Student Problems

Students up to the age of 19 are invited to send solutions to either or both of the following problems to Stan Dolan, 126A Harpenden Road, St Albans, Herts., AL3 6BZ.

Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is not necessary to submit solutions to both. Solutions should arrive by September 20th 2019. Please give your School year, the name and address of your School or College, and the name of a teacher through whom the award will be made. Please print your own name clearly! The names of all successful solvers will be published in the November edition.

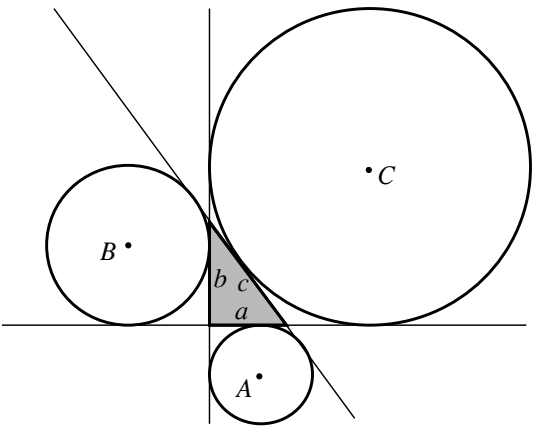
The MA and the *Gazette* comply fully with the provisions of the 2018 GDPR legislation. Submissions **must** be accompanied by the SPC permission form which is available on the Mathematical Association website

https://www.m-a.org.uk/the-mathematical-gazette

Note that if permission is not given, a pupil ***may still participate and will be eligible for a prize in the same way as others***.

Problem 2019.3 (**Paul Stephenson**)

The right-angled triangle shown shaded in the diagram has sides of lengths a, b and (hypotenuse) c. Also shown in the diagram are the three excircles of the shaded triangle and the excentres A, B and C. Find the area of triangle ABC.



Problem 2019.4 (**Stan Dolan**)

If a 5-digit number, n, is a multiple of 271 then so are all numbers given by cyclic permutations of the digits of n.

For example, 91327 is a multiple of 271 and therefore so are 79132, 27913, 32791 and 13279.

Explain this property of 271 and generalise your result.

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Solutions to 2019.1 and 2019.2

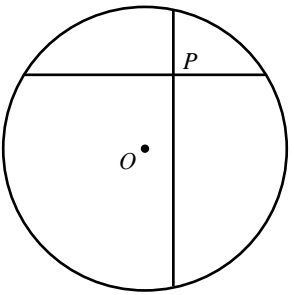
Both problems were solved by Sae Koyama (King's College London Mathematics School) and Andrew Ng (Tonbridge School). Problem 2019.1 was solved by Nathan Burn (Bishop Wordsworth's Grammar School), Isaac Harrison (The Sixth Form College, Farnborough), Shrimat Kapoor (King's College London Mathematics School) and Matthew Shipway (Nottingham High School). Problem 2019.2 was solved by Donald Flynn (Nottingham High School).

Problem 2019.1 **(Paul Stephenson)**

A unit circle has centre at the origin O. Two perpendicular chords of the circle intersect at the point $P(x, y)$, where x and y are both positive.

One of the four regions into which the chords divide the circle has area $\frac{\pi}{2}$.

Find the area of the diagonally opposite region in terms of x and y.



Solution

The required area can be found using integration to find the area underneath the curve

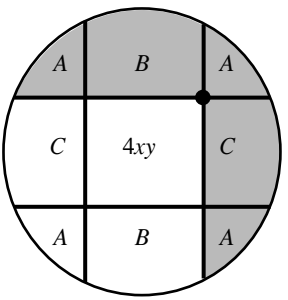
$$y = \sqrt{1 - x^2}.$$

Both Andrew and Matthew successfully carried out this calculation and Shrimat obtained the same form of answer by a more geometrical approach. Matthew also managed to simplify the final answer into the form $2xy$.

However, Isaac, Sae and Nathan cleverly used symmetry to obtain this answer with very little calculation. Nathan's argument was as follows.

The circle can be divided into regions with areas as shown.

The unshaded regions have total area $\frac{\pi}{2}$ and therefore the shaded regions also have total area $\frac{\pi}{2}$.



Then $3A + B + C = A + B + C + 4xy$ and so the required area, A, is $2xy$.

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Problem 2019.2 **(Chris Starr)**

For which integer values of n is

$$\left(2\cos\frac{\pi}{7}\right)^n + \left(2\cos\frac{3\pi}{7}\right)^n + \left(2\cos\frac{5\pi}{7}\right)^n$$

an integer?

Solution

Sae solved this problem by a sophisticated method which involved considering the 7th roots of unity as forming a cyclic group under multiplication.

Andrew and Donald both approached this problem by the same method. First, De Moivre's Theorem was used to prove that

$$\begin{aligned}\cos 7\theta &= 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta \\ &= (\cos\theta + 1)(8\cos^3\theta - 4\cos^2\theta - 4\cos\theta + 1)^2.\end{aligned}$$

Therefore $2\cos\frac{\pi}{7}$, $2\cos\frac{3\pi}{7}$ and $2\cos\frac{5\pi}{7}$ are roots of

$$x^3 - x^2 - 2x + 1 = 0 \qquad (1)$$

Now let $S_n = \left(2\cos\frac{\pi}{7}\right)^n + \left(2\cos\frac{3\pi}{7}\right)^n + \left(2\cos\frac{5\pi}{7}\right)^n$. Then, from (1)

$$S_{n+3} - S_{n+2} - 2S_{n+1} + S_n = 0,$$

for any integer n.

Therefore, if three successive values of S_n are integers, S_n is an integer for all (positive and negative) integer values of n. A simple check of $n = 0$, 1 and 2 shows that this is the case:-

$S_0 = 3$ and, applying Vieta's formulae to (1), $S_1 = 1$ and $S_2 = 1^2 - 2(-2) = 5$.

The first prize of £25 is awarded to Nathan Burn. The second prize of £20 is awarded to Donald Flynn.

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Note from the Editor

See *www.m-a.org.uk/the-mathematical-gazette* for the SPC questions.

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