) In §2, we shall be distinguishing various extensionality-like notions and in the terminology there introduced will put this by saying that all contexts are 'weakly extensional' according to $\vdash_{\mathsf{IL}}$. Appendix A (= §5), which assumes familiarity with §2 below, treats a further variation on this theme, due to Łukasiewicz.

As already remarked, Proposition 1.3 can be derived from this using the fact that $\Vdash_{\mathsf{LC}} \subseteq \Vdash_{\mathsf{CL}}$ using the observation (= Proposition 1.4 in Humberstone, 2013) that $\Vdash^{\mathrm{rep}}_0 \subseteq \Vdash^{\mathrm{rep}}_1$ whenever $\Vdash_0 \subseteq \Vdash_1$ and the language of $\Vdash_0$ coincides with that of $\Vdash_1$; however, we need the proof given above of (part (i) of) Proposition 1.3—and not just the result proved—for some parts of the discussion in §2. In fact, Proposition 1.4 of Humberstone (2013) makes the observation just cited explicitly only for the case of consequence relations $\vdash_0$, $\vdash_1$.

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Indeed, since we are especially interested in consequence relations rather than generalized consequence relations, we simultaneously re-work (i)–(iv) of Proposition 1.3 so as to avoid multiple right-hand sides, at the same avoiding multiple left-hand sides, for the sake of a symmetrical presentation, while also unpacking the "rep" superscript, and generalizing to the case in which n-fold conjunctions and disjunctions are involved; we write "$\bigwedge_i B_i$" and the like, meaning $\bigwedge_{i=1}^{n} B_i$:

COROLLARY 1.8. *For all formulas* $B_1, \ldots, B_n$ $(n \geq 1)$ *and contexts* $C(p)$*, we have:*

(i) $\bigwedge_i C(B_i) \vdash_{\mathsf{CL}} C(\bigwedge_i B_i)$; (ii) $C(\bigwedge_i B_i) \vdash_{\mathsf{CL}} \bigvee_i C(B_i)$;

(iii) $\bigwedge_i C(B_i) \vdash_{\mathsf{CL}} C(\bigvee_i B_i)$; (iv) $C(\bigvee_i B_i) \vdash_{\mathsf{CL}} \bigvee_i C(B_i)$.

Again, we could formulate these claims for LC instead of CL. A more orthodox definition of $\vdash_{\mathsf{LC}}$, equivalent to that given before the statement of Proposition 1.7, would involve the notion of an *intermediate consequence relation* as any substitution-invariant consequence relation $\vdash$ such that $\vdash_{\mathsf{IL}} \subseteq \vdash \subseteq \vdash_{\mathsf{CL}}$. $\vdash_{\mathsf{LC}}$ is then the least intermediate consequence relation $\vdash$ such that $\vdash (A \to B) \vee (B \to A)$, for all formulas A, B. We refer to the disjunctive schema just cited as Dummett's LC axiom, in the course of the proof of Proposition 1.9 below.

PROPOSITION 1.9. $\vdash_{\mathsf{LC}}$ *is the least intermediate consequence relation for which any of the four conditions (omitting the "*CL*") listed in Corollary 1.8 holds (for all contexts* $C(p)$*).*

Proof. It suffices to examine the contexts cited under Examples 1.6—in the cases in which more than one example is given, namely (ii) and (iii), the first such example. For instance, the $C(p)$ of part (ii) , namely $C(p) = p \to q$ would give $((A \wedge B) \to q) \to ((A \to q) \vee (B \to q))$, which, on substituting $A \wedge B$ for q readily yields Dummett's LC axiom. □

Picking up the reference in this proof to Examples 1.6(ii) and (iii) providing two cases each: for (ii) the second example, with $C(p)$ as $\neg p$, would not yield LC, but instead the weaker Dummett–Lemmon intermediate logic KC. The second example mentioned under (iii), that is, $C(p)$ as $(q \to p) \to q$, is in the LC-yielding camp, though this is more conveniently shown at a later stage below: see note 25.

Proposition 1.9, when taken together with Proposition 1.7, shows a conspicuous interdependence among the four parts of Proposition 1.3/Corollary 1.8 as far as intermediate logics are concerned: given any one, your get the remaining three. From the examples already seen here (or in Humberstone, 2013) the reader may already have discerned some patterns of dependence. Here we concentrate on the link between parts (i) and (ii) of the results cited, which also obtains between parts (iii) and (iv).

In the case of extensions of $\vdash_{\mathsf{CL}}$, with classical negation to hand,[4] we could easily establish the following implication for an arbitrary binary connective # (chosen as binary purely for illustrative purposes): if # is aggregative according to $\vdash$ then # is also disaggregative according to $\vdash$ (in the sense of Definition 1.4). For suppose that for all contexts $C(\cdot)$, $C(A) \wedge C(B) \vdash C(A \# B)$ for arbitrary A, B, and we have a context $C'(\cdot)$ for which we want to show that $C'(A \# B) \vdash C'(A) \vee C'(B)$, for some chosen A, B. We simply

[4] That is, consequence relations $\vdash \supseteq \vdash_{\mathsf{CL}}$ which are $\neg$-classical in the sense of Humberstone (2011). Note that the language of $\vdash$ here is not required to be that of $\vdash_{\mathsf{CL}}$: many additional connectives may be present.

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instantiate the "for all $C(\cdot)$" to the special case $\neg C'(\cdot)$, and then contrapose the result:

$$\neg C'(A) \wedge \neg C'(B) \vdash \neg C'(A \,\#\, B),$$

collapsing some double negations and exploiting some De Morgan equivalences to get the desired conclusion, $C'(A \,\#\, B) \vdash C'(A) \vee C'(B)$. A similar line of thought would establish the converse implication (disaggregativity $\Rightarrow$ aggregativity, according to $\vdash^{\mathsf{rep}}$).

But what of consequence relations (such as $\vdash_{\mathsf{IL}}$) not providing a connective with the powers $\neg$ is required to exercise in the above reasoning? We need a more general argument. Let $\vdash$ be any (substitution-invariant) consequence relation extending $\vdash_{\mathsf{IL}}$, with # again taken as binary for illustration. Suppose, as before, that for all contexts $C(\cdot)$, $C(A) \wedge C(B) \vdash C(A \,\#\, B)$ for all formulas A, B and we are now considering a context $C'(\cdot)$, wanting to show that $C'(A \,\#\, B) \vdash C'(A) \vee C'(B)$, for some chosen A, B. This time we will make our instantiation of the "for all $C(\cdot)$" depend not only on $C(\cdot)$, as before, but also on the particular A and B chosen. (As long as we can do this, *however* A and B are chosen, we still have the desired universal result.) In particular, we take $C(\cdot)$ as $C'(\cdot) \rightarrow (C'(A) \vee C'(B))$. This gives us:

$$C'(A) \rightarrow ((C'(A) \vee C'(B)), C'(B) \rightarrow ((C'(A) \vee C'(B)) \vdash C'(A \,\#\, B) \rightarrow ((C'(A) \vee C'(B)),$$

from which we would expect to be able to 'cut' the formulas on the left, giving us that on the right as a consequence of $\varnothing$, from which again we would expect to be able to to pass to $C'(A \,\#\, B) \vdash C'(A) \vee C'(B)$. Now, all these expectations are fulfilled with $\vdash$ satisfying the supposition in Proposition 1.10, in which we have relaxed the restriction to binary #:

PROPOSITION 1.10. *Suppose $\vdash$ is any substitution-invariant consequence relation such that $\vdash \supseteq \vdash_{\mathsf{IL}}$ and let # be a connective of the language of $\vdash$ (which may properly extend the language of $\vdash_{\mathsf{IL}}$). Then # is aggregative according to $\vdash$ if and only if # is disaggregative according to $\vdash$.*

Proof. The preceding discussion establishes the "only if" direction. For the "if" direction, proceed similarly but taking the instantiation of "for all $C(\cdot)$" for a given C', A, B to be: $((C'(A) \wedge C'(B)) \rightarrow C'(\cdot)$. □

Note that Proposition 1.10 does not speak of intermediate consequence relations, since we want to allow the presence of any number of new connectives—# itself perhaps being one of them—exhibiting any kind of logical behaviour. (Note also that there is no requirement that the intermediate consequence relations in question should satisfy the Deduction Theorem.)

§2. Extensional contexts. We need to begin by fixing some terminology. Consider the following pair of ('left extensionality' and 'right extensionality') conditions on a context $C(\cdot)$, formulas and a generalized consequence relation $\Vdash$, and formulas A, B (of the language of $\Vdash$):

(LE) $A, B, C(A) \Vdash C(B)$ (RE) $C(A) \Vdash C(B), A, B.$

DEFINITION 2.1.

(i) The context $C(\cdot)$ is left-extensional *according to $\Vdash$ if (LE) is satisfied for all formulas A, B (of the language of $\Vdash$; $\Vdash$ itself is* left-extensional *if every context (provided by the language of $\Vdash$) is left-extensional according to $\Vdash$.*

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(ii) *The context* $C(\cdot)$ *is* right-extensional *according to* $\Vdash$ *if (RE) is satisfied for all formulas* A, B*;* $\Vdash$ *itself is* right-extensional *if every context is right-extensional according to* $\Vdash$.

(iii) *The context* $C(\cdot)$ *is* extensional *according to* $\Vdash$ *if* $\dot{(C)}$ *is both left- and right-extensional according to* $\Vdash$*;* $\Vdash$ *itself is* extensional *if every context is extensional according to* $\Vdash$. ◀

The extensional generalized consequence relations are exactly those $\Vdash$ such that every valuation consistent with $\Vdash$ associates with each connective in the language of $\Vdash$ some truth-function (function from $\{T, F\}^n$ to $\{T, F\}$, that is, where n is the arity of the connective concerned), though for a given connective the associated truth-function may vary from valuation to valuation.[5] (See Humberstone, 2011, chap. 3, for details and substantiation of this and the corresponding remark concerning consequence relations below.) Such valuation-to-valuation variation in the associated truth-function makes no difference to matters of aggregation and disaggregation, as long as the truth-function concerned is in each case idempotent, which is the reason we are investigating extensionality in this section. Now, consequence relations $\vdash$ are notoriously less able (than their generalized cousins) to constrain the association of truth-functions with their connectives on all valuations consistent with $\vdash$, but we should like at least to have a syntactic notion of extensionality for consequence relations along the lines of Definition 2.1(iii) which coincides with the property of being determined by *some* class of valuations, over each of which every connective is associated (in the sense of note 5) with some truth-function. To that end, consider the following two conditions on contexts and formulas, relative to a consequence relation $\vdash$. The first is a modified version of the right-extensionality condition (RE) appropriate for generalized consequence relations, obtained by a conditional formulation as in Definition 1.4(ii).

(RE)′ *If* $\Gamma, C(B) \vdash D$ *and* $\Gamma, A \vdash D$ *and* $\Gamma, B \vdash D$, *then* $\Gamma, C(A) \vdash D$.

(WE) $\Gamma, A \vdash B$ *and* $\Gamma, B \vdash A$ *imply* $\Gamma, C(A) \vdash C(B)$,

DEFINITION 2.2.

(i) *A context* $C(\cdot)$ *is* left-extensional *according to a consequence relation* $\vdash$ *if the condition (LE) with "*$\Vdash$*" rewritten as "*$\vdash$*" is satisfied for all formulas* A, B *of the language of* $\vdash$*;* $\vdash$ *itself is* left-extensional *if every context provided by its language is left-extensional according to* $\vdash$.

(ii) *A context* $C(\cdot)$ *is* right-extensional *according to a consequence relation* $\vdash$ *if the condition (RE)′, above is satisfied for all sets of formulas* $\Gamma \cup \{A, B, D\}$ *of the language of* $\vdash$*; a consequence relation is* right-extensional *if every context provided by the language of* $\vdash$ *is right-extensional according to* $\vdash$.

[5] The n-ary truth-function f is *associated with* the n-ary connective # (alternatively: # is associated with f) on the valuation v (for some language) if and only if for all formulas $A_1, \ldots, A_n$ (of that language): $v(\#(A_1, \ldots, A_n)) = f(v(A_1), \ldots, v(A_n))$. An n-ary context $C(p_1, \ldots, p_n)$ can similarly be defined to be associated on a valuation v with an n-ary truth-function when for all formulas $A_1, \ldots, A_n$ we have $v(C(A_1, \ldots, A_n)) = f(v(A_1), \ldots, v(A_n))$, though what this function is will typically depend on $v(q_i)$ for q_i in $C(p_1, \ldots, p_n)$. (This is what is called the truth-function induced by $C(p)$ the context relative to v, in the proof of Proposition 1.3(i) above).

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(iii) *A context $C(\cdot)$ is* extensional *according to a consequence relation $\vdash$ if it is left- and right-extensional according to $\vdash$; a consequence relation is extensional if it is left- and right-extensional.*

(iv) *A context $C(\cdot)$ is* weakly extensional *according to a consequence relation $\vdash$ just in case the condition (WE) above is satisfied for all Γ, A, B; a consequence relation $\vdash$ is* weakly extensional *if every context its language provides is weakly extensional according to $\vdash$.* ◀

In connection with (WE), it worth noting that a generalized consequence relation is extensional in the sense of Definition 2.1(iii) if and only if $\Gamma, A \Vdash B, \Delta$ and $\Gamma, B \Vdash A, \Delta$ imply $\Gamma, C(A) \Vdash C(B), \Delta$ (for all sets of formulas $\Gamma \cup \Delta \cup \{A.B\}$ and call contexts $C(\cdot)$. The condition of weak extensionality for a consequence relation given in Definition 2.2(iv) is what we obtain from this by discarding the now supernumerary Δ (and writing "$\vdash$" for "$\Vdash$"). Note that weak extensionality (for a given context, or for all contexts) is indeed weaker than extensionality (understood correspondingly): while it is not hard to see that any context which is extensional according to a given $\vdash$ is must also satisfy (WE), one also has such examples as that of $\neg$ in intuitionistic logic: all contexts are weakly extensional according to $\vdash_{\mathsf{IL}}$ but $C(p) = \neg p$ does not satisfy (RE)$'$, and neither does $C(p) = p \to q$.[6] Thus $\vdash_{\mathsf{IL}}$ itself is weakly extensional but not extensional, in the present terminology. A further notion of extensionality, emerging from some formulations of Łukasiewicz, is discussed in Appendix A (= §5) under the rubric of Ł-extensionality.

In a suitably 'classical' setting, the contrast between weak extensionality and extensionality itself evaporates; the 'disjunctive' in the formulation of the following observation, which spells out this point about the contrast, is not redundant, since extensions of a disjunctive consequence relation need not themselves be disjunctive:

PROPOSITION 2.3. *Suppose that $\vdash$ is a disjunctive consequence relation extending $\vdash_{\mathsf{IL}}$ (though the language of $\vdash$ may have further primitive connectives as well) satisfying either of the following ('classical') conditions for all Γ, A, B, D:*

(1) $\Gamma, \neg A \vdash D$ *implies* $\Gamma \vdash A \vee D$ (2) $\Gamma, A \leftrightarrow B \vdash D$ *implies* $\Gamma \vdash D \vee (A \vee B)$.

Then any context $C(\cdot)$ which is weakly extensional according to $\vdash$, is extensional according to $\vdash$.

Proof. (LE) follows from (WE) quire generally (take $\Gamma = \{A, B\}$ in (WE)), so we need only show that under the conditions stated, (RE)$'$ also follows from (WE). Since $\vdash$ is disjunctive, it suffices to derive the "$\vee$"-version of (RE), $C(A) \vdash C(B) \vee A \vee B$ for the given $C(\cdot)$ assumed to satisfy (WE). But (WE), taking Γ as $\{\neg A, \neg B\}$, gives this conclusion by two appeals to the condition (1) in the statement of the result. On the other hand, to obtain this conclusion using (2) instead, take Γ in (WE) as $\{A \leftrightarrow B\}$. □

To speak more concisely about the matter of valuation-to-valuation variation of the truth-function associated with a connective involved in the discussion after Definitions 2.1 above, the following terminology is convenient; the definitions use the notion of association between connectives and truth-functions explained in note 5, and employ formal quantifier notation to make vivid the scope contrast between the two notions defined:

[6] Since for example $q \nvdash_{\mathsf{IL}} r \vee \neg q \vee \neg r$—here and elsewhere parentheses are dropped as associativity permits—which would yield $\vdash_{\mathsf{CL}}$: just substitute $\top$ for q. By contrast, (WE) evidently implies (LE) for any $\vdash$: see the proof of Proposition 2.3 below.

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DEFINITION 2.4.

(i) # *is* truth-functional *over* V *just in case* $\exists f \forall v \in V$*:* f *is associated with* # *on* v*. (Here "*f*" ranges over truth-functions of the same arity as the connective* #*.)*

(ii) # *is* pseudo-truth-functional *over* V *just in case* $\forall v \in V \,.\, \exists f$*:* f *is associated with* # *on* v*.* ◀

EXAMPLE 2.5. *The local consequence relation of the smallest modal logic (thought of as a set of formulas) containing all formulas of the form* $A \to \Box A$ *for all* A*, is determined by the class of all boolean valuations on which* $\Box$ *is associated either with the identity truth-function or with the (1-ary) constant true truth-function. The modal logic involved here is the normal modal logic)* KT_c *in the nomenclature of Chellas (1980). In general to obtain the* local *consequence relation* $\vdash_{\mathsf{S}}$ *from* S*, a logic in this sense, define* $\Gamma \vdash_{\mathsf{S}} A$ *to mean that* A *can be derived from the theorems of* S *together with formulas in* Γ *by means of Modus Ponens. If we allowed the rule Necessitation as well as Modus Ponens we would be defining instead the* global *consequence relation associated with* S*. (For the present choice of* S *as* KT_c *this would make no difference, but we fix the notation* $\vdash_{\mathsf{S}}$ *here in the local sense so as to be able to use it below—with* S *as* $\mathsf{S4}$ *in Example 3.2—in cases where it does make a difference.) Although, as we began by noting, over some class of valuations by which the current consequence relation ("*$\vdash_{\mathsf{KT}_c}$*") is determined,* $\Box$ *is pseudo-truth-functional, it is clear that there is no such determining class over which* $\Box$ *is truth-functional, since this would imply the existence of a* $\Box$*-free, purely boolean formula* A *for which we had* $\Box q \dashv\vdash_{\mathsf{KT}_c} A$*, and evidently there is no such* A*. (A somewhat similar modal example was given in Humberstone, 1986.)* ◀

Since $\Box$ in Example 2.5 creates an extensional context ($C(p) = \Box p$) in the sense that $\Box A \leftrightarrow \Box B$ is true on any such valuation which verifies $A \leftrightarrow B$—a more suitable definition, purified of reference to specific connectives, follows in due course—the appropriate semantic analogue of extensionality is determination by some class of valuations over which each connective is pseudo-truth-functional rather than requiring here truth-functionality.

To bring this discussion to bear on the issues of §1, let us state the obvious generalization of Proposition 1.3 (or Corollary 1.8), for which we require the notion of idempotence, suitably generalized in Definition 2.6(i), and it is convenient to introduce in the same breath the related notion of conservativity:[7]

DEFINITION 2.6. *Let* X *be a nonempty set. A function* $f : X^n \longrightarrow X$ *is*

(i) idempotent *when* $f(\overbrace{x, \ldots, x}^{n \text{ times}}) = x$*, for all* $x \in X$*; and*

(ii) conservative *when for all* $x_1, \ldots, x_n \in X$*,* $f(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$*.* ◀

There is a slight terminological hazard lurking here in the $n = 1$ case; Definition 2.6(i) rules such an f idempotent when $f(x) = x$ (all $x \in X$), whereas according to another

[7] This use of 'conservative' can be found, for example, in Goldstern & Pinsker (2008) and—as generalized in the following sentence—has been current in universal algebra for about 30 years, though the original definition of the term (as applied to functions) in Quackenbush (1972) was somewhat different. In the same author's Quackenbush (1974) the close relationship between conservative operations in the present sense—their called quasi-projections—and idempotence is evident, Quackenbush introducing the intermediate concept of k-idempotence ($k \in Nat$) to apply to any f, assumed n-ary, when $f(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$ whenever $|\{a_1, \ldots, a_n\}| \leq k$.

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commonly encountered usage 1-ary functions f are said to be idempotent when the weaker condition: $f(f(x)) = f(x)$ (all $x \in X$), is satisfied. The latter usage is a reflection of the fact that when restricted to such functions, the binary operation of functional composition is itself idempotent in the sense of Definition 2.6(i) (and so they get to be called idempotent elements of the semigroup of all functions from X to iself).

Our current interest is in the application of the properties defined in Definition 2.6 to (bivalent) *truth-functions*, that is, the special case in which $X = \{\mathrm{T}, \mathrm{F}\}$. This kind of conservativity implies idempotence for any X, by consideration of the case in $x_1 = \ldots = x_n$. The converse implication we do not have in general,[8] but it goes through for (any choice of n) whenever X has no more than two elements (as in the case of {T, F}), since if $f(x_1, \ldots, x_n) \notin \{x_1, \ldots, x_n\}$ then $x_1, \ldots, x_n$ cannot all be the same element of X (or that element would be $f(x_1, \ldots, x_n)$, by idempotence). Thus, for some i, j ($1 \leq i < j \leq n$) $x_i \neq x_j$, so since X only has two elements, $X = \{x_i, x_j\}$, and therefore $f(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$ after all. Although equivalent in the two-element case, the terminology of conservativity is often more suggestive, and the concept involved, if not the term, figured for example in the proof of Proposition 1.3 above, with the words "$a \wedge b$ must be a or else b, so…". In fact, in the more general setting needed to subsume the application to LC—in Proposition 1.7—we shall see in §3 that it is a syntactical analogue of conservativity ('componentiality', we call it) rather than idempotence itself—which by then will come in two flavours, plain and 'strong'—that is actually the relevant property when idempotence and conservativity come apart. Relevant to aggregation and disaggregation, that is. A proof of the following is straightforward, along the lines of the proof of Proposition 1.3(i) above:

PROPOSITION 2.7. *If an extensional consequence relation $\vdash$ (extensional generalized consequence relation $\Vdash$) is determined by a class of valuations on each of which some connective # in its language is associated with an idempotent truth-function, then # is aggregative and disaggregative according to $\vdash$ (resp. $\Vdash$).*

Thus parts (i)–(ii), addressing $\wedge$, and (iii)–(iv), addressing $\vee$, of Proposition 1.3 (or the corresponding parts of Corollary 1.8), which evidently do not depend on the fact that the truth-function concerned (the conjunction and disjunction truth-functions, respectively) were binary, also do not depend on the salience of a *single* truth-function in either case. In §3 of Humberstone (2013),[9] for example, a binary connective $*$ is considered with the common logical properties of conjunction and disjunction —$*$ thus being what is called in Humberstone (2011) a *hybrid* of conjunction and disjunction—along with an arbitrary choice of the familiar boolean connectives. This amounts to taking an interest in the boolean valuations on which this connective is associated with the usual conjunction truth-function as well as those on which it is associated with the disjunction

[8] For example, define a function f of two arguments on {0,1,2} by setting $f(x, y) = x$ if $x = y$ with $f(x, y) = 2$ otherwise; then f is idempotent but, since, for example, $f(0, 1) = 2$, not conservative. Pictorially, if one thinks of a 'multiplication table' for binary f, idempotence constrains only the (descending) diagonal, while conservativity rules out an entry anywhere that coincides neither with the head of its column nor with the head of its row.

[9] That discussion follows Rautenberg (1989); the topic is further treated in section 3.24 of Humberstone (2011); see also 5.35 of Humberstone (2011) for a discussion of similarly hybridizing the two binary projection connectives—that is, the consequence relation on a language with binary $\cdot$ (say) with A a consequence of Γ just in case this is so when every subformula $B_1 \cdot B_2$ (of a formula in $\Gamma \cup \{A\}$) is interpreted as B_1 and also when this is the case when $B_1 \cdot B_2$ is interpreted as B_2.

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truth-function.[10] So $*$ is pseudo-truth-functional though not truth-functional over this class of valuations—which is enough for the consequence relation thereby determined, $\vdash$ say, to be extensional. Here all the conditions of Proposition 2.7 are satisfied (whether we are consider the consequence relation or the generalized consequence relation), since in addition on every valuation in the class concerned, the associated truth-function is idempotent, semantically grounding the aggregativity and disaggregativity of $*$ (according to the $\vdash$ or $\Vdash$ in question). Indeed, these conditions would all be satisfied in the presence of further connectives which were pseudo-truth-functional but not truth-functional over any determining class of valuations: for example, in the presence of the $\Box$ of Example 2.5.[11]

EXAMPLE 2.8. *This is Proposition 1.1(ii) of Humberstone (2013):* $A, A \to B \vdash^{\mathsf{rep}}_{\mathsf{CL}} B$ *(for all* A, B*). Much as in the proof of Proposition 1.3(i), suppose, for a contradiction, that for some* $C(\cdot)$ *and some boolean valuation* v, $v(C(A)) = v(C(A \to B)) = \mathrm{T}$ *while* $v(C(B)) = \mathrm{F}$. *Let* g *be the truth-function associated with* $C(\cdot)$ *on* v, *with and use* $\to$ *to denote the truth-function associated with (the connective)* $\to$ *on any boolean valuation, with* a, $a \to b$ *and* b *for* $v(A)$, $v(A \to B)$ *and* $v(B)$, *respectively. We then have* $g(a) = g(a \to b) = \mathrm{T}$, *while* $g(b) = \mathrm{F}$. *Since* $g(b) \neq g(a)$ *and* $g(b) \neq g(a \to b)$, $b \neq a$ *and* $b \neq a \to b$, *and therefore* $a = a \to b$. *For the truth-function* $\to$, $a = a \to b$ *implies that* $a = a \to b = \mathrm{T}$ *(since* a *and* $a \to b$ *cannot both be F), so* $b = \mathrm{T}$, *contradicting the earlier finding that* $b \neq a$. *Thus* $A, A \to B \vdash^{\mathsf{rep}}_{\mathsf{CL}} B$, *as claimed. (We return to this matter in Example 4.3 below, where the status of a similar claim for* $\vdash_{\mathsf{LC}}$ *is addressed.)* ◀

More detail about the behaviour of the truth-function $\to$ than is needed for the argument of Example 2.8 has been invoked in the above discussion, however. All that is needed is the transition from $a = a \to b$ to $a = b$, so let us construct a partial truth-table indicating the constraint this imposes on valuations for a binary connective—call it #—playing the $\to$ role:

A	#	B
T	—	T
T	F	F
F	T	T
F	—	F

[10] For the benefit of any readers worried, as a referee for this journal was, that we are here sailing in a Tonkerly direction: the present account gives the hybrid the intersection rather than the union of the logical properties of the connectives hybridized. For further elaboration, see notes 9 and 11, and references there given.

[11] Of course, one can also consider hybrids against a nonclassical background setting, for example, exploring the logic of a connective with the common properties of intuitionistic $\to$ and $\leftrightarrow$, where "common properties" means those expressed by (outright, rather than conditional) $\vdash_{\mathsf{IL}}$-statements, as Rasiowa (1995) originally did for this pair of connectives in classical logic—though in fact she considered only their behaviour as $\vdash_{\mathsf{CL}}$-consequences of $\varnothing$, and also concentrated on the 'pure' case in which no other connectives were present. (Rasiowa made use of direct products of two-element matrices, which Rautenberg has observed works also for this case when the discussion is transferred to consequence relations. In this case, but not in every case—a counterexample being given at p. 532 of Rautenberg (1989), repeated in Humberstone (2011, p. 466). Because the 'product connectives' do not always yield the desired hybrids, the terminology of *product* for *hybrid* in Humberstone (1986) was an unfortunate choice.) We do not venture into such waters here, however.

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In constructing this table, where the dashes represent values we do not care about, on the basis of the constraint on valuations just mentioned, it is convenient to consider the constraint in its contraposed form: if $a \neq b$, then $a \neq a \to b$, so we enter the truth-value of $\neg A$—or equivalently, since these are the cases in which $a \neq b$, the truth-value of B—as the value of $A \to B$ in the two lines in which A's and B's values differ, leaving the others lines blank. (In the case of a binary idempotent function it would be the other way round: top and bottom lines filled out—with T and F respectively—with the middle two lines left blank.) Thus there are four truth-functions we could arrive at by filling the above blanks with choices of T and F, and the contextually embedded Modus Ponens inference from $C(A)$ and $C(A \# B)$ to $C(B)$ would go through for all $C(\cdot)$ on the consequence relation determined by V as long as # is associated on each $v \in V$ with one of these four functions, even in the ('properly') pseudo-truth-functional case in which different $v \in V$ make different such associations—for example, to take just the two here which are 'essentially binary', with $A \# B$ as hybridizing $A \to B$ with $\neg A \wedge B$.

But let us return to the issue of idempotence. What about conjunction and disjunction in intuitionistic logic? $\vdash_{\mathsf{IL}}$ is also determined by a class of valuations on all of which each of these connectives is associated with an idempotent truth-function: consider the analogues of the valuations $v_x^{\mathcal{M}}$ with $\mathcal{M}$ a Kripke model for intuitionistic logic and $v_x^{\mathcal{M}}(A) = \mathrm{T}$ is to defined to hold for any formula A if and only if $\mathcal{M} \models_x A$.[12] Yet the deliverances of Proposition 2.7 fail in this case, as we saw under Examples 1.6; so what has gone wrong? A preliminary response would be that that result has among its preconditions that the consequence relation concerned should be determined by some class of valuations over which every primitive connective is pseudo-truth-functional, and even if for some of these connectives ($\wedge$, $\vee$) we have the stronger condition satisfied without the 'pseudo', for others—$\neg$ and $\to$—which need to be taken into account because the conclusion concerns to the replacement core, and so universally quantifies over contexts involving them, not even the weaker version with the 'pseudo' is satisfied. If it were, $\vdash_{\mathsf{IL}}$ would be extensional and not just weakly extensional. This consideration notwithstanding, the preliminary response offers only provisional satisfaction: after all, as Proposition 1.7 tells us, $\vdash_{\mathsf{LC}}$ also enjoys the aggregation and disaggregation properties with which we are concerned, and $\vdash_{\mathsf{LC}}$, just like $\vdash_{\mathsf{IL}}$, is weakly extensional without being extensional (as the example in note 6 shows). To get to the bottom of this—as we shall in Proposition 3.6—and also not least because of their own intrinsic interest, we need to look further into idempotence and related properties.

§3. Idempotence and variations. While there is a single natural notion of idempotence for truth-functions, namely that given in the preceding section (Definition 2.6(i)), for *connectives* there are two immediately obvious candidates, most clearly contrasted relative to a generalized consequence relation $\Vdash$ in Definition 3.1(i), (ii). Definition 3.1(iv) adjusts (ii) this for consequence relations in the now familiar 'conditional' manner to handle the absence of multiple conclusions. The terminological contrast here explained is that of Humberstone (2011).

[12] Thus, we get a determining class of valuations by considering all such $v_x^{\mathcal{M}}$; alternatively, pick a single characteristic model—for example, the canonical model—and consider the various $v_x^{\mathcal{M}}$ arising from this as the choice of $\mathcal{M}$.

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DEFINITION 3.1.

(i) An n-ary connective # in the language of $\Vdash$ *is* idempotent *according to* $\Vdash$ *if for all A in the language of* $\Vdash$ *we have*

$$A \Vdash \#(\underbrace{A, \ldots, A}_{n \text{ times}}) \qquad \textit{and} \qquad \#(\underbrace{A, \ldots, A}_{n \text{ times}}) \Vdash A.$$

(ii) Such a # is strongly idempotent *according to* $\Vdash$ *when for all* $A_1, \ldots, A_n$ *(in the language of* $\Vdash$*), we have:*

$$A_1, \ldots, A_n \Vdash \#(A_1, \ldots, A_n) \textit{ and } \#(A_1, \ldots, A_n) \Vdash A_1, \ldots, A_n.$$

(iii) # is idempotent *according to a consequence relation* $\vdash$ *when for all A in the language of* $\vdash$*, we have* $A \dashv\vdash \#(A, \ldots, A)$ *(again, n occurrences of "A").*

(iv) # is strongly idempotent *according to a consequence relation* $\vdash$ *when for all* $A_1, \ldots, A_n$ *(in the language of* $\vdash$*),* $A_1, \ldots, A_n \vdash \#(A_1, \ldots, A_n)$ *and also for all formulas* $A_1, \ldots, A_n$, D *and all sets of formulas* Γ*, if* $\Gamma, A_i \vdash D$ *for each i* $(1 \leq i \leq n)$*, then* $\Gamma, \#(A_1, \ldots, A_n) \vdash D$*. If* $\vdash$ *is disjunctive, we have, as an equivalent, the simpler unconditional form for this second condition:* $\#(A_1, \ldots, A_n) \vdash A_1 \vee \ldots \vee A_n$.

(v) The properties defined in (i)–(iv) *can all be applied more generally to n-ary contexts* $C(p_1, \ldots, p_n)$ *by putting* $C(A_1, \ldots, A_n)$ *in place of* $\#(A_1, \ldots, A_n)$ *in the definitions.* ◀

Evidently, strong idempotence implies idempotence according to a given $\Vdash$, since we may take $A_1 = \ldots = A_n$, and it is easily verified that idempotence implies strong idempotence according to any extensional $\Vdash$ (as per Definition 2.1(iii)); not too much thought establishes the corresponding conclusions for the case of strong idempotence and idempotence for consequence relations (here using Definition 2.2(iii) for the appropriate notion of extensionality). Note that if $\wedge$ and $\vee$ are present in the language of $\vdash$ and behave as expected according to $\vdash$, then saying that binary # is strongly idempotent according to $\vdash$ is equivalent to saying that for all formulas A and B: $A \wedge B \vdash A \# B \vdash A \vee B$. ("As expected" here means disjunctivity for $\vdash$, explained in Definition 1.5, with the corresponding 'conjuctivity' property in the case of $\wedge$: $\Gamma, A \wedge B \vdash D$ iff $\Gamma, A, B \vdash D$.)

The following modal example illustrates the possibility of a connective's being idempotent but not strongly idempotent (according to a given consequence relation); here $\vdash_{\mathsf{S4}}$ is the local consequence relation of S4, as explained in Example 2.5:

EXAMPLE 3.2. *Define* $A \# B = A \wedge \Box(A \rightarrow B)$*; according to* $\vdash_{\mathsf{S4}}$ *for example, # is idempotent, since on identifying A and B the second conjunct here drops out as vacuous. But # is not strongly idempotent, since* $q, r \nvdash_{\mathsf{S4}} q \wedge \Box(q \rightarrow r)$. ◀

Thus in Example 3.2 the derived binary connective # fails to satisfy the first of the two strong idempotence conditions, involving a transition from components to compound. Consequence relations such as $\vdash_{\mathsf{S4}}$, figuring in the example, are not even weakly extensional, so it is not surprising that we can find connectives which are idempotent though not strongly idempotent here. On the other hand, $\vdash_{\mathsf{IL}}$ *is* weakly extensional, and this means we would not be able to find idempotent-but-not-strongly-idempotent connectives in quite the same way: as violating the first of the two conditions in Definition 3.1(iv). This is because, as remarked at the end of note 6, (WE) implies (LE), and one half of the condition of idempotence, namely $A \vdash A \# A$, together with (LE), taking $C(p)$ as $A \# p$:

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$A, B, A \# A \vdash A \# B$, yield the desired conclusion, $A, B \vdash A \# B$, by a cut on $A \# A$. This means that in the intuitionistic case, it will be a failure of the second condition defining strong idempotence—the one for which the $\Vdash$ version had to be reformulated—that has to be exploited (which means in fact a failure of (RE)′) to find a connective which is idempotent but not strongly so:

EXAMPLE 3.3. *Define* $A \ddot{\vee} B = (A \to B) \to B$, *as in Humberstone (2011), especially 4.22.8 and 7.22.7–8, q.v. for further information, in particular on how this connective behaves according to* $\vdash_{\mathsf{IL}}$. *On putting* $A = B$, *we see that* $\ddot{\vee}$ *is idempotent according to* $\vdash_{\mathsf{IL}}$. *But* $q \ddot{\vee} r$ *does not have* $q \vee r$ *as an intuitionistic consequence, as it would have to satisfy the second strong idempotence condition in Definition 3.1*(iv) *(We shall see more of* $\ddot{\vee}$ *in Example 4.2 below, and the discussing preceding it.)* ◀

As an indicator of the possible promise of strong idempotence of a (for simplicity) binary # as critical to the aggregativity/disaggregativity of # is the evident resemblance of the two conditions, here put in $\Vdash$-notation for conciseness:

strong idempotence	corresponding aggreg/disagg've behaviour
$A, B \Vdash A \# B$	$A, B \Vdash^{\mathsf{rep}} A \# B$
$A \# B \Vdash A, B$	$A \# B \Vdash^{\mathsf{rep}} A, B$

These formal resemblances notwithstanding, we already know that even strong idempotence is not strong enough to secure aggregation and disaggregation in general, in view of Examples 1.6, since $\wedge$ and $\vee$ are there seen not to exhibit aggregation/disaggregation, despite the strong idempotence of these connectives in IL (i.e., according to $\vdash_{\mathsf{IL}}$ or $\Vdash_{\mathsf{IL}}$).

To isolate the required further strengthening of strong idempotence, we need a suitable transformation of the notion of conservativity from Definition 2.6(ii). This defined a function or operation to be conservative when its value for a given tuple of arguments was identical with one of those arguments, so we need to simulate the idea of identity or equality within propositional logic. Such a simulation has been the *raison d'être*, within algebraic logic, of the concept of an equivalential logic, which we now define.[13]

DEFINITION 3.4.

(i) *A consequence relation* $\vdash$ *is* equivalential *if there is a set of formulas* $E(p_1, p_2)$ *in the language of* $\vdash$*—more precisely a nonempty set of 2-ary contexts in which no propositional variables other than* p_1 *and* p_2 *occur—such that* $\vdash E(A, A)$ *and* $E(A, B), C(A) \vdash C(B)$ *for all formulas* A, B, *and contexts* $C(\cdot)$.

(ii) *The set* $E(p_1, p_2)$ *in the above definition is called a* set of equivalence formulas *for* $\vdash$*; if that set is a singleton, we call* $\vdash$ singly equivalential.

(iii) *The terms introduced in* (i) *and* (ii) *will be applied also in the case of generalized consequence relations without any change to the definition other than putting* $\Vdash$ *for* $\vdash$. ◀

Note that the appearance of "E" appears on the right of the "$\vdash$", in the first of the conditions on $\vdash$ in Definition 3.4(i) does not mean that we are now forgetting to write $\Vdash$

[13] See Humberstone (2013) for the equivalence of the present definition to others in the literature, and also—in the discussion following Corollary 2.6 there—for references to that literature, on which see also Czelakowski (2001).

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for $\vdash$ there. Rather, the current notation means simply that for each $B \in E(A, A)$, we have $\vdash B$.[14] Thus in the pure implicational fragment of IL, to say that $E(A, B)$ is a consequence of some set of formulas Γ, where we have $E(p_1, p_2)$ in mind as $\{p_1 \rightarrow p_2, p_2 \rightarrow p_1\}$, amounts to saying that each of $A \rightarrow B$, $B \rightarrow A$, is a consequence of Γ. Of course, this issue does not arise in the more commonly encountered language for IL (or CL) in which we can produce a singleton $E(p_1, p_2)$ in the shape of $\{p_1 \leftrightarrow p_2\}$: whether $\leftrightarrow$ is taken as a primitive connective or whether instead $p_1 \leftrightarrow p_2$ is taken to be the formula $(p_1 \rightarrow p_2) \wedge (p_2 \rightarrow p_1)$, we are here dealing with a singly equivalential logic. In fact, though an equivalential consequence relation need not be extensional, or vice versa,[15] we pause to observe (without proof) here that if $\vdash$ is a weakly extensional substitution-invariant consequence relation extending $\vdash_{\mathsf{IL}}$, then $\vdash$ is equivalential, with set of equivalence of formulas $\{p_1 \leftrightarrow p_2\}$.

Now, the point of introducing equivalentiality here is precisely to isolate those features of $\leftrightarrow$ in the intermediate logic LC that are exploited in the proof of Proposition 1.7. In Definition 3.5, we give our syntactic analogue of conservativity of functions—which we noted coincided with idempotence when restricted to functions on a two-element set—for the case of singly equivalential consequential logics (such as $\vdash_{\mathsf{IL}}$ and $\vdash_{\mathsf{LC}}$) with this prototype in mind, though the "singly" restriction is not essential (as we note in Definition 3.5). What to call it, though? The term "conservative" could be retained for this adapted concept, the last to be considered among variations on idempotence (for connectives), but in view of its widespread use in logic for quite unrelated concepts—in particular in connection with the phrase *conservative extension* (e.g., citing an example at random: Meyer, 1973), though also its use as a condition on generalized quantifiers (originally in Keenan & Stavi, 1986, p. 275)—it seems wiser to change terminology. The term chosen here is "componential" to reflect the idea that a compound formed by a connective with this property is equivalent—in a sense clarified in the definition to follow—to one of its components (immediate proper subformulas).[16]

DEFINITION 3.5. *Let $\vdash$ be a disjunctive, equivalential consequence relation with # an n-ary connective in its language. # is* componential *according to $\vdash$ if and only if, using standard notation for iterating the binary disjunction concerned:*

$$\vdash \bigvee_{i=1}^{n} E(\#(A_1, \ldots, A_n), A_i)$$

for all formulas $A_1, \ldots, A_n$. This definition requires no further elaboration in the case in which $\vdash$ is singly equivalential (with $E(p_1, p_2)$ as its set of equivalence formulas), but requires an explanation for the general case in which there are several formulas in the smallest set of equivalence formulas: in this case we interpret $\bigvee_{i=1}^{n} E(\#(A_1, \ldots, A_n), A_i)$

[14] Observe also that if $E(p_1, p_2)$ and $E'(p_1, p_2)$ are both of them sets of equivalence formulas for a given $\vdash$, then $E(p_1, p_2) \dashv\vdash E'(p_1, p_2)$, where again an apparently multiple-conclusion claim abbreviates a multiplicity of single-conclusion claims. (Even more to the point: we have $\vdash E(E'(p_1, p_2), E(p_1, p_2))$.)

[15] For equivalentiality without extensionality, consider $\vdash_{\mathsf{S4}}$ (as in Example 3.2), with $E(p_1, p_2)$ the (unit set of) the strict equivalence of p_1, p_2; extensionality without equivalentiality is illustrated by any atheorematic fragment of $\vdash_{\mathsf{CL}}$, for example the $\{\wedge, \vee\}$-fragment.

[16] There are other uses of this adjective which are, however, unlikely to cause interference, such as in the *componential analysis* once popular in linguistic circles, or when it is misused in place of *compositional* in connection with semantic theories (as in Bigelow, 1981, for example).

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as the set of all disjunctions $\bigvee_{i=1}^{n} D_i$ *such that* $D_i \in E(\#(A_1, \ldots, A_n), A_i)$. *The definition of componentiality then requires that* $\vdash \bigvee_{i=1}^{n} D_i$ *for each such disjunction.* ◀

Some comments on the definition: If we were defining a corresponding notion for generalized consequence relations we could forget about disjunction and simply list rather than disjoin the E-formulas concerned:

$$\Vdash E(\#(A_1, \ldots, A_n), A_1), \ldots, E(\#(A_1, \ldots, A_n), A_n).$$

This makes immediate sense in the singly equivalential case but in the general case, as in Definition 3.5 a few words of explanation are in order, which we confine to a footnote.[17] If, sticking with consequence relations but wanting to drop the disjunctivity pre-condition, we would naturally adopt a conditional formulation:

$$\textit{If } \Gamma, E(\#(A_1, \ldots, A_n), A_i) \vdash D \textit{ for each } i \ (1 \leq i \leq n), \textit{ then } \Gamma \vdash D.$$

The version of componentiality provided by Definition 3.5 itself will serve us perfectly satisfactorily in what follows.

One notable consequence of the role of disjunction in Definition 3.5 is that if a disjunctive consequence relation has the Disjunction Property (in the sense that when a disjunction is a consequence of $\varnothing$, then so is one or other of its disjuncts), then componentiality takes on a degenerate aspect: if n-ary # is componential according to such a $\vdash$, then # is a projection connective: for some i $(1 \leq i \leq n)$, we have $\vdash E(\#(A_1, \ldots, A_n), A_i)$ for all $A_1, \ldots, A_n$. (This implies that $E(\#(A_1, \ldots, A_n) \dashv\vdash A_i$ for the i in question.) Indeed this phenomenon arises even if $\vdash$ is not disjunctive (e.g., the implicational fragment of $\vdash_{\mathsf{IL}}$), given a suitable understanding of the Disjunction Property for disjunction-free logics: For any formulas A, B, if for all Γ, C, for which $\Gamma, A \vdash C$ and $\Gamma, B \vdash C$ we have $\Gamma \vdash C$, then either $\vdash A$ or $\vdash B$ (alternatively put: ... then either for all Γ, C, if $\Gamma, A \vdash C$ then $\Gamma \vdash C$ or else Γ, C, if $\Gamma, B \vdash C$ then $\Gamma \vdash C$). When the crucial connection between componentiality and the aggregation/disaggregation phenomena becomes clear—in Proposition 3.6 below—this will be seen to explain the paucity of instances of that phenomenon in IL itself, which famously (and by contrast with LC and its extensions) possesses the Disjunction Property.

Moving toward that clarification, let us take stock of the three idempotence-related conditions on connectives that have been in play in this section. An arrow ($\Rightarrow$) from one property to another means that for any consequence relation $\vdash$, any connective with the first property according to $\vdash$ has the second property according to $\vdash$, though we must in the case of the first arrow below recall that componentiality has only been defined for connectives according to equivalential consequence relations (indeed by the letter of Definition 3.5, disjunctive such consequence relations, though as we have noted, this last is an easily lifted restriction):

$$\textit{Componential} \quad \Rightarrow \quad \textit{Strongly Idempotent} \quad \Rightarrow \quad \textit{Idempotent}$$

[17] The suggested explanation uses an 'order-forgetful' version of the Cartesian product construction, for which we use the notation $\cdot$, as in Van Fraassen (1969, p. 484): given two sets Γ and Δ of formulas, then $\Gamma \cdot \Delta$ is the set of all unordered pairs $\{A, B\}$ for which $A \in \Gamma$, $B \in \Delta$. Then the inset $\Vdash$-statement above, for the case in which $E(p_1, p_2)$ is not assumed to be a singleton, and taking # to be binary for simplicity, can be formulated as the condition that $\Vdash \Theta$ for each $\Theta \in E(A \# B, A) \cdot E(A \# B, B)$. Relaxing the assumption that # is binary, we may use a generalized form of $\cdot$, $\bigodot$ say, with $\bigodot(\Gamma_1, \ldots, \Gamma_n, \ldots) = \{\{A_1, \ldots, A_n, \ldots\} \mid A_i \in \Gamma_i\}$, though the ellipsis going beyond n here is not required for the present application, as we consider only finitary #.

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Both implications are proper, even restricting attention to equivalential (and disjunctive) consequence relations. Among intermediate logics, for example, the irreversibility of the first implication here is illustrated by the case of conjunction (or disjunction) in intuitionistic logic itself, in view of Examples 1.6 (taken together with Proposition 3.6 below), while that of the second implication is illustrated by Example 3.3.

Now, if one is available, a single explanation of the (dis)aggregativity is preferable to a patchwork of unrelated sufficient conditions. We do not want the explanation of why, for example, a hybrid of $\wedge$ and $\vee$ remarked on in §2. exhibits these properties, to be given in terms of idempotence (with the further presumption[18] that all contexts are extensional), while on the other hand the case of $\wedge$ and $\vee$ themselves as they behave in LC (where the further presumption is not satisfied anyway) is explained by the quite different property of componentiality (or perhaps—see Proposition 4.1—strong idempotence). Thus is of interest to observe that in a suitably classical setting the three idempotence-related properties we are considering are all equivalent, meaning that we do not need to change explanatory horses as we change logics: what matters all along is precisely componentiality, which in certain logical settings, though not in all, happens to coincide with idempotence,[19] making possible a unifying rapprochement of our various sufficient conditions for the (dis)aggregation. Not only that, but the condition in question is also necessary, as the following result assures us. The cases of aggregativity and disaggregativity are listed separately here because in the present more general setting, we can no longer rely on some of the details (concerning $\rightarrow$ and $\wedge$ in extensions of IL) appealed to in the proof of Proposition 1.10; aspects of that proof will re-appear here however, and the reader will recognise, for example, the (1) $\Rightarrow$ (2) part of the following proof as a distillation of the proof of Proposition 1.7, which itself used very little of the power of LC (all the distinctively intuitionistic properties of $\rightarrow$, for instance—the Deduction Theorem, $\rightarrow$-contraction, etc.).

PROPOSITION 3.6. *Suppose that # is an n-ary connective in the language of a disjunctive equivalential consequence relation $\vdash$. Then the following are equivalent:*

(1) *# is componential according to $\vdash$;*

(2) *# is aggregative according to $\vdash$;*

(3) *# is disaggregative according to $\vdash$.*

Proof. For simplicity, we treat explicitly only the case in which $\vdash$ is singly equivalential, and, to aid readability, write $E(D, D')$ as $D \leftrightarrow D'$ (literally, $E(D, D')$ would be $\{D \leftrightarrow D'\}$), and finally, take the $n = 2$ case as representative, to reduce subscript clutter.

First, (1) $\Leftrightarrow$ (2). (1) $\Rightarrow$ (2): Suppose we have, for $\vdash$ as described, an instance of (1):

$$\vdash ((A \,\#\, B) \leftrightarrow A) \vee ((A \,\#\, B) \leftrightarrow B) \qquad (i)$$

[18] This enters Proposition 2.7 with the reference to determination by a class of valuations over which all connectives are pseudo-truth-functional.

[19] We have already had occasion to observe that idempotence and conservativity coincide for functions on a two-element set, though not in general (note 8). This special two-element case is all that we need even for the coincidence of componentiality and idempotence of connectives in the pseudo-truth-functional setting (Definition 2.4(ii)), since valuations are considered one at a time. The componentiality of conjunction and disjunction in LC, exploited in the proof of Proposition 1.7, similarly reflects—or if you prefer, is reflected by—the conservativity of the meet and join operations in linearly ordered Heyting algebras (subdirectly irreducible L-algebras in the sense of Horn, 1969, where they amount to *min* and *max*; *cf.* also Dunn & Meyer, 1971, p. 227, paragraph preceding Theorem 4, or Bull, 1964, or the proof of Lemma 5.2, p. 120 in Köhler, 1981).

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For an arbitrary context $C(\cdot)$, equivalentiality gives (ii), (iii):

$$(A \# B) \leftrightarrow A, C(A) \vdash C(A \# B) \tag{ii}$$

$$(A \# B) \leftrightarrow B, C(B) \vdash C(A \# B) \tag{iii}$$

And from (ii) and (iii), by disjunctivity, we have (iv):

$$((A \# B) \leftrightarrow A) \vee ((A \# B) \leftrightarrow B), C(A), C(B) \vdash C(A \# B). \tag{iv}$$

Thus 'cutting' with (i) gives us

$$C(A), C(B) \vdash C(A\#B), \tag{v}$$

establishing (2).

For (2) $\Rightarrow$ (1), suppose that # is aggregative according to $\vdash$, that is, $(C(A), C(B) \vdash C(A \# B)$ for all $A, B, C(p)$. For any given A, B, choose $C(p)$ as $(p \leftrightarrow A) \vee (p \leftrightarrow B)$ to derive (1).

Next, (1) $\Leftrightarrow$ (3). (1) $\Rightarrow$ (3): We begin again with an instance of (1).

$$\vdash ((A \# B) \leftrightarrow A) \vee ((A \# B) \leftrightarrow B) \tag{i}$$

For an arbitrary context $C(\cdot)$, equivalentiality gives (ii), (iii):

$$(A \# B) \leftrightarrow A, C(A \# B) \vdash C(A) \tag{ii}$$

$$(A \# B) \leftrightarrow B, C(A \# B) \vdash C(B) \tag{iii}$$

And from (ii) and (iii), by disjunctivity, we have (iv):

$$((A \# B) \leftrightarrow A) \vee ((A \# B) \leftrightarrow B), C(A \# B) \vdash C(A) \vee C(B), \tag{iv}$$

from which a Cut with (i) delivers all we need for (3):

$$C(A \# B) \vdash C(A) \vee C(B). \tag{v}$$

For the (3) $\Rightarrow$ (1) direction, suppose that # is disaggregative according to $\vdash$, that is, $C(A \# B) \vdash C(A) \vee C(B)$ for all $A, B, C(p)$. Fixing on A, B, choose $C(p)$ as $(A \# B) \leftrightarrow p$ to obtain (1). □

Here we have completed our mission of "getting to the bottom" of how aggregation and disaggregation are related to something like idempotence, finding the something in question to be componentiality. Evidently, the distillation of the relevant considerations provided by Proposition 3.6 is rather more general than the specifically intermediate logical-characterization in terms of extensions of $\vdash_{\mathsf{LC}}$, to which we nevertheless return in §4. for some unfinished business. The greater generality alluded to is especially evident in the area of substructural logics, several of which are complete w.r.t classes of linearly ordered structures (for the significance of which in connection with LC, see note 19): RM ("R-Mingle"), Łukasiewicz's many-valued logics, Hájek's 'Basic Logic', and Meyer and Slaney's Abelian Logic coming to mind as worth investigating for the componentiality of their conjunction and disjunction connectives.[20] Only the case of RM is touched on in

[20] The Abelian case is especially interesting for the abundance of distinct and independently motivated consequence relations all agreeing on the consequences of $\varnothing$: see Paoli *et al.* (2008). The LC case may put one in mind of the local consequence relation associated with its modal companion, S4.3, but this is a blind alley for present purposes: though disjunctive and equivalential (via strict equivalence) $\wedge$ and $\vee$ are not componential according to it.

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the following remarks. The associated consequence relation $\vdash_{\mathsf{RM}}$ takes us from a Γ set of formulas to those formulas obtainable from Γ and a set of axioms for RM by means of Modus Ponens and the rule $A, B / A \wedge B$. This is a singly equivalential consequence relation with $p_1 \leftrightarrow p_2$ (abbreviating $(p_1 \rightarrow p_2) \wedge (p_2 \rightarrow p_1)$). Moreover, $\vdash_{\mathsf{RM}}$ is disjunctive, the less obvious half of which (namely, that $\Gamma, A \vdash_{\mathsf{RM}} C$ and $\Gamma, B \vdash_{\mathsf{RM}} C$ imply $\Gamma, A \vee B \vdash_{\mathsf{RM}} C$) may be established with the aid of a suitable form of the Deduction Theorem for RM—the details of which (going back to work by A. Avron and by M. Tokarz) can be found in Blok & Raftery (2004), p. 17. Thus the present case falls within the ambit of Proposition 3.6. A well-known RM-theorem schema is $(A \rightarrow B) \vee (B \rightarrow A)$—in other words, Dummett's LC-axiom again—and indeed conjunction and disjunction are both componential, and thus, by Proposition 3.6, (dis)aggregative, according to $\vdash_{\mathsf{RM}}$. That is, concentrating on the case of $\wedge$, we do have $\vdash_{\mathsf{RM}} ((A \wedge B) \leftrightarrow A) \vee ((A \wedge B) \leftrightarrow B)$ for all formulas A, B.[21] The relation between this disjunction and Dummett's axiom in RM, however, is very different from what it was in LC. In the latter case the two disjunctions are disjunctwise equivalent: in any intermediate logic, $A \rightarrow B$ is freely interreplaceable with $(A \wedge B) \leftrightarrow A$ and $B \rightarrow A$ with $(A \wedge B) \leftrightarrow B$, so Dummett's axiom is and the componentiality condition (for $\wedge$, though the same point holds for $\vee$) are simply minor reformulations of each other. In RM, by contrast, $A \rightarrow B$ is by no means equivalent to $(A \wedge B) \leftrightarrow A$, $q \rightarrow r$ not provably implying $q \rightarrow (q \wedge r)$, for example.[22]

This discussion should not create the impression that it is only $\wedge$ and $\vee$ (and their obvious n-ary generalizations) which deserve attention, in a substructural setting or otherwise. Minimally further afield, let us note that connectives derived by composing componential connectives are themselves automatically componential; thus the 'two-thirds majority' connective # defined by:

$$\#(A, B, C) = (A \wedge B) \vee (B \wedge C) \vee (A \wedge C)$$

is componential (according to $\vdash_{\mathsf{CL}}$ or $\vdash_{\mathsf{LC}}$, for instance). There are even binary connectives which are componential and sufficiently unlike conjunction and disjunction as to be non-commutative, such as the hybrid-of-projections connective mentioned in note 9. Further varying the diet of examples, not to be overlooked is the possibility of componential connectives—though an immediate logical motivation is unclear—corresponding arising from conservative commutative operations arising out of tournaments (as groupoids—see Ježek *et al.* (2000) and references) which typically fail to be associative, as when the underlying binary relation is not transitive.[23]

We conclude this section with a puzzle for the reader's amusement. Definition 3.5 as applied to the case of # as n-ary disjunction of course rules this connective to be classically componential: $\vdash_{\mathsf{CL}} \bigvee_{i=1}^{n}(\bigvee_{j=1}^{n}(A_j) \leftrightarrow A_i)$. Proposition 3.6 predicts that this choice of # is aggregative according to CL—correctly since all contexts are classically $\wedge$-aggregative.

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[21] The easiest way to see this is to put q, r, for A, B, and check that the resulting formula is valid in the four-element Sugihara matrix, conveniently displayed at p. 231 of Dunn and Hardegree Dunn & Hardegree (2001). For background and further information as to why this simple check suffices, see Dunn (1970) and Blok & Raftery (2004).

[22] Since only two variables are involved here the four-element Sugihara matrix (see note 21) can be used to check this.

[23] Think of the 'beats' relation in the game of scissors–paper–stone, giving a 3-element nonassociative groupoid whose fundamental operation takes a pair $\langle x, y\rangle$ or $\langle y, x\rangle$ to whichever of x, y beats the other when $x \neq y$, and to x (alias y) when $x = y$.

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The well-known analogy between disjunction and existential quantification is apparently borne out by the satisfaction of the corresponding componentiality-like condition in classical predicate logic (here indicated by QCL): $\vdash_{\mathsf{QCL}} \exists x((\exists y.\varphi(y)) \leftrightarrow \varphi(x))$. But the word "classically" in the reminder that all contexts are classically $\wedge$-aggregative is a reference to CL, not QCL, and, as remarked in Humberstone (2013), the presence of quantifiers *prima facie* produces contexts which are not $\wedge$-aggregative:[24] Corollary 1.8(i), for example, told us that $\bigwedge_i C(B_i) \vdash_{\mathsf{CL}} C(\bigwedge_i B_i)$, while $\exists x(\varphi(x)) \wedge \exists x(\psi x) \nvdash_{\mathsf{QCL}} \exists x(\varphi(x) \wedge \psi(x))$. "Prima facie" because it is not immediately clear how to extend the notion of a context in play in our discussion to languages with variable binding devices, which is only one issue raised by the presence of such devices (for some others, see the start of Appendix C in Blok & Pigozzi, 1989). Our point here is simply to draw to the reader's attention the desirability of accounting for the present disanalogy, rather than to urge any particular such account.

§4. Last words on LC. In this section we discuss two topics: the relation between strong idempotence and componentiality as far as LC is concerned, and the relation between the replacement cores of $\vdash_{\mathsf{LC}}$ and $\vdash_{\mathsf{CL}}$. The question naturally arises as to whether, in the conspicuously aggregation-friendly environment provided by LC itself (witness Proposition 1.7), strong idempotence implies componentiality. Here a subtlety emerges. The concepts of idempotence, strong idempotence and componentiality, can be deployed for n-ary contexts rather than just for n-ary connectives, as noted for the first two of these in Definition 3.1(v), with an analogous understanding being available for the third. With this generalization of the terminology in place—in terms of which implications such as "Componential ⇒ Strongly Idempotent ⇒ Idempotent" remain intact—an intriguing situation arises:

PROPOSITION 4.1.

(i) Every (primitive or derived) binary connective # which is strongly idempotent according to $\vdash_{\mathsf{LC}}$ *is componential according to* $\vdash_{\mathsf{LC}}$*, but:*

(ii) Not every binary context $C(p_1, p_2)$ *which is strongly idempotent according to* LC *is componential according to* $\vdash_{\mathsf{LC}}$.

The heart of the proof, which can be found in Appendix B, is what is there called the Local Supervenience Lemma, from which we extract a further observation here. Let us describe a formula as *becoming true* in the transition from x to y (with Rxy) when it is false at x and true at y—and recall that there is no question of a formula's analogously 'becoming false' in such a transition. The lemma in question—used in showing Proposition 4.1(i)—or, more accurately, the proof of the lemma, reveals that in any LC-model, for any n-ary connective # (primitive or derived) whenever $\#(A_1, \ldots, A_n)$ becomes true in a transition, some A_i ($i = 1, \ldots, n$) must also become true in that transition. This in turn assures us of the correctness for LC of some interesting principles. For example, we are assured *a priori* that any primitive or derived binary connective # we for any have for all A, B:

$$\vdash_{\mathsf{LC}} (A \,\#\, B) \vee ((A \,\#\, B) \rightarrow (A \vee B)),$$

[24] In Humberstone (2013) $\wedge$-aggregative contexts are simply called aggregative.

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where "*a priori*" means: in advance of investigating the individual choices of # one might make. For the case of binary # there are, according to the calculations of Horn (1969, p. 480), altogether 342 such choices, while the number of pairwise LC-nonequivalent ternary connectives—for which an evident variant of the above schema applies—exceeds 10^{11}.

Let us turn to the second of the two topics mentioned in the opening sentence of this section, which remains something of a loose end in the discussion to date. Nothing said in up to this point in the present paper (or anywhere in Humberstone, 2013) explicitly rules out the following hypothesis: $\vdash_{\mathsf{CL}}^{\mathsf{rep}} = \vdash_{\mathsf{LC}}^{\mathsf{rep}}$. (The possibility of distinct consequence relations having the same replacement core is illustrated in Humberstone, 2013, Example 1.3.) As already noted here after Corollary 1.8, the (dis)aggregativity phenomena which distinguish CL from IL put LC on the CL side of this division. Similarly, the example used in Humberstone (2013) that we do not have, for arbitrary A, B: $A, A \to B \vdash_{\mathsf{IL}}^{\mathsf{rep}} B$ (taking $C(p) = (p \to q) \to r$) would not have refuted the hypothesis that $A, A \to B \vdash_{\mathsf{LC}}^{\mathsf{rep}} B$. To settle the question of whether $\vdash_{\mathsf{CL}}^{\mathsf{rep}} = \vdash_{\mathsf{LC}}^{\mathsf{rep}}$ it suffices to note that we have the Ł-extensionality schema of Appendix A in the form:

$$A, \neg A \vdash^{\mathsf{rep}} B$$

for $\vdash = \vdash_{\mathsf{CL}}$ but not for $\vdash = \vdash_{\mathsf{LC}}$, since we do not have $q \to s, \neg q \to s \vdash_{\mathsf{LC}} r \to s$. (This example shows that the only Ł-extensional intermediate logic is CL, in fact, since we can substitute $q \vee \neg q$ for s and $r \to r$ for r.) However, one would like to know whether a counterexample can be given in the positive (i.e., $\{\to, \wedge, \vee\}$) fragment, and perhaps more specifically, whether $A, A \to B \vdash^{\mathsf{rep}} B$ itself already provides such a counterexample.

We may approach this question by considering the issue of monotone contexts, where $C(p)$ is *monotone* according to $\vdash$ if $C(A) \vdash C(B)$ whenever $A \vdash B$. It is not hard to see that $C(p) = (q \to p) \to p$, alias the $q \mathbin{\ddot{\vee}} p$ of Example 3.3 above, is monotone according to CL (since $A \mathbin{\ddot{\vee}} B \dashv\vdash_{\mathsf{CL}} A \vee B$) though not according to IL, but the question of current interest is whether LC again lines up with CL, or, instead, with IL in this respect. In the following we make use of facts about $\ddot{\vee}$ (in the intermediate logics) mentioned in the references cited in Example 3.3.

EXAMPLE 4.2. *For a case of $C(p)$ monotone according to $\vdash_{\mathsf{CL}}$ but not $\vdash_{\mathsf{LC}}$, take $C(p) = (q \to p) \to p$ ($= q \mathbin{\ddot{\vee}} p$). This is monotone according to $\vdash$, in which setting $\ddot{\vee}$ and $\vee$ are equivalent, but not according to $\vdash_{\mathsf{CL}}$. For this $C(p)$, an example in which $A \vdash_{\mathsf{LC}} B$ while $C(A) \nvdash_{\mathsf{LC}} C(B)$, is obtained by taking A as r and B as $r \vee s$. Certainly we have $A \vdash_{\mathsf{LC}} B$, so we need only check that $C(A) \nvdash_{\mathsf{LC}} C(B)$). Suppose otherwise, which is to say, suppose that:*

$$(q \to r) \to r \vdash_{\mathsf{LC}} (q \to (r \vee s)) \to (r \vee s).$$

Then we could substitute s for q and rapidly infer, dropping the provable $s \to (r \vee s)$ antecedent on the right that this substitution would create, that

$$(s \to r) \to r \vdash_{\mathsf{LC}} r \vee s.$$

This is obviously not the case as it would render $\ddot{\vee}$ (as it was written above) and $\vee$ LC-equivalent, the converse already holding for $\vdash_{\mathsf{IL}}$, and the only intermediate logic in which these two coincide is CL. [Recall from Dummett (1959) that $\vee$ is definable in LC, $A \vee B$ being provably LC-equivalent to the conjunction of $A \mathbin{\ddot{\vee}} B$ with $B \mathbin{\ddot{\vee}} A$, LC being the least intermediate logic to provide this equivalence—in particular the direction from the

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conjunction of $\ddot{\vee}$-formulas to the disjunction $A \vee B$.[25] *] Incidentally, we could have confined ourselves to the implicational fragment in giving a counterexample to the suggestion that the present $C(p)$ was monotone, by keeping A as above, but taking B as $(r \to s) \to s$. If the context were monotone, we should then have:*

$$(q \to r) \to r \vdash_{\mathsf{LC}} (q \to (r \to s_{\centerdot} \to s)) \to (r \to s_{\centerdot} \to s).$$

This would then simplify down, after substituting s for q, to:

$$(s \to r) \to r \vdash_{\mathsf{LC}} (r \to s) \to s.$$

But again, the only intermediate logic in which $\ddot{\vee}$ is commutative is CL. ◄

It remains to convert the above example into something which shows that we do not have $A, A \to B \vdash^{\mathsf{rep}} B$ for all A, B, with $\vdash$ as $\vdash_{\mathsf{LC}}$. No actual conversion proves necessary, just an observation about the formula $C(A \to B)$ involved:

EXAMPLE 4.3. *We use A, B, $C(p)$ from Example 4.2, which turns the claim that $C(A), C(A \to B) \vdash_{\mathsf{LC}} C(B)$ into this:*

$$(q \to (r \to {\cdot}r \vee s)) \to (r \to {\cdot}r \vee s), (q \to r) \to r \vdash_{\mathsf{LC}} (q \to (r \vee s)) \to (r \vee s).$$

But since the leftmost formula (the current $C(A \to B)$) is LC*-provable (indeed* IL*-provable), this would give us the conclusion that*

$$(q \to r) \to r \vdash_{\mathsf{LC}} (q \to (r \vee s)) \to (r \vee s),$$

seen to be false in the discussion in Example 4.2. ◄

§5. Appendix A: extensionality and Ł-extensionality. Proposition 2.3 found, for a ('classically') restricted class of consequence relations $\vdash$, a context $C(p)$ of the language of $\vdash$, the notions of extensionality and weak extensionality in play in §2 coincided. In the presence of $\to$ and $\leftrightarrow$ exhibiting similarly classical behaviour this notion of the extensionality/weak extensionality of a context $C(\cdot)$ can be embodied in a formula-schema, and while our tendency in that section was to try and avoid the dependence of definitions of extensionality and its variants upon the presence of particular connectives with prescribed logical behaviour, this 'impure' approach is all that is available to those who, like Łulkasiewicz, saw logics simply as (certain) sets of ('provable') formulas. The schema in question is $(A \leftrightarrow B) \to (C(A) \leftrightarrow C(B))$. $C(\cdot)$ is extensional according to the logic when for all A, B, the relevant instance of this schema is provable. By doing this, we shall be able to report on some interesting observations from Łukasiewicz (1953) and Smiley (1963), packaged up as Proposition 5.1 below—though these authors do not make

[25] At this point we can address the matter of showing that, as mentioned after Proposition 1.9, $\vdash_{\mathsf{LC}}$ is the least intermediate consequence relation for which $C(A \vee B)$ is a consequence of $C(A)$ and $C(B)$ for all A and B is exactly LC when $C(p)$ is taken as $(q \to p) \to q$ (the second example mentioned under (iii) of Examples 1.6). Thus we are dealing with the $\vdash$-statement:

$$(q \to A) \to q, (q \to B) \to q \vdash (q \to (A \vee B)) \to q.$$

Putting $A \vee B$ for q and making some evident simplifications, this is equivalent to: $(B \to A) \to (A \vee B), (A \to B) \to (A \vee B) \vdash A \vee B$. The formulas on the left are IL-consequences respectively of $(B \to A) \to A$ and $(A \to B) \to B$, so this would render $A \vee B$ a consequence of $\{B \ddot{\vee} A, A \ddot{\vee} B\}$ and $\vdash$ accordingly coincides with $\vdash_{\mathsf{LC}}$.

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explicit the contrast between parts (i) and (ii), as is done here. In effect these authors address Proposition 5.1(ii) and its converse, the latter following from what we isolate as Proposition 5.1(i).

Łukasiewicz, in the Appendix to Łukasiewicz (1953) used a different condition on contexts to do much the same work as extensionality—as Proposition 5.1 makes clear—and we shall call contexts satisfying this condition, namely that for all A, B:

$$\vdash (C(A) \wedge C(\neg A)) \rightarrow C(B),$$

Ł-*extensional*.[26] The proofs of parts (i) and (ii) are adapted from, respectively, Smiley (1963, p. 129) and Łukasiewicz (1953, p. 381) (see Theorem 27).

PROPOSITION 5.1. *With* $\vdash$ *satisfying the conditions of Proposition 2.3:*

(i) Any context which is extensional according to $\vdash$ *is Ł-extensional according to* $\vdash$*;*

(ii) If every context is Ł-extensional according to $\vdash$*, then every context is extensional according to* $\vdash$.

Proof. (i): Suppose $C(\cdot)$ is extensional according to $\vdash$. The conditions imposed on $\vdash$ guarantee that for all formulas A, B, we have $\vdash (A \leftrightarrow B) \vee (\neg A \leftrightarrow B)$ and that from the extensionality of $C(\cdot)$ we have $\vdash (C(A) \leftrightarrow C(B)) \vee (C(\neg A) \leftrightarrow C(B))$. But since (as $\vdash \supseteq \vdash_{\mathsf{CL}}$) we have $(q \leftrightarrow s) \vee (r \leftrightarrow s) \vdash (q \wedge r) \rightarrow s$, substituting $C(A)$, $C(\neg A)$ and $C(B)$ for q, r, s respectively, we conclude that $\vdash (C(A) \wedge C(\neg A)) \rightarrow C(B)$, showing $C(\cdot)$ to be Ł-extensional.

(ii): We show that an arbitrarily selected 1-ary context $C(\cdot)$ is extensional according to $\vdash$ on the supposition that every context is Ł-extensional according to $\vdash$. Note that for any formulas A, B:

(1) $\vdash (A \leftrightarrow A) \rightarrow (C(A) \leftrightarrow C(A))$ and (2) $\vdash (A \leftrightarrow \neg A) \rightarrow (C(A) \leftrightarrow C(\neg A))$,

in the case of (1), because the consequent of the formula concerned is provable outright and of (2) because its antecedent is refutable. Now consider, for a given formula A, the context $C'(p) = (A \leftrightarrow p) \rightarrow (C(A) \leftrightarrow C(p))$. Thus $C'(A)$ is the formula figuring in (1), and $C'(\neg A)$ that figuring in (2). So by the Ł-extensionality of $C'(\cdot)$, we conclude that $\vdash C'(B)$ for arbitrary B, i.e., that $\vdash (A \leftrightarrow B) \rightarrow (C(A) \leftrightarrow C(B))$, revealing C to be extensional according to $\vdash$. □

Note that, since under the conditions on $\vdash$ there in play, the hypothesis of Proposition 5.1(ii) could equally well be put by saying that for all $C(\cdot)$, we have $C(A), C(\neg A) \vdash C(B)$; in the notation introduced in the opening section, this appears as: $A, \neg A \vdash^{\mathsf{rep}} B$ —a contextualised version of the *Ex Falso Quodlibet* principle. Here we are temporarily changing direction and treating $\vdash$ as a consequence relation again, rather than as just indicating theoremhood. We could purify this formulation of the explicit use of negation

[26] In fact, Łukasiewicz had propositional variables rather than schematic letters A and B, and similarly used an object language one-place 'variable functor' ("δ") attached to them rather than thinking of them as embedded in schematically indicated 1-ary contexts. See Smiley (1963, p. 128). The use of the term "Ł-extensional" should not be taken as implying that this was how Łukasiewicz understood talk of extensionality: he understood it exactly as explained above, but made axiomatic use of the principle that all contexts were Ł-extensional in order to derive the result that they were all extensional. The usage should also not be taken as implying that Łukasiewicz was the first to make axiomatic use of the principle: see the reference to unpublished work of C. A. Meredith on p. 365 of Łukasiewicz (1953).

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if we went still further in this direction and used a generalized consequence relation: If $A, A' \Vdash$ and $\Vdash A, A'$, then $A, A' \Vdash^{\mathsf{rep}} B$. When this is so for all A, A', B, let us say that $\Vdash$ satisfies the *purified* Ł*-extensionality condition*. We can compare this condition with the condition of extensionality from §2, defined by (RE) and (LE):

PROPOSITION 5.2.

(i) Extensionality implies the purified Ł-extensionality condition for an arbitrary generalized consequence relation.

(ii) The purified Ł-extensionality condition does not imply extensionality for an arbitrary generalized consequence relation.

Proof. For (i): Suppose for a given $A, B, C(\cdot)$ in the language of a generalized consequence relation $\Vdash$, (1) and (2) below are satisfied, with a view to showing that we have $C(A), C(B) \Vdash C(D)$ for arbitrarily chosen D. (Here A, B, D, correspond respectively to the "A", "A'"" and "B" of the above discussion.) We then reason as in steps (3)–(11):

(1)	$A, B \Vdash$	(2) $\Vdash A, B$
(3)	$A, D, C(A) \Vdash C(D)$	(LE)
(4)	$D, C(A) \Vdash C(D), B$	2, 3, Cut A
(5)	$C(B) \Vdash C(D), B, D$	(RE)
(6)	$C(A), C(B) \Vdash C(D), B$	4, 5, Cut D
(7)	$B, D, C(B) \Vdash C(D)$	(LE)
(8)	$C(A) \Vdash C(D), A, D$	(RE)
(9)	$B, C(A), \Vdash C(D), D$	1, 8, Cut A
(10)	$B, C(A), C(B) \Vdash C(D)$	7, 9, Cut D
(11)	$C(A), C(B) \Vdash C(D)$	6, 10, Cut B.

For (ii), take $\Vdash$ as the restriction of $\Vdash_{\mathsf{S4}}$, for example, to formulas constructed using $\Box$ and $\rightarrow$, since this is not extensional in the latter sense, $\Box$ not being extensional according to it. However, this $\Vdash$ satisfies the purified Ł-extensionality condition vacuously since there are no A, A', in its language for which $A, A' \Vdash \varnothing$. □

We return from generalized consequence relation to consequence relations, and indeed to the 'sets of formulas' conception of logics favoured by Łukasiewicz, to ask, in view of the high profile played by formulas provable outright—$\vdash$-consequences of the empty set—in the proof of Proposition 5.1, about the status of a weaker version of Ł-extensionality, namely the condition:

$$\textit{For all } A, B, \textit{ if } \vdash C(A) \textit{ and } \vdash C(\neg A) \textit{ then } \vdash C(B).$$

The first thing to note about this new condition is that it is indeed a weaker condition than Ł-extensionality on individual contexts: for example, for $\vdash = \vdash_{\mathsf{S4}}$ and $C(p) = \Diamond p$ this condition is vacuously satisfied since for no A are both $\Diamond A$ and $\Diamond \neg A$ both S4-provable, but this context is not Ł-extensional according to $\vdash_{\mathsf{S4}}$, since $\nvdash_{\mathsf{S4}} (\Diamond q \wedge \Diamond \neg q) \rightarrow \Diamond r$ (alternatively put: $\Diamond q \wedge \Diamond \neg q \nvdash_{\mathsf{S4}} \Diamond r$). On the other hand, it is obvious that any $C(\cdot)$ which is Ł-extensional satisfies the condition inset above.

However, much as with Proposition 5.1(ii), if the inset condition above is satisfied by every $C(\cdot)$ according to $\vdash$ (again meeting the conditions from the start of this section), then every context is Ł-extensional. To show this let $C(\cdot)$ be an arbitrary context with a view to showing that for A, B, we have $C(A), C(\neg A) \vdash C(B)$ on the assumption that inset condition above is satisfied by all contexts. Thus in particular, for any given formula A,

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it is satisfied by the context $C'(p) = (C(A) \wedge C(\neg A)) \rightarrow C(p)$. But $\vdash C'(A)$ and $\vdash C'(\neg A)$, so by the inset condition, $\vdash C'(B)$, i.e., $\vdash (C(A) \wedge C(\neg A)) \rightarrow C(B)$, or equivalently, $C(A), C(\neg A) \vdash C(B)$. Thus $C(\cdot)$ is Ł-extensional.

As already noted, Proposition 5.1(i) implies the converse of 5.1(ii), but the converse of 5.1(i) itself is false, as we may illustrate with the example of pretty well any normal modal logic and the context $C(p) = \Box p$: we always have $(\Box A \wedge \Box\neg A) \rightarrow \Box B$ provable, though we hardly ever have $(A \leftrightarrow B) \rightarrow (\Box A \leftrightarrow \Box B)$ provable (for all A, B, that is).

To be more precise on the "pretty well any" and "hardly ever" here, one easily checks that there are just three consistent normal modal logics in which all instances of the extensionality schema just cited are provable, namely (i) the smallest (normal)[27] modal logic containing all formulas of the form $\Box A \leftrightarrow A$ (the 'trivial system' as it is sometimes called), (ii) the smallest (normal) modal logic containing all formulas of the form $\Box A$ (the 'verum system'), and (iii) the intersection of these two logics—or more explicitly, the smallest normal modal logic containing all formulas of the form $A \rightarrow \Box A$.[28] In cases (i) and (ii), $\Box$ is definable in boolean terms ($\Box A = A$ and $\Box A = A \rightarrow A$, respectively) while in case (iii), that of KT_c, as we saw in Example 2.5, this is not so.

§6. Appendix B: Proof of Proposition 4.1. PROPOSITION 4.1(i) *Every (primitive or derived) binary connective # which is strongly idempotent according to* $\vdash_{\mathsf{LC}}$ *is componential according to* $\vdash_{\mathsf{LC}}$*, but:*

(ii) *Not every binary context* $C(p_1, p_2)$ *which is strongly idempotent according to* LC *is componential according to* $\vdash_{\mathsf{LC}}$.

For the proof we need a semantic lemma of interest in its own right; the result does not hold for IL and intuitionistic Kripke models containing incomparable points.

LOCAL SUPERVENIENCE LEMMA Let $\mathcal{M} = \langle U, R, V\rangle$ be a Kripke model for LC, that is, with R being a reflexive linear order on U. Then if A is a formula in which at most the variables $q_1, \ldots, q_n$ appear and we have $x, y \in U$ which Rxy and $\mathcal{M} \models_x q_i$ iff $\mathcal{M} \models_y q_i$ for all $i \in \{1, \ldots, n\}$, then $\mathcal{M} \models_x A$ iff $\mathcal{M} \models_y A$.

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[27] The parentheses around "normal" here and in what follows mean that the presence or absence of this word makes no difference to the class of logics specified.

[28] No parentheses on "normal" in this last occurrence—as the word cannot be omitted as redundant. Kracht Kracht (2003), p. 320, discussing (as we are here) modal logics as sets of formulas, describes these three and the inconsistent modal logic as the only Fregean modal logics, where *Fregean* means *weakly extensional* (see Czelakowski & Pigozzi, 2004), which amounts to *extensional* in the present setting (by Proposition 2.3), but the word "normal" needs to be inserted after "Fregean" here; the latter term is usually applied to consequence relations, and thus presumably transferred to this usage by thinking of the associated local consequence relations. (Kracht, 2003 distinguishes normal modal logics from modal logics in general, in the usual manner, at p. 311.) Otherwise there are plenty of further cases, such as the smallest modal logic in which for all A, we have $\Box A \leftrightarrow \neg A$; for a guided tour around these further cases—though not using the "$\Box$" notation—see Humberstone (1986, 1997). It should also be mentioned that Galatos *et al.* Galatos *et al.* (2007) mistakenly use the term 'Fregean' for consequence relations which are congruential in the sense of Segerberg Segerberg (1982), which is to say: which satisfy the special $\Gamma = \varnothing$ case of the weak extensionality condition (WE) given before Definitions 2.2. (This property is referred to as self-extensionality in Czelakowski & Pigozzi, 2004 and elsewhere.)

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Proof. By induction on the complexity of A, assuming $\mathcal{M}$, x and y as in the statement of the Lemma. In the basis case, where A is one of the q_i, of course if x, y agree on the q_i, they agree on A. The induction step is immediate for $A = B \wedge C$ and for $A = B \vee C$. Now consider $A = B \rightarrow C$, with x, y agreeing on $q_1, \ldots, q_n$ and such that Rxy. If x and y do not agree on A, since Rxy it must be that (1) $\mathcal{M} \not\models_x A$ while (2) $\mathcal{M} \models_y A$. By (1), there is some $u \in U$ with Rxu, $\mathcal{M} \models_u B$ and $\mathcal{M} \not\models_u C$. Thus we do not have Ryu, so by linearity, Ruy, and accordingly (since $\mathcal{M} \models_u B$) $\mathcal{M} \models_y B$, so $\mathcal{M} \models_y C$ (by (2)). On the other hand, since Rxu and $\mathcal{M} \not\models_u C$, we have $\mathcal{M} \not\models_x C$, so C, a formula of lower complexity than the current A and containing no variables outside of the listed $q_1, \ldots, q_n$ on which x and y are presumed to agree, has gone from being false at x to being true at y, contradicting the induction hypothesis. A final inductive case with A as $\neg B$ can be taken as having been dealt with by the preceding case, thinking of $\neg B$ as $B \rightarrow C$ with C being $\bot$: here the option of going from false to true collapses even earlier, at the stage when we conclude that $\mathcal{M} \models_y C$. □

Now, for a proof of Proposition 4.1(i), suppose that # is strongly idempotent according to $\vdash_{\mathsf{LC}}$:

$$A \wedge B \underset{\text{(a)}}{\vdash_{\mathsf{LC}}} A \# B \underset{\text{(b)}}{\vdash_{\mathsf{LC}}} A \vee B \text{ (all } A, B),$$

in which the $\vdash$-conditions are labelled for back-reference below. For a contradiction, we now suppose further that # is not componential. Componentiality (according to $\vdash_{\mathsf{LC}}$) amounts to the LC-provability of $(A \# B) \leftrightarrow A$, $(A \# B) \leftrightarrow B$ (for all A, B), so the supposition that # is not componential implies that that at least one of (1)–(4) below has to be false:

(1) $\vdash_{\mathsf{LC}} ((A \# B) \rightarrow A) \vee ((A \# B) \rightarrow B)$;
(2) $\vdash_{\mathsf{LC}} ((A \# B) \rightarrow A) \vee (B \rightarrow (A \# B))$;
(3) $\vdash_{\mathsf{LC}} (A \rightarrow (A \# B)) \vee ((A \# B) \rightarrow B)$;
(4) $\vdash_{\mathsf{LC}} (A \rightarrow (A \# B)) \vee (B \rightarrow (A \# B))$.

(1) and (4), which would be correct for CL without any assumptions concerning #, remain correct for LC in the presence of the assumptions above. For (1), use the LC derivable schema $(A \vee B) \rightarrow A) \vee ((A \vee B) \rightarrow B)$, appealing to (b) to replace the $A \vee B$ in each antecedent with $A \# B$; For (2), start with $(A \rightarrow (A \wedge B)) \vee (B \rightarrow (A \wedge B))$, appealing to (a) to replace the $A \wedge B$ in each consequent with $A \# B$. This leaves (2) and (3). If (2) is false then it fails for the case in which A, B, are distinct propositional variables, q and r, say, so let us consider how things would be in a totally ordered Kripke model $\mathcal{M} = \langle U, R, V \rangle$ with $w \in U$ and $\mathcal{M} \not\models_w ((q \# r) \rightarrow q) \vee (r \rightarrow (q \# r))$, so each of these disjuncts is false at w in $\mathcal{M}$. There must accordingly be $y, x \in U$ with Rwy and Rwx, with $\mathcal{M} \models_y q \# r$ while $\mathcal{M} \not\models_y q$, and $\mathcal{M} \models_x r$ while $\mathcal{M} \not\models_x q \# r$. Since either Rxy or Ryx and $q \# r$ is true at y but not at x, it must be that Rxy, which, given that q is false at y, implies that q is false at x, and, since r is true at x, implies that r is true at y. In summary, we have the following for x and y in $\mathcal{M}$: q false at both, r true at both, $q \# r$ false at x, true at y. Since Rxy, this contradicts the Local Supervenience Lemma, showing the present situation could not arise: (2) is indeed a correct claim. A similar appeal to the Lemma establishes (3), completing the proof of Proposition 4.1(i).

For Proposition 4.1(ii), it suffices to give an example. Let $C(p_1, p_2)$ be $((p_1 \vee s) \wedge p_2$. Clearly this is strongly idempotent according to $\vdash_{\mathsf{LC}}$ (as it is for $\vdash_{\mathsf{IL}}$. But this time, the analogue of (2) above fails: we have $\not\vdash_{\mathsf{LC}} (C(q, r) \rightarrow q) \vee (r \rightarrow C(q, r))$. The previous

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considerations concerning (2) for # mean that in a countermodel we have Rxy with q false at both x and y, r true at both, and $C(q, r)$ false at x and true at y. But this does not contradict the Lemma, since q and r are not the only variables occurring in $C(q, r)$. For confirmation that $\nvdash_{\mathsf{LC}} (C(q, r) \to q) \vee (r \to C(q, r))$, let us write out the disjunction in question fully:

$$(((q \vee s) \wedge r) \to q) \vee (r \to ((q \vee s) \wedge r)),$$

and get rid of the "$\wedge r$" on the right:

$$(((q \vee s) \wedge r) \to q) \vee (r \to (q \vee s)).$$

Now substitute $\bot$ for q and simplify, giving:

$$\neg(s \wedge r) \vee (r \to s).$$

Finally, putting $\top$ for r, we get $\neg s \vee s$. So the LC-provability of the original disjunction would imply, contrary to fact, that LC = CL. ■

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