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ABATEMENT TECHNOLOGY ADOPTION UNDER UNCERTAINTY

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New technology has been credited with solving environmental problems by mitigating the effects of pollutants. We construct a general equilibrium model in which abatement technology is a real option and pollution's (negative) amenity value alters both risk aversion and the intertemporal elasticity of substitution. We derive the tax scheme such that in a decentralized economy agents adopt the abatement technology at the time that is socially optimal. We show that the higher the greenness of preferences, the earlier the adoption and the higher the optimal tax rate. We also obtain that adoption is fostered by uncertainty if the effective intertemporal elasticity of substitution is large enough, but is not affected by uncertainty if this elasticity is low. Moreover, the optimal tax rate, which only exists if the effective intertemporal elasticity of substitution is high, is an increasing function of uncertainty.

Keywords: Abatement Technology, Uncertainty, Optimal Timing, Pollution Tax

1. INTRODUCTION

New technology has been credited with solving environmental problems by mitigating the effects of pollutants. Indeed, the seeming intractability of environmental problems, such as global climate change, combined with considerable uncertainties and the long time frame over which their ultimate consequences will play out, may make the development and deployment of new technologies attractive as a major response. But adoption is costly, because firms must learn about new technology, purchase new equipment, and adapt it to their particular circumstances. Therefore, the problem is to know when to adopt a new abatement technology. In fact, this encompasses two related questions. First, what are the determinants of the

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socially optimal adoption timing? Second, how should environmental regulation be designed to allow this socially optimal timing to be chosen in a decentralized way by the agents of a polluting economy? Indeed, we cannot expect that agents will choose the right date to implement an abatement technology, because of the externalities associated with pollution. Because the polluter reaps the benefits derived from polluting, while imposing the pollution costs to others, she lacks an incentive to reduce those costs, that is, to adopt an abatement technology.

Many empirical studies confirm that the mere presence of environmental technology is not enough to encourage its use, and that even when a technology is available it will not be adopted and diffused through the economy without regulatory incentives to do so [see, for example, Jaffe and Stavins (1995), Gray and Shadbegian (1998), Popp (2001), or Kerr and Newell (2003)]. A large body of the literature, theoretical as well as empirical, is concerned with the choice of the best policy instrument: subsidy to the new technology, tax on pollution, or tradable permits. For example, in an empirical analysis of the effects of environmental policy instruments on technology diffusion, Jaffe and Stavins (1995) conduct econometric analyses of the factors that affected the adoption of thermal insulation technologies in new residential construction in the United States from 1979 to 1988. Among other things, they examine the effect of technology adoption costs on average residential energy-efficient technologies in new home construction. The effects of changes in adoption costs can be interpreted as indicating what the effects of technology-adoption subsidies would be. They find that equivalent percentage cost subsidies would have been about three times as effective as taxes in encouraging adoption.

Turning to general theoretical analysis of technology adoption, the considerable weight of adoption costs (20 times those for innovation, according to Jovanovic (1997)) has directed the interest of growth theorists toward the study of problems of diffusion and adoption of new technologies. The recent boom in information technologies and the associated employment and growth opportunities has stimulated a lot of research in this field. The objective was to treat the two following questions: Given the state of the economy, is it optimal to switch to a new technology? And if the decision is to adopt, what is the optimal timing of adoption? Indeed, a too quick adoption could be a disaster, because, among other reasons, the return to adoption is uncertain, and there is a lack of complementarity between the new technology and the actual structure of the economy [for instance, the composition of skills; see Galor and Tsiddon (1997)].

In a deterministic environment, one generally concludes that only the absence of adoption (if adoption is too costly next to its economic payoff) or an immediate adoption (if it is not) can be optimal. Nevertheless, Khan and Ravikumar (2002) and Cunha-E-Sa and Reis (2007) obtain an optimal timing of adoption through a sunk cost that is not proportional to wealth.[1] Khan and Ravikumar (2002) consider an AK growth model in which the new technology increases the productivity of capital. Their result is that the level of wealth accumulation that triggers adoption is independent of preference parameters and only depends on technological

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parameters. Cunha-E-Sa and Reis (2007) also consider an AK growth model. Utility depends on consumption, but also on a flow of environmental services, inversely proportional to production. The coefficient of proportionality is interpreted as the environmental quality of the technology, and the adoption of the (greener) new technology allows increasing this coefficient. The level of wealth accumulation which triggers adoption becomes then sensitive to preference parameters.

But the decision to adopt an abatement technology is highly sensitive to uncertainty in the environment. We consider very long-term benefits, whose size is not at all known with certainty. It is therefore legitimate to consider the technology adoption in a stochastic framework. In such a framework, optimal adoption has extensively been studied in partial equilibrium, in which technological adoption is treated as the decision to exercise a real option. Abel and Eberly (2002), (2004) study the optimal adoption of the stochastic latest technology. By constrast, in Roche (2003), it may be optimal for an upgrading firm to keep some distance with the frontier technology. The intuition runs as follows. In particular, during recessions, which reduce current cash flow, the firm has less need for highly performing technology; consequently, the manager tends to delay scrapping and adopts a less sophisticated technology during recessions. Grenadier and Weiss (1997) also focus on investment opportunities in stochastic technological innovations. In a sequential investment framework, adopting an innovation provides the firm with an option value to learn. This literature shows that with uncertainty and a sunk adoption cost, there exists a significant value in waiting for new information. As a result, investment projects should only be undertaken when the value of the new technology exceeds the adoption cost by a potentially large premium. This generates an optimal delay for technology adoption, and allows an optimal timing. Nevertheless, because policies are formulated in partial equilibrium settings, they ignore the interaction between optimal consumption and optimal adoption. Moreover, it would be natural to consider an abatement technology adoption problem in a general equilibrium framework, because such a technology would never be adopted in a firm-level setup, as pollution is not internalized by the polluter.

Theoretical analyses of technology adoption in general equilibrium and a stochastic environment are a lot more recent and very limited in number. Hugonnier et al. (2008) study the optimal adoption of a new technology that increases the productivity of capital.[2] There exists then an optimal adoption timing, and this timing is highly sensitive to the size of uncertainty as well as to the degree of agents' risk aversion.

Tackling the specific problem of abatement technology adoption is a bit different. First, the determination of the socially optimal adoption timing is affected by the existence of pollution, i.e., negative externality, in the problem. So we propose to solve the centralized program corresponding to the optimal adoption of an abatement technology. Therefore, we extend the general equilibrium model with a real option proposed by Hugonnier et al. (2008). Amenity value (which is negative) alters both risk aversion and the intertemporal elasticity of substitution. Then, following Smith and Son (2005), we define an effective risk aversion as well as

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an effective intertemporal elasticity of substitution. We find that the threshold that triggers adoption depends not only on technological parameters but on preference parameters as well. Second, we derive the tax scheme such that in a decentralized economy agents adopt the abatement technology for the same level of economic development as in the centralized economy. We especially focus on the role of uncertainty both for the optimal adoption and for the tax scheme. We obtain that because of uncertainty it may be optimal to reduce the current consumption rate in order to foster adoption. We also show that the higher the uncertainty, the earlier the adoption, which is a result strikingly different from what is obtained in partial equilibrium.[3] It follows that a larger uncertainty generates a higher tax rate in order to foster adoption in the decentralized equilibrium. We then show how the optimal timing depends on agents' sensitivity with respect to pollution, i.e., on the greenness of their preferences, and on the effective preference parameters. Indeed, we obtain that the higher the sensitivity of agents with respect to pollution, the earlier the adoption. Moreover, we also show that the effect of risk aversion on the optimal adoption is ambiguous. Finally, we study the determinants of this tax in order to address some policy recommendations with respect to abatement technology. In particular, we obtain first that the higher the sensitivity of agents with respect to pollution, the higher this tax rate, and second, the larger the effective risk aversion, the higher this tax rate as well.

The remainder of the paper is as follows. Section 2 presents the general model, explaining assumptions that will be valid for the whole paper. Because this model is to be solved backward, we start with Section 3 by developing the general equilibrium framework once the abatement technology has been adopted, and in Section 4 we solve the model before the technology adoption and derive the socially optimal adoption timing. In Section 5, we consider the competitive economy counterpart of the preceding model in order to derive the optimal taxation scheme. Section 6 concludes.

2. THE MODEL

We use a very simple growth model. The production function is of the AK type, with an uncertain marginal productivity of capital. Emissions of pollution are a by-product of capital. The (constant) coefficient of emission can be reduced by adopting an abatement technology, a scrubber, for example. The representative agent derives utility from consumption and disutility from emissions of pollution. The problem is to find the optimal timing of adoption of the abatement technology.

Output is generated from capital by the linear production technology

$$dY_t = AK_t dt + \sigma K_t dz_t. \tag{1}$$

Equation (1) states[4] that the accumulated flow of output over a time interval of length dt consists of two components, a deterministic one AK_t that represents the mean rate of output per unit of time, and a stochastic one that can be interpreted as a

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productivity shock. This shock is governed by a temporally independent, normally distributed, stochastic process with zero mean and variance $\sigma^2 dt$. The overall size of the stochastic disturbance in output varies with the existing capital stock and, hence, with the size of current output. Production can be either consumed or invested. The law of accumulation of capital is then

$$dK_t = (AK_t - C_t)\,dt + \sigma K_t dz_t, \tag{2}$$

where C is the consumption. The production process is polluting. The emissions coefficient is assumed, constant, and equal to $B_1 > 0$ in the initial situation. The economy can lower this coefficient by switching to a new technology of abatement (a new quality of scrubbers, for instance), but at the expense of a fixed cost in terms of capital $\beta > 0$. This cost cannot be recovered. This implies that once the new technology has been adopted, there is no incentive to turn back to the old one, whatever shocks the economy experiences. We then have the following flow of pollution:

$$F_t = \begin{cases} B_1 K_t & \text{for } 0 \le t < \tau, \\ B_2 K_t & \text{for } t \ge \tau, \end{cases} \tag{3}$$

where $B_2 < B_1$ and τ is the date of adoption of the technology that allows the reduction of emissions. The social preferences over consumption plans are represented by the lifetime expected utility functional

$$E_0\left[\int_0^\infty U(C_t, F_t)\,dt\right] = E_0\left[\int_0^\infty \frac{\left(C_t^a F_t^{-b}\right)^{1-\gamma}}{1-\gamma} dt\right], \tag{4}$$

where $1 \neq \gamma > 0$ is the constant relative risk aversion of the representative agent.

The social utility discount rate is equal to zero. There is a huge debate in environmental economics about the way future utilities should be discounted, especially when today's decisions have very long-term consequences. To our point of view, economic arguments in favor of a constant and strictly positive social utility discount rate are not especially more convincing than the ones in favor of a zero social utility discount rate [see, for instance, for a thorough survey of the debates, Portney and Weyant (1999)]. We rather concentrate here on the effects of uncertainty and technological change, and make the assumption of a zero discount rate for the preceeding reasons and, more technically, because introducing a positive discount rate would prevent us from analytically solving the model,[5] without changing the nature of the results [see Hugonnier et al. (2005) for simulations in a similar technical context].

The optimal switching time should maximize the intertemporal utility subject to the law of motion of capital, the non-negativity constraint, and the initial condition K_0. To facilitate the presentation, let Θ denote the set of admissible plans, that is, the set of consumption plans and dates of adoption (C, τ) such that

$$E_0\left[\int_0^\infty |U(C_s, F_s)| ds\right] < \infty.$$

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With this notation, the value function of the central planner is

$$V(K_0) = \sup_{(C,\tau)\in\Theta} E_0 \left\{ \int_0^\tau \frac{\left[C_t^a (B_1 K_t)^{-b}\right]^{1-\gamma}}{1-\gamma} dt + W(K_\tau - \beta) \right\},$$

where W is the value after the switch. This program can be solved in two stages. We first solve for the optimal consumption behavior of the representative agent assuming that the new technology has been adopted. This will provide us with a boundary condition at the time of the technology adoption. Using this condition, we determine the optimal time for adopting the new technology.

3. THE OPTIMAL PATH AFTER THE SWITCH

We assume in this section that the new abatement technology has been adopted. Then the set of admissible plans Θ collapses to the set of consumption plans such that

$$E_\tau \left[\int_\tau^\infty |U(C_s, F_s)|\, ds \right] < \infty.$$

The value function of the central planner is

$$W(K_t) = \sup_{C\in\Theta} E_t \left\{ \int_t^\infty \frac{\left[C_s^a (B_2 K_s)^{-b}\right]^{1-\gamma}}{1-\gamma} ds \right\}, \quad t \geq \tau. \tag{5}$$

The Bellman equation is written

$$W(K_t) = \max_{C_t} \left\{ \frac{\left[C_t^a (B_2 K_t)^{-b}\right]^{1-\gamma}}{1-\gamma} + E_t[W(K_{t+dt})] \right\}, \quad t \geq \tau.$$

The optimal consumption path is given by

$$C_t^* = \frac{a(1-\gamma)}{a(1-\gamma)-1} \left\{ A + \frac{\sigma^2}{2} [(a-b)(1-\gamma) - 1] \right\} K_t. \tag{6}$$

The derivation is standard and is relegated to Appendix A.

The optimal consumption is proportional to the capital stock.

We define the effective coefficient of risk aversion R and the inverse of the effective intertemporal elasticity of substitution S:

$$R = 1 - (a-b)(1-\gamma), \tag{7}$$

$$S = 1 - a(1-\gamma). \tag{8}$$

It is easy to show that the condition on parameters that ensures the concavity of the utility function is $(a-b)(1-\gamma) - 1 \leq 0$. We assume this condition fulfilled. Then the effective risk aversion is positive. The effective intertemporal elasticity of substitution must also be positive, which requires $1 - a(1-\gamma) > 0$. We also

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assume this condition fulfilled. With these notations, the optimal consumption path is given by

$$C_t^* = \frac{S-1}{S}\left(A - \frac{\sigma^2}{2}R\right)K_t, \quad t \geq \tau. \tag{9}$$

The expected optimal growth rate is

$$\frac{E_t(dK_t/K_t)}{dt} = A - E_t\left(\frac{C_t^*}{K_t}\right) = \frac{1}{S}\left[A + (S-1)\frac{\sigma^2}{2}R\right], \tag{10}$$

and the explicit expression of the value function (see Appendix A) is

$$W(K_t) = aB_2{}^{S-R}\left[\frac{S-1}{S}\left(A - \frac{\sigma^2}{2}R\right)\right]^{-S}\frac{K_t^{1-R}}{1-R}. \tag{11}$$

Consumption must be positive, which requires that the "feasibility condition" is satisfied:

$$(S-1)\left(A - \frac{\sigma^2}{2}R\right) > 0. \tag{12}$$

We impose this condition, which means that the expected productivity of capital must be high (low) enough if the inverse of the effective elasticity of intertemporal substitution is greater (less) than one.

With time-separable and isoelastic preferences depending on consumption as a single argument, we know since Merton (1969) that the feasibility condition is equivalent to the transversality condition [see also Smith (1996)]. But in our model, preferences depend on both consumption and emissions of pollution, and it is possible to show (see Appendix A) that this slightly transforms the transversality condition, which can be written

$$(R-1)\left(A - \frac{\sigma^2}{2}R\right) > 0. \tag{13}$$

The feasibility and transversality conditions reduce to

$$(\gamma-1)\left(A - \frac{\sigma^2}{2}R\right) > 0 \tag{14}$$

if and only if $a > b$, which will be the case (see below).

Note that in a deterministic setup equivalent to ours but with a positive discount rate ρ, the feasilibity/transversality condition would be [Smith (1996)]

$$\rho + (\gamma-1)A > 0.$$

It would always be satisfied for $\gamma > 1$ (intertemporal elasticity of substitution less than one, negative utility), even for a zero discount rate, because it would reduce in this case to the positiveness of the constant maginal productivity of capital.

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Conversely, it would never be satisfied for $\gamma < 1$ and a zero discount rate. In our stochastic setup, $A - (\sigma^2/2)R$ in equation (14) is the certainty equivalent of the return on production. When $\gamma > 1$, this return has to be positive; it has to be negative when $\gamma < 1$, which is now possible, if uncertainty is high enough. The condition (14) can be satisfied with $\gamma < 1$, whereas it cannot in the absence of uncertainty.

We also define the "greenness" of preferences, that is, the relative weight of the environmental variable in the utility function, as b/a.

We have

$$\frac{\partial(C_t/K_t)}{\partial\sigma^2} = -\frac{S-1}{S}\frac{R}{2}.$$

Thus an increase in risk will increase (decrease) the consumption rate when the intertemporal elasticity of substitution $(1/S)$ is greater (less) than one. Indeed, an increase in risk reduces the certainty equivalent of the return on production $[A - (\sigma^2/2)R]$; this reduction generates opposite income and substitution effects, and when the intertemporal elasticity of substitution is greater than one the substitution effect prevails [see also Weil (1990)].

Having solved the program once the abatement technology has been adopted, one can easily deduce the solution of the program of the central planner in an economy with no technological change. This will provide a nice benchmark for what follows. Denoting as $W_0(K_t)$ the value function of the central planner of the economy with no technological change, the solution is simply

$$C_t^* = \frac{S-1}{S}\left(A - \frac{\sigma^2}{2}R\right)K_t, \tag{15}$$

$$W_0(K_t) = a{B_1}^{S-R}\left[\frac{S-1}{S}\left(A - \frac{\sigma^2}{2}R\right)\right]^{-S}\frac{K_t^{1-R}}{1-R}. \tag{16}$$

Recall that $W_0(K_t)$ cannot be greater than the lifetime utility of the agent in an economy with the new technology: we must have

$$W_0(K_t) \leq W(K_t). \tag{17}$$

This condition ensures that there exists an optimal switching date; that is, in the absence of costs of switching to the new technology, the central planner would choose to immediately switch for any current level of capital accumulation. With the assumption $B_1 > B_2$, it is easy to see that a necessary and sufficient condition for equation (17) to be satisfied is

$$a > b, \tag{18}$$

which we impose. To sum up, the restrictions imposed so far on the parameters are those indicated in equations (14) and (18).

Moreover, following Cunha-E-Sa and Reis (2007), we can consider the difference between the two value functions for any level of capital

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$\Delta(K_t) = W(K_t) - W_0(K_t)$. We have

$$\frac{\partial \Delta(K_t)}{\partial K_t} = a\left(B_2^{S-R} - B_1^{S-R}\right)\left[\frac{S-1}{S}\left(A - \frac{\sigma^2}{2}R\right)\right]^{-S} K_t^{-R}$$
$$\gtreqless 0 \Leftrightarrow S - R \lesseqgtr 0 \Leftrightarrow \gamma \lesseqgtr 1.$$

Then if $\gamma < 1$, the difference between the values in the two economies increases with the stock of capital, whereas it decreases if $\gamma > 1$. This effect can be related to the way the marginal disutility of pollution varies when consumption increases. If $\gamma < 1$, the marginal disutility of pollution increases when consumption increases. It is the "disgust effect" identified by Michel and Rotillon (1995). It means here that when consumption and the stock of capital are high, the utility derived from a decrease in pollution is high, and so that adoption will be delayed until the stock of capital is high enough. Conversely, if $\gamma > 1$, the marginal disutility of pollution decreases when consumption increases ("compensation effect"), and adoption must take place as soon as possible. This effect, totally independent of uncertainty, will partly explain why the results we obtain are so different depending on the value of γ. The other part of the explanation lies in the effects of uncertainty, and must be searched for in what happens before the switch.

4. THE OPTIMAL SWITCHING TIME

The central planner's problem, that is, the choice of an optimal consumption plan and of an optimal technology adoption timing, is given by the maximization of the intertemporal utility function subject to the law of capital accumulation. Once the new abatement technology has been adopted, the central planner optimally follows the consumption plan described by equation (9). Therefore, the value function at the time of the switch is given by the following value-matching and smooth pasting conditions:

$$V(K_\tau) = W(K_\tau - \beta), \tag{19}$$

$$V_K(K_\tau) = W_K(K_\tau - \beta), \tag{20}$$

where K_τ is the level of the capital stock for which it is optimal to switch, which implicitly determines the optimal switching time τ. The value-matching condition (19) simply requires that at the time of the switch (i.e., for a level K_τ of capital) the value before the switch be equal to the value after the switch once paid the cost in terms of capital. The smooth pasting condition (20) ensures the smoothness of the value function around the switch, because it is effectively V before the switch and W after the switch. The other condition the value function has to satisfy because it is always possible for the central planner to indefinitely postpone the adoption of the new technology, is

$$W_0(K_t) \leq V(K_t) \quad \forall t. \tag{21}$$

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The central planner's program is

$$V(K_0) = \sup_{(C,\tau)\in\Theta} E_0 \left\{ \int_0^\tau \frac{\left[C_t^a (B_1 K_t)^{-b}\right]^{1-\gamma}}{1-\gamma} dt + W(K_\tau - \beta) 1_{\{\tau<\infty\}} \right\}$$
$$\text{subject to } dK_t = (AK_t - C_t)\, dt + \sigma K_t dz_t \quad \text{for } 0 \le t < \tau.$$

In the next section, we determine the expression of the marginal value of capital, which has to satisfy equation (20). Integrating this value between zero and the level of the capital stock at the optimal switching time will then allow us to use the value matching condition (19) to get the optimal switching date.

4.1. The Marginal Value of Capital

We show in Appendix B that the marginal value of capital before the switch is[6]

$$V_K(K_t) = \left(\frac{S}{S-1} \frac{a^{\frac{1}{S}} B_1^{\frac{S-R}{S}}}{A - \frac{\sigma^2}{2} R} K_t^{-\frac{R}{S}} + C_2 K_t^{-\frac{A}{\frac{\sigma^2}{2} S}} \right)^S. \tag{22}$$

C_2 is a constant that must be determined using the smooth pasting condition [equation (20)]. Note that this expression corresponds to the marginal value when there is no opportunity to switch [which can be deduced from equation (16)], extended to include a "bubble component" which represents the option value to switch. This component disappears for $C_2 = 0$.

We obtain (see Appendix B)

$$C_2 = \frac{S}{S-1} \frac{a^{\frac{1}{S}}}{A - \frac{\sigma^2}{2} R} \left[B_2^{\frac{S-R}{S}} (K_\tau - \beta)^{-\frac{R}{S}} - B_1^{\frac{S-R}{S}} K_\tau^{-\frac{R}{S}} \right] K_\tau^{\frac{A}{\frac{\sigma^2}{2} S}}. \tag{23}$$

Therefore,

$$V_K(K_t) = \left([W_{0K}(K_t)]^{\frac{1}{S}} + \underbrace{\left\{ [W_K(K_\tau - \beta)]^{\frac{1}{S}} - [W_{0K}(K_\tau)]^{\frac{1}{S}} \right\} \left(\frac{K_\tau}{K_t} \right)^{\frac{2A}{\sigma^2 S}}}_{=D, \text{ part due to the option to switch}} \right)^S. \tag{24}$$

The marginal value of capital differs significantly from that one which can be derived in the absence of technological change from equation (16); this is due to the existence of an option to switch that generates an option value taken into account in the marginal value of capital. In the absence of such an option, $D = 0$ and the marginal value of capital reduces to W_{0K}. This option value is simply the difference between the marginal value after the switch (and having paid the cost β) and the marginal value in the absence of opportunity to switch, discounted by the distance to the switch, which is related to the ratio between the actual capital and K_τ.

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Moreover,

$$C_t = \left[\frac{V_K(K_t)\,(B_1 K_t)^{R-S}}{a}\right]^{-1/S}. \tag{25}$$

Note that this expression is fully standard except that the marginal value of capital includes the option value to switch.

Equations (11) and (16) can be written as follows:

$$\left\{\begin{array}{l} W(K_t) = f(B_2)\frac{K_t^{1-R}}{1-R} \\ W_0(K_t) = f(B_1)\frac{K_t^{1-R}}{1-R} \end{array}\right\} \text{with } f(B_i) = a B_i^{S-R}\left[\frac{S-1}{S}\left(A - \frac{\sigma^2}{2}R\right)\right]^{-S},$$

and the corresponding marginal values are

$$\left\{\begin{array}{l} W_K(K_t) = f(B_2)K_t^{-R}, \\ W_{0K}(K_t) = f(B_1)K_t^{-R}. \end{array}\right.$$

If $\gamma < 1$, we have $S - R < 0$ and $f(B_1) < f(B_2)$; then we have

$$W_{0K}(K_t)^{\frac{1}{S}} < W_K(K_t)^{\frac{1}{S}} < W_K(K_t - \beta)^{\frac{1}{S}},$$

which ensures that $D > 0$. There is no problem of existence of V_K in this case. The marginal value of capital in the economy with the opportunity to adopt a new technology is greater than the marginal value of capital in an economy without this opportunity. This means that consumption at each time is smaller in an economy with an opportunity to switch to a new technology, compared to the consumption that prevails in an economy in which the opportunity does not exist. Indeed, if the intertemporal elasticity of substitution of the representative agent is greater than one, meaning that she likes to substitute in time, she is ready to reduce her current consumption in order to foster technology adoption.

In contrast, if $\gamma > 1$ we have $S - R > 0$, $f(B_1) > f(B_2)$ and $W_{0K}(K_t) > W_K(K_t)$; the sign of D is a priori ambiguous.

$D < 0$ means that the part due to the option to switch in the expression for the marginal value of capital is negative. Therefore, consumption at each time is greater in an economy with an opportunity to adopt a new technology than the consumption that prevails in an economy without such an opportunity. In this case, the agent does not like to substitute in time and the option to adopt is an incentive to increase her consumption today to smooth the path, which is expected to grow more once the technology is adopted. This of course delays adoption. Nevertheless such a consumption path cannot happen because for $K_t < K_{\text{inf}}$, i.e., for small values of K_t, which certainly need to be considered, we obtain that $V_K^{1/S}$ becomes negative and the program is no longer defined. More precisely, note that

$$K_{\text{inf}} = \left\{\frac{B_1^{\frac{S-R}{S}}}{\left[B_1^{\frac{S-R}{S}}K_\tau^{-\frac{R}{S}} - B_2^{\frac{S-R}{S}}(K_\tau - \beta)^{-\frac{R}{S}}\right](K_\tau)^{\frac{2A}{\sigma^2 S}}}\right\}^{\frac{1}{\frac{R}{S} - \frac{2A}{\sigma^2 S}}}. \tag{26}$$

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$D > 0$ implies that in integrating V_K, condition (21) can no longer be satisfied. Therefore $D > 0$ cannot be considered.

$D = 0$ ensures both that $V_K^{1/S}$ is positive, which means that the program is defined, and that condition (21) is satisfied. It is therefore the sole solution we can consider if $\gamma > 1$. This implies that consumption is unaffected by the existence of the option to switch to the new technology. We denote as K^* the special value of K_τ such that $D = 0$; that is, using (11) and (16),

$$K^* = \frac{\beta}{1 - \left(\frac{B_1}{B_2}\right)^{\frac{R-S}{R}}}. \tag{27}$$

K^* must be positive, which requires, given that $B_1 > B_2$, $(R - S)/R \leq 0$, i.e., $\gamma \geq 1$. Obviously, this threshold increases with the cost to switch and decreases with the distance between the two technologies. The better the new technology compared to the old one, the smaller the threshold.

4.2. Boundary Conditions

The stock of capital K_τ is such that, at the time of the switch, the value with the initial abatement technology is equal to the value with the new technology once the cost β is paid:

$$V(K_\tau) = W(K_\tau - \beta). \tag{28}$$

Again, two cases can occur.

If $\gamma < 1$, K_τ has to satisfy

$$\int_0^{K_\tau} V_K(K)\, dK = W(K_\tau - \beta) \tag{29}$$

because $V(0) = 0$ for $\gamma < 1$. Note that V_K is itself a function of K_τ, through C_2 (see equation (23)). Equation (29) can be solved numerically. Simulations are driven using the following values for the parameters: $A = 0.1$; $B_1 = 0.5$; $B_2 = 0.4$; $\beta = 0.1$; $\gamma = 0.5$; $a = 0.8$; $b = 0.5$; $\sigma^2/2 = 0.2$. The values of the effective coefficient of risk aversion and of the inverse of the effective intertemporal elasticity of substitution are then $R = 0.85$ and $S = 0.6$. Figure 1 shows the three value functions: before the switch $[V(K)]$, after the switch $[W(K - \beta)]$, and without the option to switch $[W_0(K)]$. The threshold that triggers the switch is $K_\tau = 0.84$ (remember that the cost of the switch in terms of capital is $\beta = 0.1$).

If $\gamma > 1$, K_τ has to be such that $D = 0$, that is $K_\tau = K^*$, given by equation (27), and the boundary condition (28) allows deriving the value of the program before the switch:

$$V(K_t) = \left[\frac{S-1}{S}\left(A - \frac{\sigma^2}{2}R\right)\right]^{-S} \times \frac{a}{1-R}\left[K_t^{1-R}B_1^{S-R} + (K_\tau - \beta)^{1-R}B_2^{S-R} - K_\tau^{1-R}B_1^{S-R}\right]. \tag{30}$$

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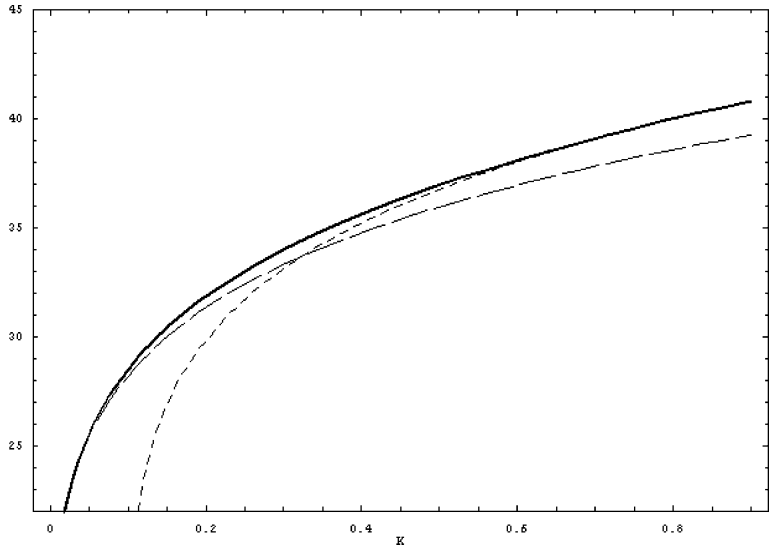


FIGURE 1. The value functions for $\gamma < 1$ (optimum). Thick line: $V(K)$, short dashes: $W(K-\beta)$, long dashes: $W_0(K)$.

Note that both K^* and $V(K_t)$ are defined if $\gamma > 1$. Simulations to draw the value functions are driven using the following values for the parameters: $A = 1$; $B_1 = 0.5$; $B_2 = 0.4$; $\beta = 0.1$; $\gamma = 2$; $a = 0.8$; $b = 0.7$; $\sigma^2/2 = 0.2$. The values of the effective coefficient of risk aversion and of the inverse of the effective intertemporal elasticity of substitution are then $R = 1.1$ and $S = 1.8$. Figure 2 shows the three value functions: before the switch $[V(K)]$, after the switch $[W(K-\beta)]$, and without the option to switch $[W_0(K)]$. The threshold that triggers the switch is $K^* = 0.76$ [see equation (27)].

4.3. Comparative Statics

For $\gamma < 1$, the level of capital that triggers adoption can only be derived numerically. Simulations are driven using the same values for the parameters as previously. We provide some figures to show the effects of some of the model parameters. The level of capital for which it is optimal to switch is a decreasing function of uncertainty (Figure 3). This means that the larger the uncertainty, the smaller the stock of capital for which adoption occurs. Note that it differs strikingly from the results obtained in partial equilibrium. Usually, in partial equilibrium, uncertainty implies the existence of an option to adopt that generates a value to wait before the adoption.

To see why things are so different in general equilibrium, note that uncertainty now has two effects.

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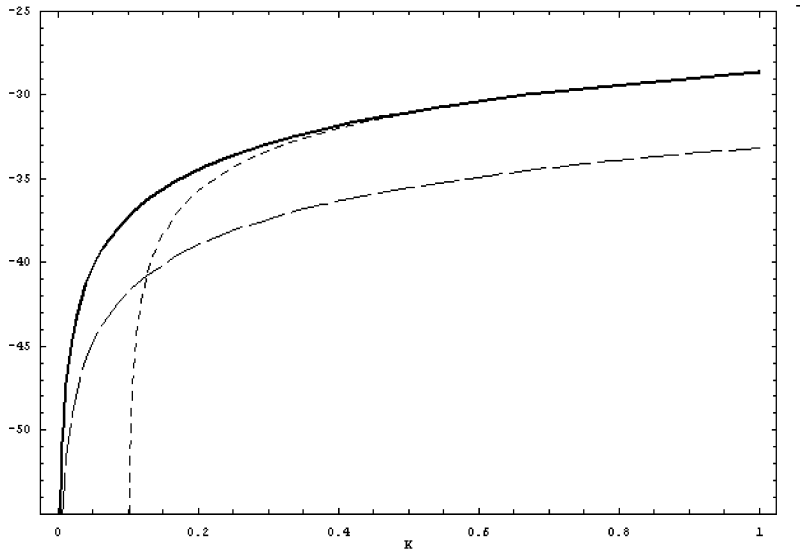


FIGURE 2. The value functions for $\gamma > 1$ (optimum). Thick line: $V(K)$, short dashes: $W(K - \beta)$, long dashes: $W_0(K)$.

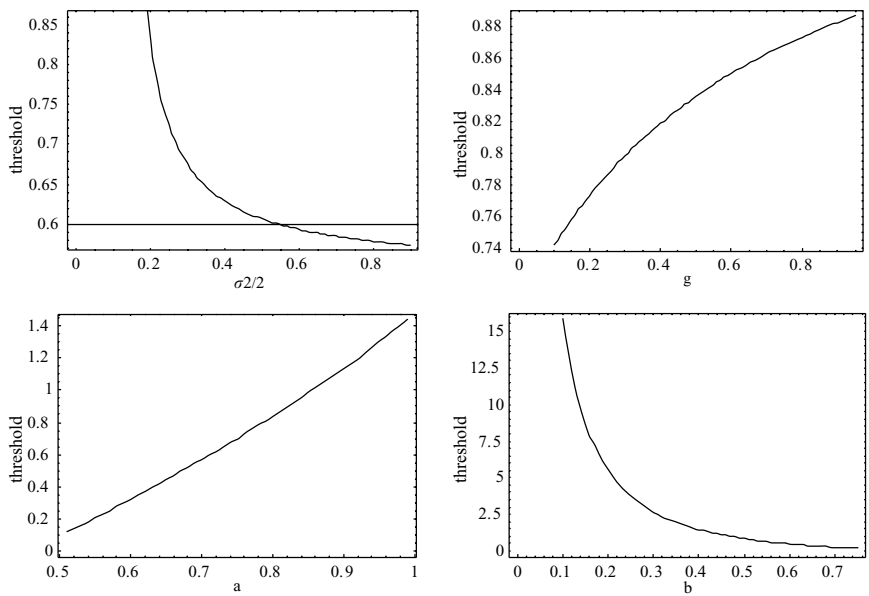


FIGURE 3. Comparative statics for $\gamma < 1$ (optimum).

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Using equations (11) and (16) and the condition (14) on parameters in the case $\gamma < 1$, we see that

$$\frac{\partial W(K_t)}{\partial \sigma^2/2} < \frac{\partial W_0(K_t)}{\partial \sigma^2/2} < 0.$$

Uncertainty reduces the value after the switch and the "nonoption part" of the value before the switch as well. Moreover, in the absence of an option to switch to the abatement technology, adoption would occur for a larger stock of capital, because uncertainty reduces value more after the switch than before it.

Effect of uncertainty on D, the part of the marginal value of capital due to the option to switch:

$$\frac{\partial D}{\partial \sigma^2/2} = \underbrace{\frac{\partial \left\{ [W_K(K_\tau - \beta)]^{\frac{1}{S}} - [W_{0K}(K_\tau)]^{\frac{1}{S}} \right\}}{\partial \sigma^2/2} \left(\frac{K_\tau}{K_t} \right)^{\frac{2A}{\sigma^2 S}}}_{<0}$$

$$+ \underbrace{\left\{ [W_K(K_\tau - \beta)]^{\frac{1}{S}} - [W_{0K}(K_\tau)]^{\frac{1}{S}} \right\} \frac{\partial \left(\frac{K_\tau}{K_t} \right)^{\frac{2A}{\sigma^2 S}}}{\partial \sigma^2/2}}_{<0} < 0$$

since

$$\frac{\partial W_K(K_\tau - \beta)}{\partial \sigma^2/2} < 0, \frac{\partial W_{0K}(K_\tau)}{\partial \sigma^2/2} < 0,$$

$$\frac{\partial W_K(K_\tau - \beta)/\partial \sigma^2/2}{\partial W_{0K}(K_\tau)/\partial \sigma^2/2} = \frac{W_K(K_\tau - \beta)}{W_{0K}(K_\tau)} > 1, \text{ and } K_t < K_\tau.$$

Uncertainty reduces the option part of the marginal value of the capital before the switch. Even if it is analytically not possible to integrate to get the value before the switch, there is no doubt that the reduction of the option part of the marginal value before the switch will contribute to the reduction of the value before the switch.

Therefore, it may well be the case that uncertainty reduces the value before the switch more than the value after it (because of the increase in the option value part), therefore reducing the level of capital that triggers the switch.

In partial equilibrium, uncertainty increases the option value and therefore the value before adoption as well. This induces a higher realization of the uncertain variable to adopt, which means that adoption is delayed. Note that in general equilibrium, uncertainty affects the adoption timing through both the optimal threshold in terms of capital and the optimal capital accumulation path, but the realization of the uncertain variable plays no role. In fact, uncertainty, by decreasing the marginal value of capital before the switch, increases current consumption and therefore reduces the economy's growth rate. Even if the threshold that triggers adoption is smaller, it may be reached later.

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The effect of the parameter γ is not straightforward, because it affects the effective relative risk aversion and the effective intertemporal elasticity of substitution. Indeed, both R and S increase with γ. The simulations show that the optimal switching time increases with γ (Figure 3). Nevertheless, this result is not robust: for the same set of parameters but $B_2 = 0.2$, the optimal switching time decreases with γ.

Effects of parameters a and b can be seen in Figure 3 as well. Recall that if b increases, this increases the effective risk aversion R but has no effect on S. In contrast, an increase in a reduces both R and S. We observe that the higher the sensitivity of agents with respect to consumption, the later the adoption. The higher the sensitivity of agents with respect to pollution, the earlier the adoption. Recall that in our model the "greenness" of preferences is given by the ratio b/a. For an effective intertemporal elasticity of substitution $1/S$ equal to 1.6667, a greenness equal to 0.25 leads to adoption for $K_\tau = 5.53$, whereas a greenness of 0.5 provides $K_\tau = 1.46$ and a greenness of 0.75 leads to $K_\tau = 0.47$.

For $\gamma > 1$, equation (27) gives

$$K^* = \frac{\beta}{1 - (B_1/B_2)^{\frac{R-S}{R}}},$$

and straightforward computations lead to

$$\frac{\partial K^*}{\partial \sigma^2} = 0, \quad \frac{\partial K^*}{\partial \gamma} < 0, \quad \frac{\partial K^*}{\partial a} > 0, \quad \frac{\partial K^*}{\partial b} < 0.$$

Note that uncertainty does not affect the adoption timing because the consumption path before adoption does not take into account the option to switch. For a given effective intertemporal elasticity of substitution, an increase in the greenness of preferences speeds up adoption, as in the case $\gamma < 1$. Finally, an increase in the intertemporal elasticity of substitution γ always speeds up adoption.

5. THE COMPETITIVE ECONOMY

We now turn to the decentralized economy. In a decentralized setup, the abatement technology will never be adopted in the absence of regulatory incentives to do so, because pollution only affects consumers. We therefore suppose that emissions of pollution are taxed[7] and search for the tax rate that incites the private agents to adopt the abatement technology at the optimal level of the stock of capital.

5.1. Firms

The flow of output of a firm i over the interval $[t, t + dt]$ is specified by a Cobb-Douglas production function,

$$dY_{i,t} = A\overline{K}_t^{\alpha} K_{i,t}^{1-\alpha}\left(dt + \frac{\sigma_i}{A} dz_t\right), \quad 0 < \alpha < 1, \tag{31}$$

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where K_i denotes the firm's capital stock and $\overline{K}$ the average economywide capital stock. The stochastic variable is temporally independent and normally distributed, with zero mean and variance $\sigma_i^2 dt$ over the instant dt. The stochastic shocks are perfectly correlated across firms. The production function represents a learning-by-doing effect, as in Romer (1986): the average capital stock yields an externality enhancing the marginal productivity of firm i's specific capital stock. We shall focus on a symmetric equilibrum in which $K_{i,t} = \overline{K}_t$. In symmetric equilibrium, the aggregate production function will be linear in the capital stock:

$$dY_t = AK_t dt + \sigma K_t dz_t. \tag{32}$$

The private return to capital is assumed to be generated by the stochastic process

$$dR_t = r_t dt + du_t.$$

The firm pays a tax at a constant rate θ on its pollution emissions. The superscript $j = 1, 2$ indicates the abatement technology. Pollution emissions of firm i using the abatement technology j are then $F_{i,t}^j = B_j K_{i,t}^j$, and the tax paid by firm i is $\theta F_{i,t}^j$. The tax revenue is redistributed to households as a lump-sum transfer. The private user cost of capital is equal to the sum of the private rate of return and the tax paid. Also, we assume that the deterministic component of the private user cost of capital is equal to its expected marginal productivity. We then have

$$r_t^j + \theta B_j = \frac{\partial \left[A \overline{K}_t^{j\,\alpha} \quad K_{i,t}^{j\,1-\alpha} \right]}{\partial K_{i,t}^j} = (1-\alpha) A \, \overline{K}_t^{j\,\alpha} K_{i,t}^{j\,-\alpha}. \tag{33}$$

Firm i with abatement technology j chooses the optimal stock of capital

$$K_{i,t}^j = \left[\frac{(1-\alpha)A}{r_t^j + \theta B_j} \right]^{\frac{1}{\alpha}} \overline{K}_t^j, \tag{34}$$

from which we deduce its expected production and instantaneous profit:

$$E_t\left[dY_{i,t}^j\right] = A \left[\frac{(1-\alpha)A}{r_t^j + \theta B_j} \right]^{\frac{1-\alpha}{\alpha}} \overline{K}_t^j dt, \tag{35}$$

$$\pi_t^j = \alpha A \left[\frac{(1-\alpha)A}{r_{j,t} + \theta B_j} \right]^{\frac{1-\alpha}{\alpha}} \overline{K}_t^j. \tag{36}$$

The firm chooses to adopt the new abatement technology at the date s at which the change in its expected discounted sum of profits becomes positive, the firm taking the rate of return r_t^j and the path of aggregate capital $\overline{K}_t^j$ as given. We use the fact that in equilibrium

$$r_t^j + \theta B_j = (1-\alpha)A. \tag{37}$$

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The deterministic part of the private rate of return is constant at equilibrium. It must be positive, which requires $\theta \leq (1-\alpha)A/B_1$ (because $B_1 > B_2$). The adoption date s is then defined by

$$\int_s^\infty e^{-r^2(t-s)}\pi_t^2\,dt - \int_s^\infty e^{-r^1(t-s)}\pi_t^1\,dt = 0, \quad \text{with } \overline{K}_s^2 = \overline{K}_s^1 - \beta.$$

We denote by g^j the constant expected growth rate of aggregate capital when abatement technology j is used. The preceeding equation then reduces to

$$\left(\overline{K}_s^1 - \beta\right)\int_s^\infty e^{(g^2-r^2)(t-s)}dt = \overline{K}_s^1 \int_s^\infty e^{(g^1-r^1)(t-s)}\,dt,$$

and we obtain

$$\overline{K}_s^1 = \frac{\beta}{1 - \dfrac{r^2-g^2}{r^1-g^1}}, \tag{38}$$

under the condition of convergence $r^j - g^j > 0$.

Note that the date s is easily deduced from the value of $\overline{K}_s^1$: we have $\overline{K}_s^1 = \overline{K}_0 e^{g^1 s}$ because technology 1 is used before date s.

It remains to find the equilibrium expected growth rate of aggregate capital.

5.2. Households

The representative household seeks to maximize its expected intertemporal utility:

$$\max_{C_t} E_0\left[\int_0^\infty \frac{\left[C_t^a F_t^{-b}\right]^{1-\gamma}}{1-\gamma}dt\right].$$

F, the flow of pollution, is an externality.

Its instantaneous budget constraint is

$$dX_t = (r_t X_t + T_t - C_t)dt + \sigma_X X_t dz_t, \tag{39}$$

where X is the agent's wealth, r the deterministic part of the return of assets and $\sigma_X dz$ its stochastic part, and T a lump-sum transfer, the counterpart of the tax on emissions paid by the firms. T is a state variable evolving stochastically, proportional to the aggregate capital stock. The household's program then exhibits two state variables, X, which is under direct control of the household, and T, which is not. As in Turnovsky and Smith (2006), the solution takes advantage of the fact that at equilibrium T and X evolve proportionately, and so reduce to a single state variable. Of course, both X and T depend at each date of the technology used at this date. We make the assumption that the households do not anticipate the switch to the new technology.

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The solution of the household's program is relegated to Appendix C. We prove that the optimal equilibrium consumption path is

$$\widetilde{C}_t = \frac{S-1}{S}\left[(1-\alpha)A + \frac{1}{S-1}\theta B - \frac{\sigma_X^2}{2}S\right]X_t. \tag{40}$$

The marginal propensity to consume has to be strictly positive.

5.3. Equilibrium

In equilibrium we have

$$\sigma_X = \sigma, \tag{41}$$

$$K_t = X_t, \tag{42}$$

$$g^j = \frac{E_t(dK_t/K_t)}{dt} = r^j + \frac{T_t}{X_t} - \frac{\widetilde{C}_t}{K_t},$$

$$= \frac{1}{S}\left[(1-\alpha)A - \theta B_j + (S-1)\frac{\sigma^2}{2}S\right]. \tag{43}$$

The transversality condition (which is equivalent to the feasibility condition since pollution is now an externality for the consumers) becomes

$$(S-1)\left[(1-\alpha)A + \frac{1}{S-1}\theta B_j - \frac{\sigma^2}{2}S\right] > 0. \tag{44}$$

Moreover, $\overline{K}_s^1$ must be positive, and we see easily, using equation (38), that this requires $S < 1$ because $B_1 > B_2$. Then the transversality condition reduces to

$$\theta > 1/B_2\left[(1-\alpha)A - \frac{\sigma^2}{2}S\right].$$

To sum up, the restrictions on the tax rate are the following:

$$\frac{(1-\alpha)A - \frac{\sigma^2}{2}S}{B_2} < \theta \leq \frac{(1-\alpha)A}{B_1} \quad \text{with } S < 1. \tag{45}$$

We have shown that a necessary condition for the existence of a finite positive date of adoption is $S < 1$ (i.e., $\gamma < 1$). So when $S > 1$ (i.e., $\gamma > 1$), adoption of the new abatement technology never occurs, even with a tax on emissions. An explanation would be the following. As the switch occurs, the deterministic part of the rate of return increases, because the emission coefficient decreases from B_1 to B_2 and the user cost of capital remains constant. It positively affects the expected consumption rate $E(\widetilde{C}/K)$ only if the intertemporal elasticity of substitution is larger than unity ($S < 1$); see equation (40). Therefore, if $S > 1$, switching

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generates a fall in the firm's gains, which prevails over the fall in its costs coming from the fall in θB.

5.4. Comparative Statics for a Given Tax Rate

Using equations (37) and (43), which define r^j and g^j as functions of the parameters and the tax rate, equation (38) allows us to analyze the sensitivity of the adoption timing with respect to the parameters for a given tax rate, in the case $S < 1$ (i.e., $\gamma < 1$). We obtain that the threshold that triggers adoption is increasing with the size of uncertainty:

$$\frac{\partial \overline{K}_s^1}{\partial \dfrac{\sigma^2}{2}} = \frac{\beta(1-S)}{(r^1 - g^1) - (r^2 - g^2)} > 0.$$

Besides, regarding the sensitivity of the threshold with respect to γ, we have

$$\frac{\partial \overline{K}_s^1}{\partial \gamma} = \frac{\beta a \left(\dfrac{S-1}{S}\right)^2 \dfrac{\sigma^2}{2} \theta \, (B_1 - B_2)}{[(r^1 - g^1) - (r^2 - g^2)]^2} > 0.$$

Note, moreover, that the sign of the sensitivity of the threshold with respect to parameter a is the opposite of the sign of its sensitivity with respect to γ. Thus the threshold that triggers adoption increases with the inverse of the intertemporal elasticity of substitution and decreases with the consumers' sensitivity with respect to consumption a. Finally, the threshold is not affected at all by the consumers' sensitivity with respect to pollution b.

5.5. The Optimal Tax Rate

The centralized and decentralized programs having been solved, it is then possible to derive a second-best tax level. Such a tax is aimed at providing an incentive for firms to adopt the new technology in an optimal way, with respect to the development level of the economy. Nevertheless, this tax does not correct for all market failures, i.e., for both the pollution externality and the learning-by-doing externality, by ensuring that the path of the competitive economy replicates that of the centralized economy. It simply ensures that the decentralized economy switches to the abatement technology for a level of capital that corresponds to the one chosen by the central planner. In what follows, we refer to such a tax as being the "optimal tax rate," even if it only allows reaching a second-best economy.

To appraise the sensitivity of this tax with respect to parameters' values, we have to rely on numerical simulations again. Baseline parameters are the same as previously. Results for the optimal tax level are provided in Table 1. The optimal tax is an increasing function of the size of uncertainty: the more uncertain the environment, the higher the tax rate. Table 1 shows that for the range of parameters for which a numerical solution exists, the optimal tax increases with the relative

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TABLE 1. Sensitivity of the optimal tax rate

$\sigma^2/2$	0.16	0.18	0.20	0.22	0.24
θ^*	1%	4.2%	9%	15%	n.s.
γ	0.3	0.4	0.5	0.6	0.7
θ^*	n.s.	4.4%	9%	13.1%	n.s.
a	0.7	0.75	0.8	0.9	1
θ^*	n.s.	17.5%	9%	3.2%	1%
b/a	0.25	0.5	0.625	0.8	
θ^*	0.6%	3.1%.	9%	n.s.	

Note: n.s. stands for "no solution."

risk aversion coefficient γ and decreases with parameter a. Finally, the higher the greenness of preferences, the higher the optimal tax rate.

6. CONCLUSION

In this paper we tackle the specific problem of abatement technology adoption under uncertainty in general equilibrium. Amenity value (which is negative) alters both risk aversion and the intertemporal elasticity of substitution; therefore, following Smith and Son (2005), we define an effective risk aversion as well as an effective intertemporal elasticity of substitution. We show that the higher the greenness of preferences, the earlier the adoption and the higher the optimal tax rate that induces agents to adopt the abatement technology at the socially optimal level of development. We also obtain that adoption is fostered by uncertainty if the effective intertemporal elasticity of substitution is large enough, but is not affected by uncertainty if this elasticity is low. Moreover, the optimal tax rate, which only exists if the effective intertemporal elasticity of substitution is high, is an increasing function of uncertainty.

NOTES

1. Learning by doing, as in Boucekkine et al. (2004), is also a way to obtain an optimal adoption timing.

2. This paper, together with Hugonnier et al. (2005), provides the resolution for the optimal threshold that triggers an irreversible decision in a general equilibrium framework for the first time in the literature.

3. Note that we only consider irreversibility on the adoption side. Irreversibility on the pollution side could also be introduced, and it would presumably induce uncertainty to foster adoption. This is done by Pindyck (2000) in a partial equilibrium framework, but, due to the specifications, the pollution irreversibility plays in fact no role.

4. See Turnovsky (2000) for other applications using this equation.

5. More precisely, with a zero discount rate, it is possible to solve analytically the model for conditions on the parameters that are not too strong under uncertainty [see below in the resolution or in Kocherlakota (1990)], except in the case in which the utility function is logarithmic (for $\gamma \to 1$).

6. A formal proof of existence of the solution can be found in Hugonnier et al. (2008).

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7. Other instruments could be considered; we study here the case of the simplest one in terms of modeling.

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APPENDIX A: SOLVING THE OPTIMAL PROGRAM AFTER THE SWITCH

A.1. SOLVING THE PROGRAM

The Bellman equation is written

$$W(K_t) = \max_{C_t} \left\{ \frac{\left[C_t^a (B_2 K_t)^{-b}\right]^{1-\gamma}}{1-\gamma} + E_t[W(K_{t+dt})] \right\}.$$

Using Itô's lemma, the Bellman equation corresponding to the problem after the switch is

$$\max_{C_t} \left\{ \frac{\left[C_t^a (B_2 K_t)^{-b}\right]^{1-\gamma}}{1-\gamma} + W_K(AK_t - C_t) + \frac{\sigma^2}{2} K_t^2 W_{KK} \right\} = 0.$$

The first-order condition yields

$$W_K = a {C_t}^{a(1-\gamma)-1} (B_2 K_t)^{-b(1-\gamma)} \Leftrightarrow C_t = \left[\frac{W_K (B_2 K_t)^{b(1-\gamma)}}{a} \right]^{\frac{1}{a(1-\gamma)-1}}.$$

Replacing consumption by its optimal expression in the Bellman equation gives

$$\frac{1}{1-\gamma} \left[\frac{W_K (B_2 K_t)^{b(1-\gamma)}}{a} \right]^{\frac{a(1-\gamma)}{a(1-\gamma)-1}} (B_2 K_t)^{-b(1-\gamma)}$$
$$+ W_K \left\{ AK_t - \left[\frac{W_K (B_2 K_t)^{b(1-\gamma)}}{a} \right]^{\frac{1}{a(1-\gamma)-1}} \right\} + \frac{\sigma^2}{2} K_t^2 W_{KK} = 0.$$

We postulate

$$W(K_t) = M \frac{K_t^{(a-b)(1-\gamma)}}{1-\gamma},$$

with M an unknown constant, which we calculate now.

With this expression of the value function we have

$$W_K = M(a-b) K_t^{(a-b)(1-\gamma)-1}$$

and

$$W_{KK} = M(a-b)\left[(a-b)(1-\gamma)-1\right] K_t^{(a-b)(1-\gamma)-2}.$$

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We replace in the Bellman equation and get

$$\frac{1-a(1-\gamma)}{a(1-\gamma)}\left(M\frac{a-b}{a}{B_2}^{b(1-\gamma)}\right)^{\frac{1}{a(1-\gamma)-1}}+A+\frac{\sigma^2}{2}\left[(a-b)(1-\gamma)-1\right]=0,$$

which allows us to obtain M :

$$M=\frac{a}{a-b}{B_2}^{-b(1-\gamma)}\left(\frac{a(1-\gamma)}{a(1-\gamma)-1}\left\{A+\frac{\sigma^2}{2}\left[(a-b)(1-\gamma)-1\right]\right\}\right)^{a(1-\gamma)-1}.$$

Then the optimal consumption path is given by

$$\begin{aligned}C_t^* &= \left[\frac{M(a-b){K_t}^{(a-b)(1-\gamma)-1}(B_2K_t)^{b(1-\gamma)}}{a}\right]^{\frac{1}{a(1-\gamma)-1}}\\ &= \frac{a(1-\gamma)}{a(1-\gamma)-1}\left\{A+\frac{\sigma^2}{2}\left[(a-b)(1-\gamma)-1\right]\right\}K_t,\end{aligned}$$

which is equation (9) in the text.

A.2. THE TRANSVERSALITY CONDITION

The transversality condition requires the convergence of the value function, i.e.,

$$\lim_{t\to\infty}E_0[W(K_t)]=0.$$

It is satisfied if the lifetime utility of capital does not grow "too fast" in expectation [see Smith (1996)]. Applying Itô's lemma to $W(K_t)$, this requires that

$$\begin{aligned}E(dW) &= W_K E(dK)+\tfrac{1}{2}W_{KK}E(dK^2)<0\\ &\Leftrightarrow (R-1)\left(A-R\frac{\sigma^2}{2}\right)>0,\end{aligned}$$

which is condition (13) in the text.

This condition can be obtained directly, by computing $W(K_t)$. Indeed, it can be written, along the optimal path, with R as defined in equation (7),

$$\begin{aligned}W(K_t) &= \frac{M}{1-\gamma}E_t\left[\int_t^\infty K_s^{1-R}ds\right]\\ &= \frac{\left[m^a B_2^{-b}\right]^{1-\gamma}}{1-\gamma}\int_t^\infty E_t\left(K_s^{1-R}\right)ds,\end{aligned}$$

where

$$m=\frac{a(1-\gamma)}{a(1-\gamma)-1}\left\{A+\frac{\sigma^2}{2}\left[(a-b)(1-\gamma)-1\right]\right\}$$

is the marginal propensity to consume. We must then compute $E_t(K_s^{1-R})$.

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We can write the differential stochastic equation (2)

$$\frac{dK_t}{K_t} = (A-m)\,dt + \sigma dz_t,$$

which states that K follows a geometric Brownian motion. We then have

$$\frac{dK_t}{K_t} = d\ln K_t = \ln K_t - \ln K_\tau \sim N\left[\left(A-m-\frac{\sigma^2}{2}\right)(t-\tau), \sigma^2(t-\tau)\right]$$

$$\Leftrightarrow \ln K_t \sim N\left[\left(A-m-\frac{\sigma^2}{2}\right)(t-\tau) + \ln K_\tau, \sigma^2(t-\tau)\right].$$

The density function of K_t is given by

$$f_K(k) = \frac{1}{\sqrt{2\pi\sigma^2(t-\tau)}}\exp\left\{-\frac{\left[\ln k - \left(A-m-\frac{\sigma^2}{2}\right)(t-\tau) - \ln K_\tau\right]^2}{2\sigma^2(t-\tau)}\right\}, \quad k>0.$$

We have

$$E(K_t^{1-R}) = E\left\{\exp\left[(1-R)\ln K_t\right]\right\}, = \int_{-\infty}^{+\infty}\exp\left[(1-R)\ln k\right]$$

$$\times \frac{1}{\sqrt{2\pi\sigma^2(t-\tau)}}\exp\left\{-\frac{\left[\ln k - \left(A-m-\frac{\sigma^2}{2}\right)(t-\tau) - \ln K_\tau\right]^2}{2\sigma^2(t-\tau)}\right\}dk$$

$$= \exp\left\{(1-R)\left[\left(A-m-\frac{\sigma^2}{2}\right)(t-\tau) + \ln K_\tau\right] + \frac{1}{2}(1-R)^2\sigma^2(t-\tau)\right\}$$

$$\times \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi\sigma^2(t-\tau)}}\exp\left\{-\frac{1}{2\sigma^2(t-\tau)}\left[\ln k - \left(A-m-\frac{\sigma^2}{2}\right)(t-\tau) - \ln K_\tau\right]^2\right.$$

$$\left.-2(1-R)\left[\ln k - \left(A-m-\frac{\sigma^2}{2}\right)(t-\tau) - \ln K_\tau\right]\sigma^2(t-\tau) + (1-R)^2\sigma^4(t-\tau)^2\right\}dk$$

$$= \exp\left\{(1-R)\left[\left(A-m-\frac{\sigma^2}{2}\right)(t-\tau) + \ln K_\tau\right] + \frac{1}{2}(1-R)^2\sigma^2(t-\tau)\right\}$$

$$\times \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi\sigma^2 t}}\exp\left\{-\frac{1}{2\sigma^2(t-\tau)}\left[\ln k - \left(A-m-\frac{\sigma^2}{2}\right)\right.\right.$$

$$\left.\left.\times(t-\tau) - \ln K_\tau - (1-R)\sigma^2(t-\tau)\right]^2\right\}dk.$$

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Using the fact that

$$\frac{1}{\sqrt{2\pi\sigma^2 t}}\exp\left\{-\frac{1}{2\sigma^2(t-\tau)}\left[\ln k-\left(A-m-\frac{\sigma^2}{2}\right)(t-\tau)\right.\right.$$
$$\left.\left.-\ln K_\tau-(1-R)\sigma^2(t-\tau)\right]^2\right\}$$

is the density function of a random variable distributed according to

$$N\left[\left(A-m-\frac{\sigma^2}{2}\right)(t-\tau)+\ln K_\tau+(1-R)\sigma^2(t-\tau),\sigma^2(t-\tau)\right]$$

(hence its integral between $-\infty$ and $+\infty$ is equal to 1), we finally obtain

$$E\left(K_t^{1-R}\right)=\exp\left\{(1-R)\left[\left(A-m-\frac{\sigma^2}{2}\right)(t-\tau)+\ln K_\tau\right]+\frac{1}{2}(1-R)^2\sigma^2(t-\tau)\right\}$$
$$=\exp\left\{(1-R)\left[\left(A-m-R\frac{\sigma^2}{2}\right)(t-\tau)+\ln K_\tau\right]\right\}.$$

Then the value function after τ can be written, along the optimal path,

$$W(K_t)=\frac{\left[m^a B_2^{-b}\right]^{1-\gamma}}{1-\gamma}K_t^{1-R}\int_t^\infty e^{(1-R)\left(A-m-\frac{\sigma^2}{2}R\right)(s-t)}ds$$
$$=\frac{\left[m^a B_2^{-b}\right]^{1-\gamma}}{1-\gamma}K_t^{1-R}\int_t^\infty e^{\frac{1-R}{S}\left(A-\frac{\sigma^2}{2}R\right)(s-t)}ds.$$

The integral converges if and only if the condition

$$(1-R)\left(A-\frac{\sigma^2}{2}R\right)<0$$

is satisfied, which is equation (13). Then we have

$$W(K_t)=-\frac{\left[m^a B_2^{-b}\right]^{1-\gamma}}{(1-\gamma)\frac{1-R}{S}\left(A-\frac{\sigma^2}{2}R\right)}K_t^{1-R}$$
$$=am^{a(1-\gamma)-1}B_2^{-b(1-\gamma)}\frac{K_t^{1-R}}{1-R}$$
$$=am^{-S}B_2^{S-R}\frac{K_t^{1-R}}{1-R},$$

which is equation (11) in the text.

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APPENDIX B: SOLVING THE OPTIMAL PROGRAM BEFORE THE SWITCH

Using Itô's lemma, the Bellman equation that the value function before technology adoption $V(K_t)_{\{t<\tau\}}$ has to satisfy is

$$0 = \max_{C_t} \left\{ \frac{\left[C_t^a (B_1 K_t)^{-b}\right]^{1-\gamma}}{1-\gamma} + V_K (AK_t - C_t) + \frac{\sigma^2}{2} K_t^2 V_{KK} \right\}.$$

The first-order condition allows us to obtain the optimal consumption,

$$C_t = \left[\frac{V_K (B_1 K_t)^{b(1-\gamma)}}{a} \right]^{\frac{1}{a(1-\gamma)-1}}.$$

Replacing consumption by its optimal expression in the Bellman equation gives

$$\frac{1-a(1-\gamma)}{1-\gamma} \left(\frac{V_K}{a} \right)^{\frac{a(1-\gamma)}{a(1-\gamma)-1}} (B_1 K_t)^{\frac{b(1-\gamma)}{a(1-\gamma)-1}} + AK_t V_K + \frac{\sigma^2}{2} K_t^2 V_{KK} = 0,$$

or, using the effective coefficient of risk aversion and intertemporal elasticity of substitution,

$$\frac{aS}{1-S} \left(\frac{V_K}{a} \right)^{\frac{S-1}{S}} (B_1 K_t)^{\frac{S-R}{S}} + AK_t V_K + \frac{\sigma^2}{2} K_t^2 V_{KK} = 0.$$

We multiply this equation by $V_K^{(1-S)/S} K_t^{(R-S)/S}$ and obtain

$$\frac{aS}{1-S} a^{\frac{1-S}{S}} B_1^{\frac{S-R}{S}} + A V_K^{\frac{1}{S}} K_t^{\frac{R}{S}} + \frac{\sigma^2}{2} V_{KK} V_K^{\frac{1-S}{S}} K_t^{\frac{R+S}{S}} = 0.$$

We make the following change of variables: $f(K_t) = V_K^{1/S} K_t^{R-S/S}$. We then have

$$f'(K_t) = \frac{1}{S} V_K^{\frac{1-S}{S}} K_t^{\frac{R-S}{S}} V_{KK} + \frac{R-S}{S} V_K^{\frac{1}{S}} K_t^{\frac{R-S}{S}-1},$$

and the preceeding equation can be written

$$\frac{aS}{1-S} a^{\frac{1-S}{S}} B_1^{\frac{S-R}{S}} + \left[A - \frac{\sigma^2}{2}(R-S) \right] K_t f(K_t) + \frac{\sigma^2}{2} S K_t^2 f'(K_t) = 0.$$

We postulate

$$f(K_t) = \frac{C_1}{K_t} + C_2 K_t^{C_3},$$

where, C_1, C_2, and C_3 are three constants to be determined. Then

$$f'(K_t) = -\frac{C_1}{K_t^2} + C_2 C_3 K_t^{C_3 - 1},$$

and we obtain

$$\frac{aS}{1-S} a^{\frac{1-S}{S}} B_1^{\frac{S-R}{S}} + \left(A - \frac{\sigma^2}{2} R \right) C_1 + \left[A - \frac{\sigma^2}{2}(R-S) + C_3 \frac{\sigma^2}{2} S \right] C_2 K_t^{C_3+1} = 0,$$

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which must be verified $\forall K_t > 0$. Then

$$C_3 = \frac{R-S}{S} - \frac{A}{\frac{\sigma^2}{2}S},$$

and

$$C_1 = \frac{S}{S-1}\frac{a^{\frac{1}{S}}B_1^{\frac{S-R}{S}}}{A-\frac{\sigma^2}{2}R}.$$

This leads to the expression for the marginal value of capital (equation (22) of the text):

$$V_K = \left(\frac{S}{S-1}\frac{a^{\frac{1}{S}}B_1^{\frac{S-R}{S}}}{A-\frac{\sigma^2}{2}R}K_t^{-\frac{R}{S}} + C_2 K_t^{-\frac{A}{\frac{\sigma^2}{2}S}}\right)^S.$$

C_2 is a constant that must be determined by using the smooth pasting condition

$$V_K(K_\tau) = W_K(K_\tau - \beta).$$

Using equations (22) and (11), we easily obtain equation (23) of the text.

APPENDIX C: SOLVING THE HOUSEHOLD'S PROGRAM

The Bellman equation is written

$$W^c(X_t, T_t) = \max_{C_t}\left\{\frac{\left[C_t^a F_t^{-b}\right]^{1-\gamma}}{1-\gamma}dt + E_t[W^c(X_{t+dt}, T_{t+dt})]\right\},$$

$$\text{with } dX_t = (r_t X_t + T_t - C_t)dt + \sigma_X X_t dz_t.$$

This program exhibits two state variables: X_t and T_t. We assume that

$$dT_t = \mu_t dt + \sigma_T dz_t.$$

We denote the covariance of the growth rate of wealth with the growth rate of the lump-sum transfer by $E_t[(dX/X)(dT/T)] = \sigma_{XT}dt$.

Using Itô's lemma we can write

$$E_t[W^c(X_{t+dt}, T_{t+dt})] = W^c(X_t, T_t) + W^c_X E_t(dX) + W^c_T E_t(dT) + \tfrac{1}{2}W^c_{XX}E_t(dX^2) + \tfrac{1}{2}W^c_{TT}E_t(dT^2) + W^c_{XT}E_t(dXdT).$$

Replacing in the Bellman equation leads to

$$\max_{C_t}\left\{\frac{\left[C_t^a F_t^{-b}\right]^{1-\gamma}}{1-\gamma} + W^c_X(r_t X_t + T_t - C_t) + \mu_t W^c_T + \frac{\sigma_X^2}{2}X_t^2 W^c_{XX} + \frac{\sigma_T^2}{2}T_t^2 W^c_{TT} + \sigma_{XT}T_t X_t W^c_{XT} = 0\right\}.$$

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The first-order condition yields

$$W_X^c = aC_t{}^{a(1-\gamma)-1}F_t{}^{-b(1-\gamma)} \Leftrightarrow C_t = \left(\frac{W_X^c F_t{}^{b(1-\gamma)}}{a}\right)^{\frac{1}{a(1-\gamma)-1}}.$$

Making use of the envelope theorem, we differentiate the Bellman equation with respect to X_t to obtain

$$r_t W_X^c + \left(r_t X_t + T_t - C_t + \sigma_X^2 X_t\right) W_{XX}^c + \left(\mu_t + \sigma_{XT} T_t\right) W_{XT}^c + \frac{\sigma_X^2}{2} X_t^2 W_{XXX}^c + \frac{\sigma_T^2}{2} T_t^2 W_{XTT}^c + \sigma_{XT} T_t X_t W_{XXT}^c = 0.$$

Applying Itô's lemma to W_X^c yields

$$E_t\left(dW_X^c\right) = W_{XX}^c E_t(dX) + W_{XT}^c E_t(dT) + \tfrac{1}{2} W_{XXX}^c E_t(dX^2) + \tfrac{1}{2} W_{XTT}^c E_t(dT^2) + W_{XXT}^c E_t(dXdT),$$

and replacing in the differentiated Bellman equation gives

$$r_t + \frac{E_t\left(dW_X^c\right)/W_X^c}{dt} + \sigma_X^2 X_t \frac{W_{XX}^c}{W_X^c} + \sigma_{XT} T_t \frac{W_{XT}^c}{W_X^c} = 0.$$

We postulate

$$W^c(X_t, T_t) = M\frac{X_t^{a(1-\gamma)-\eta} T_t^{\eta}}{1-\gamma},$$

where M and η are unknown constants.

With this expression of the value function we have

$$W_X^c = \frac{M}{1-\gamma}\left[a(1-\gamma)-\eta\right] X_t^{a(1-\gamma)-\eta-1} T_t^{\eta},$$

$$W_{XX}^c = \frac{M}{1-\gamma}\left[a(1-\gamma)-\eta\right]\left[a(1-\gamma)-\eta-1\right] X_t^{a(1-\gamma)-\eta-2} T_t^{\eta},$$

$$W_{XT}^c = \frac{M}{1-\gamma}\left[a(1-\gamma)-\eta\right]\eta X_t^{a(1-\gamma)-\eta-1} T_t^{\eta-1},$$

$$W_{XXX}^c = \frac{M}{1-\gamma}\left[a(1-\gamma)-\eta\right]\left[a(1-\gamma)-\eta-1\right]\left[a(1-\gamma)-\eta-2\right] X_t^{a(1-\gamma)-\eta-3} T_t^{\eta}$$

$$W_{XTT}^c = \frac{M}{1-\gamma}\left[a(1-\gamma)-\eta\right]\eta(\eta-1) X_t^{a(1-\gamma)-\eta-1} T_t^{\eta-2}$$

$$W_{XXT}^c = \frac{M}{1-\gamma}\left[a(1-\gamma)-\eta\right]\left[a(1-\gamma)-\eta-1\right]\eta X_t^{a(1-\gamma)-\eta-2} T_t^{\eta-1}.$$

Then the expected growth rate of the marginal value function is

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$$\frac{E_t\left(dW_X^c\right)}{W_X^c} = [a(1-\gamma)-\eta-1]\frac{E_t(dX)}{X_t} + \eta\frac{E_t(dT)}{T_t}$$
$$+\frac{1}{2}[a(1-\gamma)-\eta-1][a(1-\gamma)-\eta-2]\frac{E_t(dX^2)}{X_t^2}$$
$$+\frac{1}{2}\eta(\eta-1)\frac{E_t(dT^2)}{T_t^2} + [a(1-\gamma)-\eta-1]\eta\frac{E_t(dXdT)}{X_tT_t}.$$

We now focus on the equilibrium growth path. We drop for expositional ease the superscript j denoting the technology.

Using the fact that, at equilibrium,

$$T_t = \theta B K_t,$$

and then

$$\frac{dT_t}{T_t} = \frac{dK_t}{K_t} = \frac{dX_t}{X_t},$$

the expression for the expected growth rate of the marginal value function simplifies to

$$\frac{E_t\left(dW_X^c\right)}{W_X^c} = [a(1-\gamma)-1]\frac{E_t(dX)}{X_t} + [a(1-\gamma)-1][a(1-\gamma)-2]\frac{\sigma_X^2}{2}dt.$$

We replace in the differentiated Bellman equation and get

$$r_t + [a(1-\gamma)-1]\left(r_t + \frac{T_t}{X_t} - \frac{C_t}{X_t}\right) + [a(1-\gamma)-1]a(1-\gamma)\frac{\sigma_X^2}{2} = 0,$$

from which we deduce the propensity to consume,

$$\frac{C_t}{X_t} = \frac{a(1-\gamma)}{a(1-\gamma)-1}r_t + \frac{T_t}{X_t} + a(1-\gamma)\frac{\sigma_X^2}{2}$$
$$= \frac{S-1}{S}r_t + \frac{T_t}{X_t} - (S-1)\frac{\sigma_X^2}{2}.$$

Finally, using $r_t + \theta B = (1-\alpha)A$ and $T_t/X_t = \theta B$, we get equation (40) in the text,

$$\frac{\widetilde{C}_t}{X_t} = \frac{S-1}{S}\left[(1-\alpha)A + \frac{1}{S-1}\theta B - \frac{\sigma_X^2}{2}S\right],$$

from which we deduce the expected growth rate of wealth:

$$g = r_t + \frac{T_t}{X_t} - \frac{\widetilde{C}_t}{K_t} = \frac{1}{S}(1-\alpha)A - \frac{1}{S}\theta B + (S-1)\frac{\sigma_X^2}{2}.$$

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