A PROBLEM CONCERNING THREE-DIMENSIONAL CONVEX BODIES

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1. The problem considered in this paper arose during an investigation of what Chabauty† has called the 'anomaly' of convex bodies.

Throughout the paper $\mathfrak{K}$ denotes a closed bounded convex body in three-dimensional Euclidean space $\mathfrak{R}^3$, which is symmetric in the origin O and which contains O as an interior point. Such a body determines uniquely a distance-function $f(x, y, z)$ which is defined and finite for each point (x, y, z) of $\mathfrak{R}^3$ and possesses the following properties:

(i) $f(x, y, z) > 0$ for $(x, y, z) \neq (0, 0, 0)$.

(ii) $f(\alpha x, \alpha y, \alpha z) = \alpha f(x, y, z)$ for all $\alpha \geqslant 0$.

(iii) $f(x_1 + x_2, y_1 + y_2, z_1 + z_2) \leqslant f(x_1, y_1, z_1) + f(x_2, y_2, z_2)$.

(iv) $f(x, y, z) = f(-x, -y, -z)$.

(v) $\mathfrak{K}$ consists of all these points (x, y, z) of $\mathfrak{R}^3$ for which $f(x, y, z) \leqslant 1$.

Conversely, if $f(x, y, z)$ is any function which possesses the properties (i)–(iv), then the body $\mathfrak{K}$ defined by (v) is a closed bounded convex body which is symmetric in O and contains O as an interior point. We note, further, that it can be deduced from the properties of $f(x, y, z)$ listed above that $f(x, y, z)$ is a continuous function of x, y and z.

If the distance-function f is associated with $\mathfrak{K}$ as just described, we denote by $\mu\mathfrak{K}$, where $\mu \geqslant 0$, the set of points (x, y, z) for which $f(x, y, z) \leqslant \mu$, i.e. $\mu\mathfrak{K}$ is the body obtained by magnifying $\mathfrak{K}$ μ times about the origin.

I consider the following problem, which I state first in a somewhat loose form in order to help the reader to visualize it geometrically. Let μ be a fixed number greater than unity and let L_1, L_2 be any two different lines through O such that the z-axis, L_3, does not lie in the plane $\mathfrak{P}$ which they determine. We now dilate $\mathfrak{K}$ about O, the magnification varying with the direction so that in the three independent directions L_1, L_2, L_3 the magnifications are λ, λ, 1, respectively. We take the maximum value of λ, $\lambda(\mathfrak{P})$ say, for which this new body $\mathfrak{K}(\lambda, \mathfrak{P})$ is contained in $\mu\mathfrak{K}$, and try to make $\lambda(\mathfrak{P})$ as large as possible by choosing $\mathfrak{P}$ suitably‡.

More precisely, let $\mathfrak{K}$ and $f(x, y, z)$ be given, and suppose that $\mu \geqslant 1$ is fixed. For each $\lambda \geqslant 1$ and each pair of real numbers p, q we denote by $\mathfrak{K}(\lambda, p, q)$ the convex body

$$f\left(\frac{x}{\lambda}, \frac{y}{\lambda}, z - \frac{\lambda - 1}{\lambda}(px + qy)\right) \leqslant 1.$$

Thus $\mathfrak{K}(1, p, q) = \mathfrak{K}$ for all p, q. The plane $\mathfrak{P}$ of the preceding paragraph is $z = px + qy$.

† See following paper (2) for references.

‡ Note that $\lambda(\mathfrak{P})$ depends only on $\mathfrak{P}$ and μ and is the same for different L_1, L_2 lying in the same plane $\mathfrak{P}$.

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We define $$\lambda_{p,q} = \sup \lambda \quad [\mathfrak{R}(\lambda, p, q) \subseteq \mu\mathfrak{R}],$$

and write $$\kappa = \sup_{p,q} \lambda_{p,q}.$$

I prove

THEOREM 1. $\frac{1}{4}(1+3\mu) \leqslant \kappa \leqslant \mu$.

I also show that Theorem 1 is best possible. That is, bodies $\mathfrak{R}$ exist for which $\kappa = \frac{1}{4}(1+3\mu)$ and others for which $\kappa = \mu$. The statement and proof of these results is deferred to §§4 and 5.

2. *Notation.* Small Greek letters, with the exception of ξ, η, ζ, are reserved for positive numbers. Capital italic letters denote points of $\mathfrak{R}^3$ and German capitals are used for sets of points. Dashes denote images in the origin; this applies both to points and sets of points. Thus, if P is the point (x, y, z), P' denotes the point $(-x, -y, -z)$. Also tP denotes the point (tx, ty, tz).

If $\mathfrak{A}$ is any set of points in $\mathfrak{R}^3$, $\mathfrak{CA}$ denotes the complementary set and $\mathfrak{FA}$ the boundary of $\mathfrak{A}$, i.e. $\mathfrak{FA} = \overline{\mathfrak{A}} \cap \overline{\mathfrak{CA}}$, where the bar denotes closure. Also, $\mathfrak{A}^\circ$ denotes the interior of $\mathfrak{A}$, i.e. $\mathfrak{A}^\circ = \overline{\mathfrak{A}} - \mathfrak{FA}$.

Particular notations are the following:

$\mathfrak{O}$ is the null set.

$\mathfrak{S}(P, \delta)$ denotes the interior of the sphere of centre P and radius δ.

$\mathfrak{Z}$ denotes the plane $z = 0$.

For each pair p, q, $\Omega_{p,q}$ denotes the set of all $\lambda \geqslant 1$ for which $\mathfrak{R}(\lambda, p, q) \subseteq \mu\mathfrak{R}$. The set $\Omega_{p,q}$ is never null since $1 \in \Omega_{p,q}$ for all p, q.

The set of all convex bodies in $\mathfrak{R}^3$ forms a space into which a metric can be introduced (cf. (1)). Hence continuity of convex bodies can be defined. Since $f(x, y, z)$ is a continuous function of x, y and z, it is clear that in the space of all convex bodies in $\mathfrak{R}^3$ the body $\mathfrak{R}(\lambda, p, q)$ is a continuous function of λ for $\lambda \geqslant 1$ (in fact for $\lambda > 0$). This fact we shall frequently use.

3. *Proof of Theorem* 1. We observe first that when we are considering fixed values of p and q we may assume that $p = q = 0$. For, if we introduce new coordinates x', y', z' by putting $$x' = x, \quad y' = y, \quad z' = z - px - qy, \tag{3·1}$$ and write $f(x, y, z) = g(x', y', z')$, then $\mathfrak{R}$ becomes the body $g(x', y', z') \leqslant 1$ and $\mathfrak{R}(\lambda, p, q)$ is the body $g\left(\frac{x'}{\lambda}, \frac{y'}{\lambda}, z'\right) \leqslant 1$. The function $g(x', y', z')$ is, of course, a distance-function.

LEMMA 1. *If* $\lambda \in \Omega_{p,q}$, *then* $\lambda \leqslant \mu$.

Proof. As remarked above we may assume that $p = q = 0$. Then, if $$(x, y, z) \in \mathfrak{F}\mathfrak{R}(\lambda, p, q) \cap \mathfrak{Z}$$ and $\lambda \in \Omega_{p,q}$, we have $$1 = f\left(\frac{x}{\lambda}, \frac{y}{\lambda}, z\right) = \frac{1}{\lambda} f(x, y, z) \leqslant \frac{\mu}{\lambda}.$$

The lemma shows that if $\mu = 1$ then $\kappa = 1$; hence Theorem 1 is true for $\mu = 1$. Accordingly we assume for the rest of this section that $\mu > 1$. Then $\mathfrak{R} = \mathfrak{R}(1, p, q) \subseteq \mu\mathfrak{R}^\circ$, and therefore, since $\mathfrak{R}(\lambda, p, q)$ is continuous with respect to λ, it follows that, for each p, q there exists a $\lambda \in \Omega_{p,q}$ with $\lambda > 1$. Hence we have

LEMMA 2. *For each pair of values* p, q *we have* $1 < \lambda_{p,q} \leqslant \mu$. *In particular,* $1 < \kappa \leqslant \mu$.

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LEMMA 3. *Suppose that* $\lambda \in \Omega_{p,q}$ *and that* $1 \leqslant \lambda' \leqslant \lambda$. *Then* $\lambda' \in \Omega_{p,q}$. *In fact,*

$$\mathfrak{R}(\lambda', p, q) \subseteq \mu' \mathfrak{R},$$

where

$$\mu' = \frac{\mu(\lambda'-1)+\lambda-\lambda'}{\lambda-1} \leqslant \mu.$$

Proof. As before, we take $p = q = 0$. Suppose that $(x, y, z) \in \mathfrak{R}(\lambda', 0, 0)$ so that

$$f\left(\frac{x}{\lambda'}, \frac{y}{\lambda'}, z\right) \leqslant 1.$$

We deduce that the point $(\lambda x/\lambda', \lambda y/\lambda', z) \in \mathfrak{R}(\lambda, 0, 0)$ and therefore, if $\lambda \in \Omega_{0,0}$,

$$f\left(\frac{\lambda}{\lambda'}x, \frac{\lambda}{\lambda'}y, z\right) \leqslant \mu.$$

Hence

$$f\left(x, y, \frac{\lambda'}{\lambda}z\right) \leqslant \frac{\lambda'}{\lambda}\mu.$$

Now take $\vartheta = \dfrac{\lambda(\lambda'-1)}{\lambda'(\lambda-1)}$, so that $1-\vartheta = \dfrac{\lambda-\lambda'}{\lambda'(\lambda-1)}$.

Then $0 \leqslant \vartheta \leqslant 1$, and therefore, by the convexity property,

$$f(x, y, z) \leqslant \vartheta f(x, y, \lambda' z/\lambda) + (1-\vartheta) f(x, y, \lambda' z),$$

since $\vartheta\lambda'/\lambda + (1-\vartheta)\lambda' = 1$. Thus

$$f(x, y, z) \leqslant \vartheta\lambda'\mu/\lambda + (1-\vartheta)\lambda' = \mu',$$

and the lemma follows.

We deduce from Lemma 3 that $\lambda \in \Omega_{p,q}$ for $1 \leqslant \lambda < \lambda_{p,q}$ and, since $\mathfrak{R}(\lambda, p, q)$ is continuous with respect to λ, we obtain

LEMMA 4. $\lambda_{p,q} \in \Omega_{p,q}$.

Further, if $\mathfrak{E}_{p,q}$ denotes the intersection of the sets $\mathfrak{R}(\lambda_{p,q}, p, q)$ and $\mathfrak{F}(\mu\mathfrak{R})$ we deduce from the continuity of $\mathfrak{R}(\lambda, p, q)$

LEMMA 5. $\mathfrak{E}_{p,q} \neq \mathfrak{O}$.

LEMMA 6. $\lambda_{p,q}$ *is a continuous function of* p *and* q.

Proof. Write $x_0 = \sup x$, $y_0 = \sup y$ for $(x, y, z) \in \mathfrak{R}$. Then it is enough to prove that

$$|\lambda_{p',q'} - \lambda_{p,q}| \leqslant \mu(\mu-1)\{|p-p'|\,x_0 + |q-q'|\,y_0\} f(0, 0, 1).$$

For brevity we write $\lambda = \lambda_{p,q}$, $\lambda' = \lambda_{p',q'}$, $f_0 = f(0, 0, 1)$, and

$$r = px + qy, \quad r' = p'x + q'y.$$

Also, we may, without loss of generality, assume that $\lambda' \geqslant \lambda$. Let $(x, y, z) \in \mathfrak{E}_{p,q}$, so that

$$f\left(\frac{x}{\lambda}, \frac{y}{\lambda}, z - \frac{\lambda-1}{\lambda}r\right) = 1, \quad f(x, y, z) = \mu. \tag{3·2}$$

It follows that the point $(\lambda' x/\lambda, \lambda' y/\lambda, z + \{(\lambda'-1)r' - (\lambda-1)r\}/\lambda) \in \mathfrak{R}(\lambda', p', q')$, and is therefore in $\mu\mathfrak{R}$. Thus

$$f\left(x, y, \frac{\lambda}{\lambda'}z + \frac{(\lambda'-1)r' - (\lambda-1)r}{\lambda'}\right) \leqslant \frac{\lambda\mu}{\lambda'}. \tag{3·3}$$

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From (3·2) and (3·3) we deduce that

$$\lambda(\lambda'-1)\mu-\lambda'(\lambda-1)\frac{\lambda\mu}{\lambda'}\leqslant\lambda(\lambda'-1)f(x,y,z)-\lambda'(\lambda-1)f\left(x,y,\frac{\lambda}{\lambda'}z+\frac{(\lambda'-1)r'-(\lambda-1)r}{\lambda'}\right),$$

i.e. $$\begin{aligned}\lambda\mu(\lambda'-\lambda)&\leqslant f\{\lambda(\lambda'-1)x,\lambda(\lambda'-1)y,\lambda(\lambda'-1)z\}\\&\quad-f\{\lambda'(\lambda-1)x,\lambda'(\lambda-1)y,\lambda'(\lambda-1)z+(\lambda-1)[(\lambda'-\lambda)r+(\lambda'-1)(r'-r)]\}\\&\leqslant f\{(\lambda'-\lambda)x,(\lambda'-\lambda)y,(\lambda'-\lambda)[\lambda z-(\lambda-1)r]-(\lambda-1)(\lambda'-1)(r'-r)\}\\&\leqslant(\lambda'-\lambda)f(x,y,\lambda z-(\lambda-1)r)+(\lambda-1)(\lambda'-1)|r'-r|f_0\\&=\lambda(\lambda'-\lambda)+(\lambda-1)(\lambda'-1)|r'-r|f_0.\end{aligned}$$

Hence $$\lambda'-\lambda\leqslant\frac{(\lambda-1)(\lambda'-1)}{\lambda(\mu-1)}|r'-r|f_0\leqslant(\mu-1)|r'-r|f_0$$
$$\leqslant(\mu-1)f_0\{|p-p'|\mu x_0+|q-q'|\mu y_0\}.$$

LEMMA 7. *$\lambda_{p,q}\to1$ as $r=(p^2+q^2)^{\frac{1}{2}}\to\infty$.*

Proof. Define $a_{p,q}$ to be $\sup f(0,0,z-px-qy)$ for $(x,y,z)\in\mathfrak{K}$. Then

$$\begin{aligned}a_{p,q}&=\sup f(0,0,z-px-qy)\quad\text{for}\quad(\lambda x,\lambda y,z+(\lambda-1)(px+qy))\in\mathfrak{K}(\lambda,p,q)\\&=\sup f(0,0,\zeta-p\xi-q\eta)\quad\text{for}\quad(\xi,\eta,\zeta)\in\mathfrak{K}(\lambda,p,\eta).\end{aligned}$$

Now write $f_0=f(0,0,1)$ and put $\Delta=\inf(x^2+y^2)^{\frac{1}{2}}$ for $(x,y,z)\in\mathfrak{F}(\mathfrak{K})\cap\mathfrak{Z}$. Then $\Delta>0$ since O is an interior point of $\mathfrak{K}$. Thus the point (x_1,y_1,z_1), where

$$x_1=-\frac{p}{r}\Delta,\quad y_1=-\frac{q}{r}\Delta,\quad z_1=0,$$

lies in $\mathfrak{K}$, and $z_1-px_1-qy_1=r\Delta$. Hence

$$(\lambda-1)f_0r\Delta=(\lambda-1)f(0,0,z_1-px_1-qy_1)\leqslant(\lambda-1)a_{p,q}.$$

Now put $\lambda=\lambda_{p,q}$ and suppose that $(\xi,\eta,\zeta)\in\mathfrak{K}(\lambda,p,q)$. Then

$$f(0,0,(\lambda-1)(\zeta-p\xi-q\eta))\leqslant f(\xi,\eta,\lambda\zeta-(\lambda-1)(p\xi+q\eta))+f(\xi,\eta,\zeta)\leqslant\lambda+\mu.$$

Hence $(\lambda-1)a_{p,q}\leqslant\lambda+\mu$, and we deduce that

$$(\lambda-1)f_0r\Delta\leqslant\lambda+\mu.$$

Accordingly, if $r>(\Delta f_0)^{-1}$, we have

$$1\leqslant\lambda\leqslant\frac{r\Delta f_0+\mu}{r\Delta f_0-1},$$

which shows that $\lambda\to1$ as $r\to\infty$.

Since we know from Lemma 2 that $\kappa>1$, it follows from Lemmas 6 and 7 that $\lambda_{p,q}$ is a continuous function of p and q which attains its maximum in a closed bounded region of the p, q plane. From this we deduce†

LEMMA 8. *There exist real numbers p_0 and q_0 such that $\kappa=\lambda_{p_0,q_0}$.*

By the remark at the beginning of §3, we may assume without loss of generality that $p_0=q_0=0$. For brevity we denote $\mathfrak{E}_{0,0}$ by $\mathfrak{E}_0$. Also we assume for the remainder of §3 that $\kappa<\mu$, since Theorem 1 is satisfied if $\kappa=\mu$.

We consider now the set $\mathfrak{E}_0$ in greater detail. We have

$$\mathfrak{E}_0\cap\mathfrak{Z}=\mathfrak{D}.\tag{3·4}$$

† Lemma 8 can also be proved by an application of Blaschke's theorem (see (1)), as has been pointed out by the referee.

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For if (x, y, z) were a common point of $\mathfrak{E}_0$ and $\mathfrak{Z}$, then we should have

$$\kappa = f(x, y, \kappa z) = f(x, y, z) = \mu.$$

We write $\mathfrak{E}_0 = \mathfrak{E} \cup \mathfrak{E}'$, where $\mathfrak{E}$ consists of those points (x, y, z) of $\mathfrak{E}_0$ for which $z > 0$; $\mathfrak{E}'$ has the meaning explained in § 2. From (3·4) it follows that both $\mathfrak{E}$ and $\mathfrak{E}'$ are closed, since $\mathfrak{E}_0$ being the intersection of two closed sets, is closed.

Further, let $\mathfrak{T}_0$ be the set of points of $\mathfrak{F}(\mu\mathfrak{R})$ at which there exists a tac-line parallel to the z-axis. Then we have

$$\mathfrak{E} \cap \mathfrak{T}_0 = \mathfrak{D}. \tag{3·5}$$

For if (x, y, z) were a common point of $\mathfrak{E}$ and $\mathfrak{T}_0$ we should have, since $(x, y, \kappa z) \in \mathfrak{C}(\mu\mathfrak{R}^\circ)$,

$$\kappa = f(x, y, \kappa z) \geqslant f(x, y, z) = \mu.$$

Now let $\mathfrak{T}$ be that part of $\mathfrak{F}(\mu\mathfrak{R})$ which has the property that if $(x, y, z) \in \mathfrak{T}$ then $(x, y, z') \in \mathfrak{C}(\mu\mathfrak{R}^\circ)$ for all $z' \leqslant z$. Then $\mathfrak{T}'$ consists of points $(x, y, z) \in \mathfrak{F}(\mu\mathfrak{R})$ with the property that $(x, y, z') \in \mathfrak{C}(\mu\mathfrak{R}^\circ)$ for all $z' \geqslant z$, and we have, since $\mathfrak{R}$ is convex,

$$\mathfrak{F}(\mu\mathfrak{R}) = \mathfrak{T} \cup \mathfrak{T}', \quad \mathfrak{T}_0 = \mathfrak{T} \cap \mathfrak{T}'. \tag{3·6}$$

We now show that $\mathfrak{E} \subseteq \mathfrak{T}$. It suffices to show that if the point $P(x, y, z) \in \mathfrak{E}$ then $Q(x, y, z') \in \mathfrak{C}(\mu\mathfrak{R})$ for all $z' < z$. Suppose that this is not true. Then for some $P \in \mathfrak{E}$ there exists a $Q \in \mu\mathfrak{R}$ with $z' < z$. Now $P^*(x, y, \kappa z) \in \mu\mathfrak{R}^\circ$, since $f(x, y, \kappa z) = \kappa < \mu$. Since $z > 0$ and $\kappa > 1$, P lies strictly between P^* and Q on the line P^*Q and is therefore an interior point of $\mu\mathfrak{R}$; this is false. Hence $\mathfrak{E} \subseteq \mathfrak{T}$, and it follows from (3·5) and (3·6) that

$$\mathfrak{E} \cap \mathfrak{T}' = \mathfrak{D}. \tag{3·7}$$

It is easy to see that the sets $\mathfrak{T} - \mathfrak{T}_0$, $\mathfrak{T}' - \mathfrak{T}_0$ are open, and it follows that $\mathfrak{T}'$ and $\mathfrak{T}$ are closed. Hence, by (3·7), $\mathfrak{E}$ and $\mathfrak{T}'$ possess a positive distance apart which we call δ_1; we have

$$\delta_1 = \inf P_1P_2 \quad [P_1 \in \mathfrak{E}, P_2 \in \mathfrak{T}'].$$

Also, since $O \in \mu\mathfrak{R}^0$ there exists a positive δ_2 such that $\mathfrak{S}(O, \delta_2) \subseteq \mu\mathfrak{R}$.

LEMMA 9. *Real numbers a and b do not exist with the property that $ax + by > 0$ for all points (x, y, z) of $\mathfrak{E}$.*

Proof. We suppose that, for some a, b, we have $ax + by > 0$ for all $(x, y, z) \in \mathfrak{E}$, and deduce a contradiction. We may clearly suppose that $a^2 + b^2 = 1$.

Since $ax + by$ is a continuous function of (x, y, z), and since $\mathfrak{E}$ is closed, it follows that

$$\delta_3 = \inf (ax + by) \quad [(x, y, z) \in \mathfrak{E}]$$

is positive.

Now let $P(\xi, \eta, \zeta)$ denote any point of $\mathfrak{E}$. The distance of P from the plane $ax + by = 0$ is $a\xi + b\eta \geqslant \delta_3 > 0$. We take any fixed positive number δ satisfying

$$0 < \delta < \min (\delta_1, \delta_2, \delta_3). \tag{3·8}$$

Then $ax + by > 0$ for each point (x, y, z) of $\mathfrak{S}(P, \delta)$, since $\delta < \delta_3$. Also, since $\delta < \delta_1$, at no point (x, y, z) of $\mathfrak{F}(\mu\mathfrak{R}) \cap \mathfrak{S}(P, \delta)$ is there a tac-line parallel to the axis; further $(x, y, z') \in \mathfrak{C}(\mu\mathfrak{R})$ for $z' < z$.

Take any $\rho \geqslant 0$ and write

$$p = a\rho, \quad q = b\rho. \tag{3·9}$$

$$\mathfrak{G}_{p,q} = \mathfrak{R}(\kappa, p, q) - \bigcup_{P \in \mathfrak{E}} \mathfrak{S}(P, \delta). \tag{3·10}$$

Then, if $\rho = 0$, $\mathfrak{G}_{p,q} \subseteq \mu\mathfrak{R}^\circ$ and since $\mathfrak{G}_{p,q}$ is closed and is continuous with respect to ρ it follows that there exists a $\rho_1 > 0$ such that $\mathfrak{G}_{p,q} \subseteq \mu\mathfrak{R}^\circ$ for all $0 \leqslant \rho < \rho_1$.

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Since $\delta < \delta_2$, O is outside $\mathfrak{S}(R, \delta)$ if $R \in \mathfrak{F}(\mu\mathfrak{K})$ and hence OR cuts $\mathfrak{F}\mathfrak{S}(R, \delta)$ in a point Q which is interior to $\mu\mathfrak{K}$. Let OR, produced if necessary, cut $\mathfrak{F}\mathfrak{K}(\kappa, p, q)$ in R_ρ. Because of the continuity of $\mathfrak{K}(\kappa, p, q)$ with respect to ρ there exists a $\rho_2 > 0$ such that $R_\rho \in \mathfrak{S}(R, \delta)$ for all $R \in \mathfrak{F}(\mu\mathfrak{K})$ if $0 \leqslant \rho < \rho_2$. Thus Q lies between R_ρ and O and so $Q \in \mathfrak{K}(\kappa, p, q)$ if $0 \leqslant \rho < \rho_2$.

We now take any fixed value of ρ which satisfies

$$0 < \rho < \min(\rho_1, \rho_2).$$

With this value of ρ we have $\mathfrak{K}(\kappa, p, q) \subseteq \mu\mathfrak{K}^\circ$. Since $\mathfrak{G}_{p,q} \subseteq \mu\mathfrak{K}^\circ$, we need only show that if the point $S \in \mathfrak{K}(\kappa, p, q) \cap \mathfrak{S}(P, \delta)$ for some $P \in \mathfrak{E}$, then $S \in \mu\mathfrak{K}^\circ$. Suppose therefore that $S \in \mathfrak{K}(\kappa, p, q) \cap \mathfrak{S}(P, \delta)$ for $P \in \mathfrak{E}$ and that $S \in \mathfrak{C}(\mu\mathfrak{K}^\circ)$.

Since $\rho < \rho_2$, OP cuts $\mathfrak{F}\mathfrak{S}(p, \delta)$ in a point $Q \in \mathfrak{K}(\kappa, p, q) \cap \mathfrak{K}(\kappa, 0, 0)$, and QS cuts $\mathfrak{F}(\mu\mathfrak{K})$ in a point $T(x, y, z)$. Then

$$T \in \mathfrak{K}(\kappa, p, q) \cap \mathfrak{F}(\mu\mathfrak{K}) \cap \mathfrak{S}(P, \delta).$$

Since $f(x, y, \kappa z - (\kappa - 1)\rho(ax + by)) \leqslant \kappa$ it follows that the point (x, y, z'), where

$$z' = z - (\kappa - 1)\rho(ax + by)/\kappa,$$

belongs to $\mathfrak{K}(\kappa, 0, 0)$ and therefore to $\mu\mathfrak{K}$. Since $\delta < \delta_1$ and $z' < z$ this is a contradiction.

Thus we have shown that $(\mathfrak{K}\kappa, p, q) \subseteq \mu\mathfrak{K}^\circ$. Now $\mathfrak{K}(\lambda, p, q)$ is continuous with respect to λ and so there exists a $\lambda > \kappa$ such that $\mathfrak{K}(\lambda, p, q) \subseteq \mu\mathfrak{K}$. Thus $\kappa < \lambda_{p,q}$, which is a contradiction. This completes the proof of the lemma.

Let $\mathfrak{E}^*$ denote the projection of $\mathfrak{E}$ on $\mathfrak{Z}$ in a direction parallel to the z-axis. Since $\mathfrak{E} \subseteq \mathfrak{T} - \mathfrak{T}_0$, each point of $\mathfrak{E}^*$ corresponds to exactly one point of $\mathfrak{E}$.

LEMMA 10. *O is not a point of $\mathfrak{E}^*$. If P^* and Q^* are points of $\mathfrak{E}^*$ then O does not lie on the finite linear segment P^*Q^*.*

Proof. In the first place, O is not a point of $\mathfrak{E}^*$, since if $(0, 0, z) \in \mathfrak{E}$, $1 = f(0, 0, z) = \mu$, which is not so.

Suppose that P^* and Q^* are points of $\mathfrak{E}^*$ and that O is an interior point of P^*Q^*. Then if we take P^* to be the further of the two points from O we may take the corresponding points P and Q of $\mathfrak{E}$ in the form

$$P(\xi, \eta, \zeta), \quad Q(-\sigma\xi, -\sigma\eta, \zeta'),$$

where $0 \leqslant \sigma \leqslant 1, \zeta > 0, \zeta' > 0$. Since $Q \in \mathfrak{T}$, $Q'(\sigma\xi, \sigma\eta, -\zeta') \in \mathfrak{T}'$, and therefore $Q' \in \mathfrak{T}' - \mathfrak{T}_0$. It follows that $(\sigma\xi, \sigma\eta, 0) \in \mathfrak{C}(\mu\mathfrak{K})$. On the other hand, since P and Q' are points of $\mu\mathfrak{K}$ so is the point

$$R\left(\xi\frac{\zeta' + \sigma\zeta}{\zeta' + \zeta}, \eta\frac{\zeta' + \sigma\zeta}{\zeta' + \zeta}, 0\right),$$

which is where PQ' cuts $\mathfrak{Z}$. But $(\sigma\xi, \sigma\eta, 0)$ lies on the segment OR since $0 \leqslant \sigma \leqslant 1$ and is therefore in $\mu\mathfrak{K}$; this is a contradiction.

LEMMA 11. *Among the points of $\mathfrak{E}^*$ there are three P_i^* with coordinates*

$$(x_i, y_i, 0) \quad (i = 1, 2, 3)$$

such that O is an interior point of the triangle $P_1^*P_2^*P_3^*$ whose vertices are arranged anticlockwise about O, i.e.

$$u_1 = x_2y_3 - x_3y_2 > 0, \quad u_2 = x_3y_1 - x_1y_3 > 0, \quad u_3 = x_1y_2 - x_2y_1 > 0. \tag{3·11}$$

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Proof. We know from Lemmas 5, 9 and 10 that $\mathfrak{E}^*$ is not empty, that it does not contain O, that $\mathfrak{E}^*$ does not lie entirely on one side of any line in $\mathfrak{Z}$ through O, and that if two points of $\mathfrak{E}^*$ are collinear with O then they lie on the same side of O.

It follows at once that $\mathfrak{E}^*$ contains two points $Q_i^*(\xi_i, \eta_i, 0)\,(i = 1, 2)$ such that $\xi_1\eta_2 - \xi_2\eta_1 > 0$, i.e. the angle $Q_1^*OQ_2^*$ measured anticlockwise, is less than 180°. The lines OQ_1^*, OQ_2^* when produced in both directions divide $\mathfrak{Z}$ into four sectors. Let $\mathfrak{H}$ be that sector which contains the image in O of the mid-point of $Q_1^*Q_2^*$. We take $\mathfrak{H}$ to be open in $\mathfrak{Z}$, so that it does not contain any points lying on the two infinite lines.

If $\mathfrak{H}$ contains a point Q_3^* of $\mathfrak{E}^*$ then O is an interior point of the triangle $Q_1^*Q_2^*Q_3^*$, and by renaming Q_i^* as P_i^* we satisfy the conditions of the lemma.

We therefore suppose that $\mathfrak{H} \cap \mathfrak{E}^* = \mathfrak{O}$. Since $\mathfrak{E}^*$ is closed and bounded we can therefore find points Q_3^* and Q_4^* of $\mathfrak{E}^*$ such that (i) $Q_1^*\hat{O}Q_3^* = \sup Q_1^*\hat{O}Q^*$ for points Q^* of $\mathfrak{E}^*$ lying on the same side of OQ_1^* as Q_2^*, (ii) $Q_4^*\hat{O}Q_2^* = \sup Q^*\hat{O}Q_2^*$ for points Q^* of $\mathfrak{E}^*$ lying on the same side of OQ_2^* as Q_1^*. All angles are measured in the anti-clockwise direction. It follows that $\mathfrak{E}^*$ is contained in the sector formed by rotation about O in an anti-clockwise direction from OQ_4^* to OQ_3^*. The origin must lie on the same side of $Q_3^*Q_4^*$ as Q_1^* and Q_2^*. If we take $P_2^* = Q_3^*$, $P_3^* = Q_4^*$ and $P_1^* = Q_1^*$ or Q_2^* the conditions of the lemma are satisfied.

LEMMA 12. *Let α, u_i and $z_i (i = 1, 2, 3)$ be positive and*

$$u_1z_1 + u_2z_2 + u_3z_3 \leqslant \alpha \min\{(u_2 + u_3 - u_1)z_1, (u_3 + u_1 - u_2)z_2, (u_1 + u_2 - u_3)z_3\}.$$

Then $\alpha \geqslant 3$, and equality occurs only when

$$u_1 = u_2 = u_3, \quad z_1 = z_2 = z_3.$$

Proof. Write $v_i = u_iz_i\,(i = 1, 2, 3)$. Then

$$v_1 + v_2 + v_3 \leqslant \alpha \frac{v_1}{u_1}(u_2 + u_3 - u_1),$$

i.e.
$$\alpha(u_2 + u_3 - u_1) \geqslant \frac{u_1}{v_1}(v_1 + v_2 + v_3).$$

Similarly
$$\alpha(u_3 + u_1 - u_2) \geqslant \frac{u_2}{v_2}(v_1 + v_2 + v_3).$$

By addition we deduce that

$$2\alpha u_3 \geqslant \left(\frac{u_1}{v_1} + \frac{u_2}{v_2}\right)(v_1 + v_2 + v_3)$$

$$\geqslant 2\left\{\frac{u_1u_2}{v_1v_2}\right\}^{\frac{1}{2}} 3(v_1v_2v_3)^{\frac{1}{3}},$$

by the inequality of the arithmetic and geometric means. Hence

$$\alpha u_3 \geqslant 3\left\{\frac{u_1u_2}{v_1v_2}\right\}^{\frac{1}{2}}(v_1v_2v_3)^{\frac{1}{3}},$$

and there are two other similar inequalities obtained by permuting the suffixes cyclically. By multiplication we find that

$$\alpha^3u_1u_2u_3 \geqslant 27\frac{u_1u_2u_3}{v_1v_2v_3}v_1v_2v_3 = 27u_1u_2u_3,$$

from which it follows that $\alpha \geqslant 3$. It is clear that equality can occur only under the conditions stated.

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We are now in a position to prove Theorem 1. We observe that if $(x, y, z) \in \mathfrak{E}$ then $f(x, y, \kappa z) = \kappa$ so that $(\mu x/\kappa, \mu y/\kappa, \mu z) \in \mathfrak{F}(\mu\mathfrak{K})$. Accordingly, we deduce from Lemma 11 that $\mu\mathfrak{K}$ contains on its boundary twelve points

$$P_i(x_i, y_i, z_i), \quad Q_i(\xi_i, \eta_i, \zeta_i). \quad P_i', \quad Q_i' \quad (i = 1, 2, 3),$$

where (3·11) holds and

$$z_i > 0, \quad \xi_i = \frac{\mu x_i}{\kappa}, \quad \eta_i = \frac{\mu y_i}{\kappa}, \quad \zeta_i = \mu z_i \quad (i = 1, 2, 3). \tag{3·12}$$

Now write

$$u = z_1\{u_1 + (\kappa - 1)(u_2 + u_3)\} + \kappa(u_2 z_2 + u_3 z_3),$$

$$t_1 = \frac{1}{u}\{u_1 z_1 + \kappa(u_2 z_2 + u_3 z_3)\},$$

$$t_2 = \frac{1}{u}(\kappa - 1) u_2 z_1,$$

$$t_3 = \frac{1}{u}(\kappa - 1) u_3 z_1,$$

$$t = \frac{1}{u}\mu(u_1 z_1 + u_2 z_2 + u_3 z_3).$$

Then we see that all five quantities u, t_1, t_2, t_3 and t are positive and that

$$t_1 + t_2 + t_3 = 1.$$

Since Q_1, Q_2' and Q_3' are points of $\mu\mathfrak{K}$ it follows that the point

$$P = t_1 Q_1 + t_2 Q_2' + t_3 Q_3'$$

is also a point of $\mu\mathfrak{K}$, since $\mu\mathfrak{K}$ is convex. But $P = tP_1$ and hence, since $P_1 \in \mathfrak{F}(\mu\mathfrak{K})$ it follows that $t \leqslant 1$, i.e.

$$\mu(u_1 z_1 + u_2 z_2 + u_3 z_3) \leqslant \{u_1 + (\kappa - 1)(u_2 + u_3)\} z_1 + \kappa(u_2 z_2 + u_3 z_3).$$

Hence

$$(\mu - \kappa)(u_1 z_1 + u_2 z_2 + u_3 z_3) \leqslant (\kappa - 1)(u_2 + u_3 - u_1) z_1.$$

There are two other similar inequalities obtained by permuting the suffixes in cyclic order. We deduce from Lemma 12 that

$$\frac{\kappa - 1}{\mu - \kappa} \geqslant 3,$$

i.e.

$$\kappa \geqslant \tfrac{1}{4}(1 + 3\mu).$$

From this and Lemma 2, Theorem 1 follows.

4. *The case* $\kappa = \mu$. If $\mu = 1$ then $\kappa = \mu$ for every $\mathfrak{K}$ so that we suppose that $\mu > 1$. Also, by Lemma 8, $\kappa = \lambda_{p,q}$ for some p, q and we may suppose that the coordinates are chosen so that $p = q = 0$. If $\kappa = \mu$ then $\mathfrak{K}(\mu, 0, 0) \subseteq \mu\mathfrak{K}$. We deduce that if $f(x, y, \mu z) = \mu$ then $f(x, y, z) \leqslant \mu$. Thus if $(\xi, \eta, \zeta) \in \mathfrak{F}(\mu\mathfrak{K})$ then $(\xi, \eta, \zeta/\mu) \in \mu\mathfrak{K}$. Define the sets $\mathfrak{T}_0$, $\mathfrak{T}$ and $\mathfrak{T}'$ as in §3. Then it follows that if $(\xi, \eta, \zeta) \in \mathfrak{F}(\mu\mathfrak{K})$, $(\xi, \eta, \zeta) \in \mathfrak{T}'$ for $\zeta > 0$, and $(\xi, \eta, \zeta) \in \mathfrak{T}$ for $\zeta < 0$. Hence $\mathfrak{T}_0 = \mathfrak{Z} \cap \mathfrak{F}(\mu\mathfrak{K})$. Conversely, if $\mathfrak{T}_0 = \mathfrak{Z} \cap \mathfrak{F}(\mu\mathfrak{K})$ it follows that $\mathfrak{K}(\mu, 0, 0) \subseteq \mu\mathfrak{K}$. Thus we have proved

THEOREM 2. *Suppose that* $\mu > 1$. *Then* $\kappa = \mu$ *if and only if the set of points of* $\mathfrak{F}(\mathfrak{K})$, *at which there exists a tac-line parallel to the z-axis, lies in a plane through the origin.*

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For we could take $z - px - qy = 0$ to be the equation of this plane, as it clearly cannot pass through the z-axis, and change coordinates by a transformation of the type (3·1).

We note as a corollary that if $\mathfrak{R}$ is an ellipsoid then $\kappa = \mu$.

5. *The case* $\kappa = \frac{1}{4}(1+3\mu)$. Denote by Γ the group of all transformations of the form (3·1). The points of the z-axis are fixed points of all transformations of Γ. Each transformation $\mathscr{T} \in \Gamma$ transforms $\mathfrak{R}$ into a new convex body $\mathscr{T}\mathfrak{R}$.

We suppose that $\mu > 1$, as otherwise $\kappa = \frac{1}{4}(1+3\mu)$ for every convex body $\mathfrak{R}$. Let $Q_i(\xi_i, \eta_i, \zeta_i)\,(i = 1, 2, 3)$ be any three points such that

$$\zeta_1 = \zeta_2 = \zeta_3 > 0, \tag{5·1}$$

$$\xi_2\eta_3 - \xi_3\eta_2 = \xi_3\eta_1 - \xi_1\eta_3 = \xi_1\eta_2 - \xi_2\eta_1 > 0. \tag{5·2}$$

It follows from (5·2) that

$$0 = \begin{vmatrix} -\xi_1 & -\xi_2 & \xi_1+\xi_2 \\ \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & \xi_1+\xi_2+\xi_3 \\ \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \end{vmatrix} = (\xi_1+\xi_2+\xi_3)(\xi_1\eta_2 - \xi_2\eta_1).$$

Thus we have, by (5·2), $\xi_1+\xi_2+\xi_3 = 0, \quad \eta_1+\eta_2+\eta_3 = 0.$ (5·3)

the second equation being deduced in a similar manner.

The three points Q_1, Q_2 and Q_3 lie above $\mathfrak{Z}$ in a plane $\mathfrak{Z}_1$ parallel to $\mathfrak{Z}$, and $\mathfrak{Z}_1$ cuts the z-axis at a point G which is the centroid of the triangle $Q_1Q_2Q_3$, by (5·3). We denote by $\mathfrak{F}_1, \mathfrak{F}_2$ and $\mathfrak{F}_3$ the plane sets of points in or on the triangles $Q_1Q_2'Q_3'$, $Q_2Q_3'Q_1'$ and $Q_3Q_1'Q_2'$, respectively. We denote by $\mathfrak{R}^*$ any symmetrical convex body which contains $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \mathfrak{F}_1', \mathfrak{F}_2', \mathfrak{F}_3'$ as part of its boundary. The class of all such bodies we denote by Σ, and $\Gamma\Sigma$ denotes the class of all bodies $\mathscr{T}\mathfrak{R}^*$ where $\mathscr{T} \in \Gamma$, $\mathfrak{R}^* \in \Sigma$. We note that if (x, y, z) is a point of $\mathfrak{F}_1$, $\mathfrak{F}_2$ or $\mathfrak{F}_3$ then (x, y, z') can be a point of $\mathfrak{R}^*$ only if $z' \geqslant z$.

THEOREM 3. *Suppose that* $\mu > 1$. *Then* $\kappa = \frac{1}{4}(1+3\mu)$ *if and only if* $\mathfrak{R} \in \Gamma\Sigma$.

Proof. Write $\kappa' = \frac{1}{4}(1+3\mu)$. Suppose that $\mathfrak{R}$ is a body for which $\kappa = \kappa'$. By a transformation of Γ we may arrange that $\kappa = \lambda_{0,0}$. Then, as in §3, $\mathfrak{F}(\mu\mathfrak{R})$ contains on its boundary twelve points $P_i(x_i, y_i, z_i)$, $Q_i(\xi_i, \eta_i, \zeta_i)$, P_i' and $Q_i'\,(i = 1, 2, 3)$, whose coordinates satisfy (3·11) and (3·12). Also, from the last stages of the proof of Theorem 1 and from Lemma 12, it follows that

$$u_1 = u_2 = u_3, \quad z_1 = z_2 = z_3,$$

and that P_i is an interior point of the triangle $Q_iQ_j'Q_k'$, where i, j, k denotes a permutation of 1, 2, 3. Hence (5·1) and (5·2) hold, and the faces $\mathfrak{F}_i, \mathfrak{F}_i'\,(i = 1, 2, 3)$ form part of the boundary of $\mu\mathfrak{R}$. It follows that $\mu\mathfrak{R} \in \Sigma$ and so $\mathfrak{R} \in \Sigma$.

To complete the proof of the theorem it is necessary to show that if $\mathfrak{R} \in \Gamma\Sigma$ then $\kappa = \kappa'$. Suppose therefore that $\mathfrak{R} \in \Gamma\Sigma$; hence $\mu\mathfrak{R} \in \Gamma\Sigma$. By a transformation of Γ we may suppose that $\mu\mathfrak{R} \in \Sigma$. We show first that $\lambda_{p,q} < \kappa'$ for all p, q not both zero. For suppose that $\lambda_{p,q} \geqslant \kappa'$ for some $(p, q) \neq (0, 0)$. Then it follows from Lemma 3 that $\mathfrak{R}(\kappa', p, q) \subseteq \mu\mathfrak{R}$.

Now $\mathfrak{F}(\mu\mathfrak{R})$ contains points $Q_i(\xi_i, \eta_i, \zeta_i)\,(i = 1, 2, 3)$ which satisfy (5·1), (5·2) and (5·3). Also, part of $\mathfrak{F}(\mu\mathfrak{R})$ is composed of six triangular faces $\mathfrak{F}_i, \mathfrak{F}_1'\,(i = 1, 2, 3)$ con-

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taining $Q_i, Q_i' (i = 1, 2, 3)$ as vertices. Since Q_1 lies on $\mathfrak{F}(\mu\mathfrak{K}), f(\xi_1, \eta_1, \zeta_1) = \mu$; it follows that the point

$$Q\left(\frac{\kappa'}{\mu}\xi_1, \frac{\kappa'}{\mu}\eta_1, \frac{\zeta_1}{\mu} + \frac{\kappa'-1}{\mu}(p\xi_1 + q\eta_1)\right)$$

belongs to $\mathfrak{K}(\kappa', p, q)$ and therefore to $\mu\mathfrak{K}$. Now the point $P_1(\kappa'\xi_1/\mu, \kappa'\eta_1/\mu, \zeta_1/\mu)$ is an interior point of the triangle $Q_1Q_2'Q_3'$ since, by (5·1) and (5·3),

$$P_1 = \frac{\mu+1}{2\mu}Q_1 + \frac{\mu-1}{4\mu}Q_2' + \frac{\mu-1}{4\mu}Q_3'.$$

Thus P_1 is a point of $\mathfrak{F}_1$, and therefore

$$p\xi_1 + q\eta_1 \geqslant 0.$$

Similarly, we have $\quad p\xi_2 + q\eta_2 \geqslant 0, \quad p\xi_3 + q\eta_3 \geqslant 0.$

From these last three inequalities, (5·2) and (5·3) we easily deduce that $p = q = 0$, which contradicts our original assumption.

Hence $\lambda_{p,q} < \kappa'$ for all $(p, q) \neq (0, 0)$. We deduce from Lemma 6 and Theorem 1 that $\kappa = \lambda_{0,0} = \kappa'$. This completes the proof of Theorem 3.

We note, in particular, that $\kappa = \frac{1}{4}(1 + 3\mu)$ when $\mathfrak{K}$ is a cube with two opposite corners on the z-axis, or when $\mathfrak{K}$ is a regular octahedron with the centroids of two opposite faces on the z-axis.

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