A UNIQUENESS THEOREM CONCERNING GRAVITY FIELDS

By RUSSELL A. SMITH

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1. *Introduction.* It is well known that a uniform spherical shell produces the same gravitational field outside itself as that of a concentric uniform solid sphere with the same total mass. A given external field may therefore be produced by a variety of mass systems even under the restriction that the bodies of the system should have a prescribed uniform density. It can be asked what further conditions on the system are sufficient to ensure its uniqueness. The problem is of interest in connexion with the geological interpretation of local anomalies in the earth's gravitational field and work has already been done on it by several authors, (3), (6), (7), (8), (9). The results proved so far have all been restricted to systems consisting of a single body and the purpose of the present paper is to extend one of these results to include more general mass systems. The example of the spherical shell, mentioned above, might suggest that the absence of cavities in the bodies is a sufficient additional condition to ensure uniqueness. This is not so however. Celmiņš (3) gives an example of two different mass systems which have the same external field, each system consisting of three bodies of the same uniform density without any cavities in them.

Let ϑ be a plane and $\rho(x, y, z)$ a bounded, measurable, strictly positive function which is constant along each line perpendicular to ϑ. We shall say that a mass system is of type (ϑ, ρ) if it has the following properties:

(i) it is formed by filling a bounded open set G with mass so that the density at each point of it is $\rho(x, y, z)$;

(ii) each line perpendicular to ϑ either does not meet G at all or intersects it in a single line segment;

(iii) if G^* is the orthogonal projection of G on ϑ then its frontier $\mathfrak{F}G^*$ has zero area;

(iv) the lines perpendicular to ϑ which meet $\mathfrak{F}G$ in more than two points, intersect ϑ in a set whose closure has zero area;

(v) the closure of the projection of Γ on ϑ does not intersect the closure of the projection of $(\mathscr{C}\Gamma) \cap (\mathscr{I}\bar{G})$ on ϑ for each connected component Γ of the interior $\mathscr{I}\bar{G}$ of $\bar{G}$.

If two systems having property (i) occupy open sets G_1, G_2 and G_1 differs from G_2 by no more than a set of zero volume then it is clear that they produce the same external field. We shall say that such systems are *essentially the same*. The following result is proved:

THEOREM. *If ϑ and ρ are given then two essentially different mass systems of type (ϑ, ρ) cannot produce the same external gravity field.*

For systems consisting of a single body (i.e. G connected), a version of this theorem with extra restrictive conditions was given by Sretenskiĭ (8). A practical application

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of the above result has already been made by Bott (1). A part (§ 4) of the proof below involves a modification of Sretenskiĭ's argument used by Todorov and Zidarov (9). Both of the theorems stated in (9) appear to be false however.

2. *Point set lemmas.* Suppose the same external field is produced by two systems of type (ϑ, ρ) occupying open sets G_1, G_2 respectively. We aim to prove that G_1 and G_2 differ by no more than a set of zero volume. Take rectangular axes $Oxyz$ having ϑ as the plane $z = 0$ and let G_i^* $(i = 1, 2)$ be the orthogonal projection of G_i on ϑ. With respect to these axes the density ρ is a function of (x, y) only. For convenience we shall say that Oz points *vertically upwards*. If any points which lie on vertical lines through the closed set $\mathfrak{F}G_1^* \cup \mathfrak{F}G_2^*$ are deleted from G_i, the external field produced by G_i is unaltered since the deleted points form a set of zero volume by hypothesis (iii). We can therefore suppose that

$$G_i^* \cap \mathfrak{F}G_j^* = 0. \tag{1}$$

Using hypothesis (ii), let the vertical line through a point $(x, y, 0)$ in G_i^* meet G_i in the segment $a_i < z < b_i$. Then a_i, b_i are functions of (x, y) defined in G_i^*. If E_i is the union of all vertical lines which meet $\mathfrak{F}G_i$ in more than two points then its closure $\bar{E}_i$ has zero volume by hypothesis (iv). Any points of G_i which lie in the closed set $\bar{E}_1 \cup \bar{E}_2$ can therefore be deleted without disturbing the external field produced by G_i. Assuming that this has been done, the following holds:

LEMMA 1. *a_i and b_i are continuous functions of (x, y) in G_i^*.*

Proof. Suppose that $P^* = (x_0, y_0, 0)$ lies in G_i^* and let P be the point $(x_0, y_0, b_i(x_0, y_0))$. Given any $\epsilon > 0$, choose a point Q of G_i within a distance $\frac{1}{2}\epsilon$ vertically below P and a point R of $\mathscr{C}\bar{G}_i$ within a distance $\frac{1}{2}\epsilon$ vertically above P. Now choose δ in $0 < \delta < \frac{1}{2}\epsilon$ so that the open sets G_i, $\mathscr{C}\bar{G}_i$ contain the closed spheres of radius δ with centres at Q, R respectively. Then $|b_i(x, y) - b_i(x_0, y_0)| < \epsilon$ whenever $(x, y, 0)$ lies within a distance δ of P^*. Thus b_i is continuous at P^* and similarly a_i is continuous also.

Let H be the set of all points (x, y, z) for which $(x, y, 0)$ lies in $G_1^* \cup G_2^*$ and

$$\min(a_1, a_2) = \alpha < z < \beta = \max(b_1, b_2),$$

where $\min(\lambda, \mu)$ and $\max(\lambda, \mu)$ are to be interpreted as λ when μ does not exist. Then H contains $G_1 \cup G_2$. Let A_i be the set of all points (x, y, a_i) for which either $(x, y, 0)$ lies in $G_i^* \cap \mathscr{C}G_j^*$ or $(x, y, 0)$ lies in $G_i^* \cap G_j^*$ and $a_i < a_j$. Similarly let B_i be the set of all points (x, y, b_i) for which either $(x, y, 0)$ lies in $G_i^* \cap \mathscr{C}G_j^*$ or $(x, y, 0)$ lies in $G_i^* \cap G_j^*$ and $b_i > b_j$. Then A_1, A_2, B_1, B_2 are disjoint subsets of $\mathfrak{F}H$ and also of $\mathfrak{F}\bar{H}$.

LEMMA 2. *The projections A_i^*, B_i^* of A_i, B_i on ϑ are open plane sets (possibly empty).*

Proof. A_i^* is the union of $G_i^* \cap \mathscr{C}G_j^*$ which is open by (1) and those parts of $G_i^* \cap G_j^*$ in which $a_i < a_j$. The latter are open by Lemma 1. Thus A_i^* is open and similarly B_i^* is open.

LEMMA 3. *If $A_i \cup B_i$ is empty then $H = G_j$ where $j \neq i$.*

Proof. If $A_i \cup B_i$ is empty then $G_i^* \cap \mathscr{C}G_j^*$ is empty and $a_i \geqslant a_j$, $b_i \leqslant b_j$ throughout $G_i^* \cap G_j^*$. Thus $G_1^* \cup G_2^* = G_j^*$ and $\alpha = a_j$, $\beta = b_j$ throughout it. Hence $H = G_j$.

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LEMMA 4. *Almost all vertical lines meet $\mathfrak{F}\bar{H}$ in boundary points of $\mathscr{I}\bar{H}$ which are regular for Dirichlet's problem, where 'almost all' means that the union of the exceptional lines has cross-section of zero area.*

Proof. A refinement of Kellogg's lemma (see (2), p. 120) states that the set of irregular boundary points is the union of a denumerable family of closed sets of zero capacity, and a closed set of zero capacity is known (4) to be null with respect to two-dimensional Hausdorff measure. The projection of the irregular set on ϑ is therefore of zero area.

3. *The basic equations.* Let $U(x, y, z)$ be a function which is harmonic in an open set D containing $\bar{H}$. Let S be a sufficiently smooth closed surface which lies wholly in D and has $\bar{H}$ in its interior. The value of U at a point $P = (x, y, z)$ inside S is given by

$$U(x,y,z) = \frac{1}{4\pi}\iint_S (R^{-1}\nabla U - U\nabla R^{-1}).d\mathbf{s},$$

where R is the distance between P and the variable point Q on S (see (5), p. 219). Multiply this through by $\rho(x, y)$ and integrate it over G_i. It becomes

$$\iiint_{G_i} \rho U\,dx\,dy\,dz = \frac{1}{4\pi k}\iint_S (V_i\nabla U - U\nabla V_i).d\mathbf{s},$$

where k is the gravitational constant and V_i is the potential at Q due to the mass system G_i. Since the two systems G_1, G_2 produce the same external field, $V_1 = V_2$ and $\nabla V_1 = \nabla V_2$ on S and the right-hand side of this equation is the same for $i = 1, 2$. Hence

$$\iiint_{G_1} \rho U\,dx\,dy\,dz = \iiint_{G_2} \rho U\,dx\,dy\,dz.$$

In this equation U can be replaced by $\partial U/\partial z$ since this is also harmonic in D. Because ρ is independent of z, the equation can then be integrated with respect to z to give

$$\iint_{G_1^*} \rho[U(x,y,b_1) - U(x,y,a_1)]\,dx\,dy = \iint_{G_2^*} \rho[U(x,y,b_2) - U(x,y,a_2)]\,dx\,dy. \tag{2}$$

If $r = [(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{-\frac{1}{2}}$ then $\partial r/\partial z = -\partial r/\partial\zeta$, and the vertical component of the field due to G_i at the point (ξ, η, ζ) is

$$\begin{aligned}\frac{\partial V_i}{\partial\zeta} &= -\iiint_{G_i} \frac{\partial}{\partial z}\left(\frac{k\rho}{r}\right) dx\,dy\,dz \\ &= -\iint_{G_i^*} k\rho\left[\frac{1}{r(x,y,b_i)} - \frac{1}{r(x,y,a_i)}\right] dx\,dy.\end{aligned}$$

This, together with (2), gives

$$\begin{aligned}\frac{\partial(V_1 - V_2)}{\partial\zeta} = &-\iint_{G_1^*} k\rho[\phi(x,y,b_1) - \phi(x,y,a_1)]\,dx\,dy \\ &+\iint_{G_2^*} k\rho[\phi(x,y,b_2) - \phi(x,y,a_2)]\,dx\,dy,\end{aligned} \tag{3}$$

where $\phi = r^{-1} + U$.

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4. *The function* U_0. If n is a positive integer let D_n be the set of all points which lie within a distance $1/n$ of H and let Δ_n be the union of those parts of D_n which lie either vertically below A_1 or vertically above B_2. If $m > n$ let $U_{m,n}$ be the lower harmonic envelope in $D_m \cup \Delta_n$ corresponding to the boundary function which is zero at all frontier points of $D_m \cup \Delta_n$ except those lying vertically below A_1 and vertically above B_2 where it has the value 1 (see (2), p. 114). Then $U_{m,n}$ is harmonic and satisfies $0 \leqslant U_{m,n} \leqslant 1$ throughout $D_m \cup \Delta_n$. Since $D_m \cup \Delta_n$ is an open set containing $\bar{H}$, (2) holds with $U = U_{m,n}$. As $m \to \infty$ keeping n fixed $D_m \cup \Delta_n$ decreases to the limit set $\bar{H} \cup \Delta_n$ and $U_{m,n}$ decreases monotonically to a limit function U_n. Then as $n \to \infty$, $\bar{H} \cup \Delta_n$ decreases to the limit set $\bar{H}$ and U_n increases monotonically to a limit function U_0 which is defined and satisfies $0 \leqslant U_0 \leqslant 1$ throughout $\bar{H}$. Harnack's theorem (see (5), p. 263) shows that U_0 is harmonic throughout $\mathscr{I}\bar{H}$. Since (2) holds with $U = U_{m,n}$ the theorem of bounded convergence proves at once the following lemma:

LEMMA 5. *Equation* (2) *holds with* $U = U_0$.

LEMMA 6. $U_0 = 1$ *at all regular frontier points of* $\mathscr{I}\bar{H}$ *which lie in* $A_1 \cup B_2$ *and* $U_0 = 0$ *at all regular frontier points in* $A_2 \cup B_1$.

Proof. Suppose that P lies in A_1 and let P^* be its projection in A_1^* which is open by Lemma 2. Let Σ^* be a closed circular disk in A_1^* with centre at P^* and let Σ be the part of A_1 which projects onto Σ^*. Let C be the set of all points which lie vertically above Σ but within a vertical distance δ of it, where δ is chosen small enough to ensure that $C \subset H$. Let $W(x, y, z)$ be the lower harmonic envelope in C corresponding to the boundary function which is zero everywhere on $\mathfrak{F}C$ except on Σ where it has the value 1. Since P is a regular frontier point $W(x, y, z) \to 1$ as $(x, y, z) \to P$ through C. If $0 < t < \delta$, form sets Σ_t and C_t by displacing Σ and C through a distance t vertically downwards. Then $W(x, y, z+t)$ is the lower harmonic envelope in C_t corresponding to the boundary function which is zero everywhere on $\mathfrak{F}C_t$ except on Σ_t where it has the value 1. If μ is the shortest distance between the disjoint compact sets Σ_t and $\bar{H}$ then $\mu > 0$ and Σ_t is outside D_n for all $n > 1/\mu$. If $n > m > 1/\mu$ then Σ_t is outside $D_m \cup \Delta_n$ and from a consideration of the boundary values of $U_{m,n}$ and $W(x, y, z+t)$ it follows that $U_{m,n}(x, y, z) \geqslant W(x, y, z+t)$ throughout $C_t \cap (D_m \cup \Delta_n)$. In this make $m \to \infty$ and then $n \to \infty$. The result is $U_0(x, y, z) \geqslant W(x, y, z+t)$ throughout $C_t \cap \bar{H}$. Insert the coordinates of P in this and then make $t \to 0$. Since $W \to 1$ and $1 \geqslant U_0 \geqslant W$ it gives $U_0 = 1$ at P. A similar argument deals with the regular points in B_1, A_2 and B_2.

LEMMA 7. $U_0 = 1$ *throughout each connected component of* $\mathscr{I}\bar{H}$ *whose frontier meets* $A_1 \cup B_2$ *and* $U_0 = 0$ *throughout each component whose frontier meets* $A_2 \cup B_1$.

Proof. Since $0 \leqslant U_0 \leqslant 1$ throughout $\bar{H}$ it follows with the help of Lemma 4 and Lemma 6 that

$$U_0(x, y, a_1) \geqslant U_0(x, y, a_2) \quad \text{and} \quad U_0(x, y, b_1) \leqslant U_0(x, y, b_2)$$

for almost all vertical lines through $G_1^* \cap G_2^*$. These give

$$U_0(x, y, b_1) - U_0(x, y, a_1) \leqslant U_0(x, y, b_2) - U_0(x, y, a_2). \tag{4}$$

For almost all vertical lines through $G_1^* \cap \mathscr{C}G_2^*$, the left-hand side of (4) is -1 and the right-hand side does not exist. Also, for almost all lines through $G_2^* \cap \mathscr{C}G_1^*$, the

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right-hand side is 1 and the left-hand side does not exist. It follows that the contribution of a line to the left-hand side of (2) with $U = U_0$ does not exceed its contribution to the right-hand side for almost all vertical lines.

Now suppose that $\mathscr{I}\bar{H}$ has a component K whose frontier meets A_1. Since $U_0 \leqslant 1$ the maximum principle for harmonic functions shows that either $U_0 = 1$ throughout K or $U_0 < 1$ throughout K. If the latter holds then (4) is strict (i.e. equality does not occur) for almost all lines through $A_1 \cap \mathfrak{F}K$. By Lemma 2 the union of all vertical lines through $A_1 \cap \mathfrak{F}K$ has positive area of cross-section, from which it follows that the left-hand side of (2) is strictly less than the right-hand side when $U = U_0$. Since this contradicts Lemma 5 we conclude that $U_0 = 1$ throughout K. A similar argument deals with components of $\mathscr{I}\bar{H}$ whose frontiers meet any of the other sets B_1, A_2 and B_2.

LEMMA 8. $G_1^* = G_2^*$.

Proof. If $G_1^* \cap \mathscr{C}G_2^* \neq 0$ then any vertical line through it meets both A_1 and B_1 and passes through a single component K of $\mathscr{I}\bar{H}$. Thus $\mathfrak{F}K$ meets both $A_1 \cup B_2$ and $A_2 \cup B_1$ which is impossible by Lemma 7. Hence $G_1^* \cap \mathscr{C}G_2^* = 0$ and similarly $G_2^* \cap \mathscr{C}G_1^* = 0$ also.

5. *The function* ϕ_0. Let ϕ_n be the Green's function of the set D_n having its pole (ξ, η, ζ) in $\mathscr{I}\bar{H}$. Then (3) holds with $\phi = \phi_n$. As $n \to \infty$ keeping (ξ, η, ζ) fixed D_n decreases to the limit set $\bar{H}$ and ϕ_n decreases monotonically to a limit function ϕ_0 defined and satisfying $\phi_0 \geqslant 0$ throughout $\bar{H}$. Harnack's theorem shows that ϕ_0 is harmonic in $\mathscr{I}\bar{H}$ except for its pole at (ξ, η, ζ) and the theorem of dominated convergence shows that (3) holds with $\phi = \phi_0$. A slight modification of the proof of Lemma 6 gives the following:

LEMMA 9. $\phi_0 = 0$ *at all regular frontier points of* $\mathscr{I}\bar{H}$ *in* $A_1 \cup A_2 \cup B_1 \cup B_2$.

LEMMA 10. $\phi_0 = 0$ *throughout each component of* $\mathscr{I}\bar{H}$ *except that which contains the pole of* ϕ_0, *in which* $\phi_0 > 0$.

Proof. The second part is obvious since ϕ_0 is harmonic and $\phi_0 \geqslant 0$. Let K be a component of $\mathscr{I}\bar{H}$ not containing the pole of ϕ_0 and write $L = \mathscr{I}\bar{H} \cap \mathscr{C}K$. If K^*, L^* are the projections of K, L on ϑ then Lemma 8 and hypothesis (v) of § 1 show that $\bar{K}^* \cap \bar{L}^* = 0$. If ν is the shortest distance between $\bar{K}^*$ and $\bar{L}^*$ then $\nu > 0$ and any point which lies within a distance $\frac{1}{2}\nu$ of K must lie at a distance greater than $\frac{1}{2}\nu$ from L. Hence, the component of D_n which contains $\bar{K}$ does not contain the pole of ϕ_n if $n > 2/\nu$. Thus $\phi_n = 0$ throughout $\bar{K}$ for all $n > 2/\nu$. Lemma 10 now follows.

Proof of Theorem. It is sufficient to prove that $A_1 \cup B_2 = 0$, since $A_2 \cup B_1 = 0$ follows by interchanging G_1 and G_2 and then $G_1 = H = G_2$ by Lemma 3. Suppose that $A_1 \cup B_2 \neq 0$ and let the pole (ξ, η, ζ) of ϕ_0 lie in a component N of $\mathscr{I}\bar{H}$ for which $(A_1 \cup B_2) \cap \mathfrak{F}N \neq 0$. When $\phi = \phi_0$ in (3), the parts of the integrals arising outside N can be ignored by Lemma 10. By Lemma 8, $G_1^* \cap N^* = G_2^* \cap N^*$ and throughout this set $a_1 \leqslant a_2$ and $b_2 \geqslant b_1$ since $\mathfrak{F}N$ does not meet $A_2 \cup B_1$ by Lemma 7. Lemmas 4, 9 and 10 now show that

$$\phi_0(x, y, a_1) \leqslant \phi_0(x, y, a_2) \quad \text{and} \quad \phi_0(x, y, b_2) \leqslant \phi_0(x, y, b_1) \tag{5}$$

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for almost all vertical lines through $G_1^* \cap N^*$, so that the right-hand side of (3) is not positive. But Lemmas 9 and 10 show that at least one of the inequalities (5) is strict for almost all vertical lines through $(A_1 \cup B_2) \cap \mathfrak{F}N$. The union of such lines has positive area of cross-section by Lemma 2, so that the right-hand side of (3) is strictly negative when $\phi = \phi_0$. Hence $V_1 - V_2$ decreases strictly as the point (ξ, η, ζ) passes upwards through N along a vertical line. This contradicts $V_1 - V_2 = 0$ everywhere in $\mathfrak{F}N$, which is so because V_1, V_2 are continuous everywhere and equal outside $\bar{H}$. This contradiction shows that $A_1 \cup B_2 = 0$ and the theorem follows.

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UNIVERSITY LABORATORIES
SOUTH ROAD, DURHAM

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