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Wave turbulence in incompressible Hall magnetohydrodynamics

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Abstract. We investigate the steepening of the magnetic fluctuation power law spectra observed in the inner Solar wind for frequencies higher than 0.5 Hz. This high frequency part of the spectrum may be attributed to dispersive nonlinear processes. In that context, the long-time behavior of weakly interacting waves is examined in the framework of three-dimensional incompressible Hall magnetohydrodynamic (MHD) turbulence. The Hall term added to the standard MHD equations makes the Alfvén waves dispersive and circularly polarized. We introduce the generalized Elsässer variables and, using a complex helicity decomposition, we derive for three-wave interaction processes the general wave kinetic equations; they describe the nonlinear dynamics of Alfvén, whistler and ion cyclotron wave turbulence in the presence of a strong uniform magnetic field $B_0\hat{\mathbf{e}}_\parallel$. Hall MHD turbulence is characterized by anisotropies of different strength: (i) for wavenumbers $kd_\mathrm{i} \gg 1$ (d_i is the ion inertial length) nonlinear transfers are *essentially* in the direction perpendicular ($\perp$) to $\mathbf{B}_0$; (ii) for $kd_\mathrm{i} \ll 1$ nonlinear transfers are *exclusively* in the perpendicular direction; (iii) for $kd_\mathrm{i} \sim 1$, a moderate anisotropy is predicted. We show that electron and standard MHD turbulence can be seen as two frequency limits of the present theory but the standard MHD limit is singular; additionally, we analyze in detail the ion MHD turbulence limit. Exact power law solutions of the master wave kinetic equations are given in the small- and large-scale limits for which we have, respectively, the total energy spectra $E(k_\perp, k_\parallel) \sim k_\perp^{-5/2}|k_\parallel|^{-1/2}$ and $E(k_\perp, k_\parallel) \sim k_\perp^{-2}$. An anisotropic phenomenology is developed to describe continuously the different scaling laws of the energy spectrum; one predicts $E(k_\perp, k_\parallel) \sim k_\perp^{-2}|k_\parallel|^{-1/2}(1 + k_\perp^2 d_\mathrm{i}^2)^{-1/4}$. Non-local interactions between Alfvén, whistler and ion cyclotron waves are investigated; a non-trivial dynamics exists only when a discrepancy from the equipartition between the large-scale kinetic and magnetic energies happens.

1. Introduction

Spacecraft observations of the inner Solar wind, i.e. for heliocentric distances less than 1 AU, show magnetic and velocity fluctuations over a broad range of frequencies, from 10^{-5} Hz up to several hundred Hz (see, e.g., Coleman 1968; Belcher and Davis 1971; Matthaeus and Goldstein 1982; Roberts et al. 1987; Grappin et al. 1990; Burlaga, 1991; Leamon et al. 1998). These fluctuations possess many properties expected of fully developed weakly compressible magnetohydrodynamic (MHD)

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turbulence (Goldstein and Roberts 1999). For that reason the interplanetary medium is often seen as a vast laboratory for studying many fundamental questions about turbulent plasmas.

Compressed interaction regions produced by velocity differences are clearly observed in the outer Solar wind. In regions where fast streams overtake slow streams, the spectrum of the large-scale magnetic field fluctuations follows a f^{-2} frequency power law (Burlaga and Mish 1987) which was shown to be a spectral signature of jumps (Roberts and Goldstein, 1987). Compressive effects are, however, weaker in the inner Solar wind. For example, the normalized density fluctuations near the current sheet at 0.3 AU are often smaller than 5% (Bavassano et al. 1997). This tendency is confirmed by indirect measurements through radio wave interplanetary scintillation observations at heliocentric distances of 16–26 Solar radius (Spangler 2002). Waves and turbulence—the subject of this paper—are better observed in the pure/polar wind where generally the density fluctuations are weaker than in the current sheet. Therefore, the inner Solar wind may be seen mainly as a weakly compressible medium.

The turbulent state of the Solar wind was suggested by Coleman (1968) who reported a power law behavior for energy spectra with spectral indices lying between -1 and -2. The original signals being measured in time, these observational spectra are measured in frequency. Since the Solar wind is super-sonic and super-Alfvénic, the Taylor 'frozen-in flow' hypothesis is usually used to connect directly a frequency to a wavenumber which allows eventually a comparison with theoretical predictions. Note, however, that for anisotropic turbulence a relevant comparison with theoretical predictions, such as those made in this paper, only possible if a three-dimensional (3D) energy spectrum is accessible by *in situ* measurements, a situation that is not currently achieved (see, however, Matthaeus et al. 2005; see also Sec. 10). More precise measurements (see, e.g., Matthaeus and Goldstein 1982) revealed that the spectral index at low frequency is often about -1.7 which is closer to the Kolmogorov prediction (Kolmogorov 1941) for neutral fluids ($-5/3$) rather than the Iroshnikov–Kraichnan prediction (Iroshnikov 1963; Kraichnan 1965) for magnetized fluids ($-3/2$). Both predictions are built, in particular, on the hypothesis of isotropic turbulence. However, the presence of Alfvén waves in the fast Solar wind attests that the magnetic field has a preferential direction which is likely at the origin of anisotropic turbulence provided that the amount of counter-propagating Alfvén waves is enough. Indeed, *in situ* measurements of cross-helicity show clearly that outward propagative Alfvén waves are the main component of the fast Solar wind at short radial distances (Belcher and Davis 1971) but they become less dominant beyond 1 AU. Since pure Alfvén waves are exact solutions of the ideal incompressible MHD equations (see, e.g., Pouquet 1993), nonlinear interactions should be suppressed if only one type of wave is present. The variance analysis of the magnetic field components and magnitude shows clearly that the magnetic field vector of the fast Solar wind has a varying direction but with only a weak variation in magnitude (see, e.g., Forsyth et al. 1996a, b). Typical values give a ratio of the normalized variance of the field magnitude smaller than 10% whereas for the components it can be as large as 50%. In these respects, the interplanetary magnetic field may be seen as a vector lying approximately around the Parker spiral direction with only weak magnitude variations (Barnes 1981). Solar wind anisotropy with more power perpendicular to the mean magnetic field than that parallel is pointed out by data analyses (Belcher and Davis 1971;

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Klein et al. 1993) with a ratio of power up to 30. From single-point spacecraft measurements it is, however, not possible to specify the exact 3D form of the spectral tensor of the magnetic or velocity fluctuations. In the absence of such data, Bieber et al. (1996) with a quasi two-dimensional (2D) model, in which wavevectors are nearly perpendicular to the large-scale magnetic field, argued that about 85% of Solar wind turbulence possesses a dominant 2D component. In addition Solar wind anisotropies are detected through radio wave scintillations which reveal that density spectra close to the Sun are highly anisotropic with irregularities stretched out mainly along the radial direction (Armstrong et al. 1990).

Most of the papers dealing with interplanetary turbulence tend to focus on a frequency inertial range where the MHD approximation is well satisfied. It is the domain where the power spectral index is often found to be around the Kolmogorov index. During the past few decades several properties of the Solar wind turbulence have been understood in this framework. Less understood is what happens outside of this range of frequencies. At lower frequencies (less than 10^{-5} Hz) flatter power laws are found in particular for the fast Solar wind with indices close to -1 and even less (in absolute value) for the smallest frequencies. These large scales are often interpreted as the energy containing scales. But the precise role of the low Solar corona in the generation of such scales and the origin of the evolution of the spectral index when the distance from the Sun increases are still not well understood (see, e.g., Velli et al. 1989; Horbury 1999). For frequencies higher than 0.5 Hz a steepening of the magnetic fluctuation power law spectra is observed over more than two decades (Coroniti et al. 1982; Denskat et al. 1983; Leamon et al. 1998) with a spectral index on average around -3. Note that the latest analysis made with the Cluster spacecraft data reveals a less steep spectral index of about -2.12 (Bale et al. 2005). This new range, exhibiting a power law, is characterized by a bias of the polarization suggesting that these fluctuations are likely to be right-hand polarized, outward propagating waves (Goldstein et al. 1994). Various indirect lines of evidence indicate that these waves propagate at large angles to the background magnetic field and that the power in fluctuations parallel to the background magnetic field is much less than that in the perpendicular one (Coroniti et al. 1982; Leamon et al. 1998). For these reasons, it is thought (e.g. Stawicki et al. 2001) that Alfvén—left circularly polarized—fluctuations are suppressed by proton cyclotron damping and that the high frequency power law spectra are likely to consist of whistler waves. This scenario proposed is supported by direct numerical simulations of compressible two-and-a-half-dimensional ($2\frac{1}{2}$D) Hall MHD turbulence (Ghosh et al. 1996) where a steepening of the spectra is found and associated with the appearance of right circularly polarized fluctuations. It is plausible that what has been conventionally thought of as a dissipation range is actually a dispersive or inertial range and that the steeper power law may be due to nonlinear wave processes rather than dissipation (see, e.g., Krishan and Mahajan 2004). Under this new interpretation, the resistive dissipation range of frequencies may be moved to frequencies higher than the electron cyclotron frequency. A recent study (Stawicki et al. 2001) suggests that the treatment of the Solar wind dispersive range should include magnetosonic/whistler waves since it is often observed that the high frequency fluctuations of the magnetic field are much smaller than the background magnetic field (see also Forsyth et al. 1996a, b). It seems therefore that a nonlinear theory built on weak wave turbulence may be a useful point of departure for understanding the detailed physics of Solar wind turbulence.

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It is well known that the presence of a mean magnetic field plays a fundamental role in the behavior of compressible or incompressible MHD turbulence (Montgomery and Turner 1981; Shebalin et al. 1983; Oughton et al. 1994; Matthaeus et al. 1996; Kinney and McWilliams 1998; Cho and Vishniac 2000; Oughton and Matthaeus 2005). The main effect is that the mean magnetic field renders the turbulence quasi-bidimensional with a nonlinear transfer essentially perpendicular to its direction. Note that in such a situation, it is known that compressible MHD behaves very similarly to reduced MHD (see, e.g., Dmitruk et al. 2005). This property is generalized to $2\frac{1}{2}$D compressible Hall MHD for high and low β plasma simulations (Ghosh and Goldstein 1997) for which strong anisotropies are also found when a strong mean magnetic field is present. It was suggested that the action of the Hall term is to provide additional suppression of energy cascades along the mean field direction and incompressibility may not be able to reproduce the dynamics seen in the simulations. This conclusion is slightly different to what we find in the present study where we show that incompressibility in Hall MHD allows us to reproduce several observed properties such as anisotropic turbulence. Incompressible Hall MHD is often used to understand, for example, the main impact of the Hall term on flowing plasmas (Ohsaki 2005), or in turbulent dynamos (Mininni et al. 2003a, b). In the present paper, we derive a weak wave turbulence formalism for incompressible Hall MHD in the presence of a strong external mean magnetic field, where Alfvén, ion cyclotron and whistler/electron waves are taken into account. One of the main results is the derivation of the wave kinetic equations at the lowest order, i.e. for three-wave interaction processes. For such a turbulence, it is possible to show, in particular, a global tendency towards anisotropy with nonlinear transfers preferentially in the direction perpendicular to the external magnetic field. Hall MHD wave turbulence theory describes a wide range of frequencies, from the low-frequency limit of pure Alfvénic turbulence to the high-frequency limit of whistler wave turbulence for which the asymptotic theories have been derived recently (Galtier et al. 2000, 2002; Galtier and Bhattacharjee 2003). By recovering both theories as two particular limits, we recover all the well-known properties associated. Ion cyclotron wave turbulence appears as a third particular limit for which we report a detailed analysis. The energy spectrum of Hall MHD is characterized by two inertial ranges, which are exact solutions of the wave kinetic equations, separated by a knee. The position of the knee corresponds to the scale where the Hall term becomes (sub-) dominant. We develop a single anisotropic phenomenology that recovers the power law solutions found and makes the link continuously in wavenumbers between the two scaling laws. A non-local analysis performed on the wave kinetic equations reveals that a non-trivial dynamics between Alfvén, whistler and ion cyclotron waves happens only when a discrepancy from the equipartition between the large scale kinetic and magnetic energies exists. We believe that the description given here may help to better understand the inner Solar wind observations and, in particular, the existence of dispersive/non-dispersive inertial ranges.

The organization of the paper is as follows: in Sec. 2, we discuss about the approximation of incompressible Hall MHD and the existence of transverse, circularly polarized waves; we introduce the generalized Elsässer variables and the complex helical decomposition. In Sec. 3, we develop the wave turbulence formalism and we derive the wave kinetic equations for three-wave interaction processes. Section 4 is devoted to the general properties of Hall MHD wave turbulence. Section 5 deals with the small- and large-scale limits respectively of the master equations

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of wave turbulence. In particular, we give a detailed study of the ion cyclotron wave turbulence. In Sec. 6, we derive an anisotropic heuristic description which, in particular, makes the link between the previous predictions. In Sec. 7, we analyze non-local interactions between Alfvén, whistler and ion cyclotron waves. Section 8 is devoted to the possible sources of anisotropy. In Sec. 9, we discuss the domain of validity of the theory and the difference between wave and strong turbulence. Finally, we conclude with a summary and a general discussion in Sec. 10.

2. Hall MHD approximation

2.1. Generalized Ohm's law

Hall MHD is an extension of the standard MHD where the ion inertia is retained in Ohm's law. The generalized Ohm's law, in SI units, is then given by

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} - \frac{\mathbf{J} \times \mathbf{B}}{ne} = \mu_0 \eta \mathbf{J}, \tag{2.1}$$

where $\mathbf{E}$ is the electric field, $\mathbf{V}$ is the plasma flow velocity, $\mathbf{B}$ is the magnetic field, $\mathbf{J}$ is the current density, n is the electron density, e is the magnitude of the electron charge, μ_0 is the permeability of free space and η is the magnetic diffusivity. The Hall effect, represented by the last term in the left-hand side of the generalized Ohm's law, becomes relevant when we intend to describe the plasma dynamics up to length scales shorter than the ion inertial length d_i ($d_\mathrm{i} = c/\omega_\mathrm{pi}$, where c is the speed of light and ω_pi is the ion plasma frequency) and time scales of the order of, or shorter than, the ion cyclotron period ω_ci^{-1}. It is one of the most important manifestations of the velocity difference between electrons and ions when kinetic effects are not taken into account. The importance of the Hall effect in astrophysics has been pointed out to understand, for example, the presence of instabilities in protostellar disks (Balbus and Terquem 2001), the magnetic field evolution in neutron star crusts (Goldreich and Reisenegger 1992; Cumming et al. 2004), impulsive magnetic reconnection (see, e.g., Bhattacharjee 2004) or the formation of filaments (see, e.g., Passot and Sulem 2003; Dreher et al. 2005).

2.2. Incompressible Hall MHD equations

The inclusion of the Hall effect in Ohm's law leads, in the incompressible case, to the following Hall MHD equations:

$$\nabla \cdot \mathbf{V} = 0, \tag{2.2}$$

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla P_* + \mathbf{B} \cdot \nabla \mathbf{B} + \nu \nabla^2 \mathbf{V}, \tag{2.3}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{V} - d_\mathrm{i} \nabla \times [(\nabla \times \mathbf{B}) \times \mathbf{B}] + \eta \nabla^2 \mathbf{B}, \tag{2.4}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.5}$$

where $\mathbf{B}$ has been normalized to a velocity ($\mathbf{B} \to \sqrt{\mu_0 n m_\mathrm{i}}\,\mathbf{B}$, with m_i the ion mass), P_* is the total (magnetic plus kinetic) pressure and ν is the viscosity. The Hall effect appears in the induction equation as an additional term proportional to the ion inertial length d_i which means that it is effective when the dynamical scale is small enough. In other words, for large scale phenomena this term is negligible and we recover the standard MHD equations. In the opposite limit, e.g. for very fast

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time scales ($\ll\omega_{\rm ci}^{-1}$), ions do not have time to follow electrons and they provide a static homogeneous background on which electrons move. Such a model where the dynamics is entirely governed by electrons is called the electron MHD (EMHD) model (Kingsep et al. 1990; Shukla and Stenflo 1999). It can be recovered from Hall MHD by taking the limits of small velocity $\mathbf{V}$ and large $d_{\rm i}$. The electron and Hall MHD approximations are particularly relevant in the context of collisionless magnetic reconnection where the diffusion region develops multiscale structures corresponding to ion and electron characteristic lengths (Huba 1995; Biskamp 1997). For example, it is often considered that whistler/EMHD turbulence may act as a detector for magnetic reconnection at the magnetopause (Cai et al. 2001).

2.3. 3D inviscid invariants

The three inviscid ($\nu = \eta = 0$) quadratic invariants of incompressible Hall MHD are the total energy

$$E = \frac{1}{2}\int (\mathbf{V}^2 + \mathbf{B}^2)\, d\mathscr{V}, \tag{2.6}$$

the magnetic helicity

$$H_{\rm m} = \frac{1}{2}\int \mathbf{A}\cdot\mathbf{B}\, d\mathscr{V}, \tag{2.7}$$

and the generalized hybrid helicity

$$H_{\rm G} = \frac{1}{2}\int (\mathbf{A} + d_{\rm i}\mathbf{V})\cdot(\mathbf{B} + d_{\rm i}\nabla\times\mathbf{V})\, d\mathscr{V}, \tag{2.8}$$

with $\mathbf{A}$ the vector potential ($\mathbf{B} = \nabla\times\mathbf{A}$). The third invariant generalizes the cross-helicity, $H_{\rm c} = (1/2)\int \mathbf{V}\cdot\mathbf{B}\, d\mathscr{V}$, which is no longer conserved when the Hall term is present in the MHD equations. The generalized hybrid helicity can be seen as the product of two generalized quantities, a vector potential $\boldsymbol{\Upsilon} = \mathbf{A} + d_{\rm i}\mathbf{V}$ and a vorticity $\boldsymbol{\Omega} = \mathbf{B} + d_{\rm i}\nabla\times\mathbf{V}$. The role played by the generalized vorticity is somewhat equivalent to that played by the magnetic field in standard MHD (Woltjer 1958). Indeed, both quantities obey the same Lagrangian equation, which is for $\boldsymbol{\Omega}$,

$$\frac{d\boldsymbol{\Omega}}{dt} = \boldsymbol{\Omega}\cdot\nabla\mathbf{V}. \tag{2.9}$$

By applying Helmholtz's law (see, e.g., Davidson 2001) to Hall MHD, we see that the generalized vorticity lines are frozen into the plasma (see, e.g., Sahraoui et al. 2003). The presence of the Hall term breaks such a property for the magnetic field which is, however, still frozen but only in the electron flow: the introduction of the electron velocity $\mathbf{V}_e$, with $\mathbf{V}\times\mathbf{B} - \mathbf{J}\times\mathbf{B}/ne \simeq \mathbf{V}_e\times\mathbf{B}$, leads to

$$\frac{d\mathbf{B}}{dt} = \mathbf{B}\cdot\nabla\mathbf{V}_e, \tag{2.10}$$

which proves the statement. We will see below that the detailed conservation of invariants is the first test that the wave kinetic equations have to satisfy.

2.4. Incompressible Hall MHD waves

One of the main effects produced by the presence of the Hall term is that the linearly polarized Alfvén waves, solutions of the standard MHD equations, become circularly polarized and dispersive (see, e.g., Mahajan and Krishan 2005; Sahraoui et al. 2005). Indeed, if we linearize (2.2)–(2.5) around a strong uniform magnetic

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field $\mathbf{B}_0$ such that,

$$\mathbf{B}(\mathbf{x}) = B_0\hat{\mathbf{e}}_\parallel + \epsilon\mathbf{b}(\mathbf{x}), \tag{2.11}$$

$$\mathbf{V}(\mathbf{x}) = \epsilon\mathbf{v}(\mathbf{x}), \tag{2.12}$$

with ϵ a small parameter ($0 < \epsilon \ll 1$), $\mathbf{x}$ a 3D displacement vector, and $\hat{\mathbf{e}}_\parallel$ a unit vector ($|\hat{\mathbf{e}}_\parallel| = 1$), then we obtain the following inviscid equations in Fourier space:

$$\mathbf{k}\cdot\mathbf{v}_\mathbf{k} = 0, \tag{2.13}$$

$$\partial_t\mathbf{v}_\mathbf{k} - ik_\parallel B_0\mathbf{b}_\mathbf{k} = \epsilon\{-\mathbf{v}\cdot\nabla\mathbf{v} - \nabla P_* + \mathbf{b}\cdot\nabla\mathbf{b}\}_\mathbf{k}, \tag{2.14}$$

$$\partial_t\mathbf{b}_\mathbf{k} - ik_\parallel B_0\mathbf{u}_\mathbf{k} - d_\mathrm{i}B_0k_\parallel\mathbf{k}\times\mathbf{b}_\mathbf{k} = \epsilon\{-\mathbf{v}\cdot\nabla\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{v} - d_\mathrm{i}\nabla[(\nabla\times\mathbf{b})\times\mathbf{b}]\}_\mathbf{k}, \tag{2.15}$$

$$\mathbf{k}\cdot\mathbf{b}_\mathbf{k} = 0, \tag{2.16}$$

where the wavevector $\mathbf{k} = k\hat{\mathbf{e}}_k = \mathbf{k}_\perp + k_\parallel\hat{\mathbf{e}}_\parallel$ ($k = |\mathbf{k}|$, $k_\perp = |\mathbf{k}_\perp|$, $|\hat{\mathbf{e}}_k| = 1$) and $i^2 = -1$. The index $\mathbf{k}$ denotes the Fourier transform, defined by the relation

$$\mathbf{v}(\mathbf{x}) \equiv \int \mathbf{v}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\,d\mathbf{k}, \tag{2.17}$$

where $\mathbf{v}(\mathbf{k}) = \mathbf{v}_\mathbf{k} = \tilde{\mathbf{v}}_\mathbf{k}e^{-i\omega t}$ (the same notation is used for the magnetic field). The linear dispersion relation ($\epsilon = 0$) reads

$$\omega^2 - (\Lambda d_\mathrm{i}B_0k_\parallel k)\omega - B_0^2k_\parallel^2 = 0, \tag{2.18}$$

with

$$\begin{Bmatrix}\tilde{\mathbf{v}}_\mathbf{k}\\ \tilde{\mathbf{b}}_\mathbf{k}\end{Bmatrix} = \Lambda i\hat{\mathbf{e}}_k\times\begin{Bmatrix}\tilde{\mathbf{v}}_\mathbf{k}\\ \tilde{\mathbf{b}}_\mathbf{k}\end{Bmatrix}. \tag{2.19}$$

We obtain the solutions

$$\omega \equiv \omega_\Lambda^s = \frac{sk_\parallel kd_\mathrm{i}B_0}{2}\left(s\Lambda + \sqrt{1 + \frac{4}{d_\mathrm{i}^2k^2}}\right), \tag{2.20}$$

where the value (± 1) of s defines the directional wave polarity. In other words, we have $sk_\parallel \geqslant 0$ and ω_Λ^s is a positive frequency. The Alfvén wave polarization Λ tells us if the wave is right ($\Lambda = s$) or left ($\Lambda = -s$) circularly polarized. In the first case, we are dealing with whistler waves, whereas in the latter case with ion cyclotron waves. We see that the transverse circularly polarized Alfvén waves are dispersive and we note that we recover the two well-known limits, i.e. the pure whistler waves ($\omega = sk_\parallel kd_\mathrm{i}B_0$) in the high-frequency limit ($kd_\mathrm{i} \to \infty$), and the standard Alfvén waves ($\omega = sk_\parallel B_0$) in the low-frequency limit ($kd_\mathrm{i} \to 0$). The Alfvén waves become linearly polarized only when the Hall term vanishes: when the Hall term is present, whatever its magnitude is, the Alfvén waves are circularly polarized. Note that this situation is different from the compressible case for which the Alfvén waves are elliptically polarized. As expected, it is possible to show (Hameiri et al. 2005; Sahraoui et al. 2005) that the ion cyclotron wave has a resonance at the frequency $\omega_\mathrm{ci}k_\parallel/k$, where $\omega_\mathrm{ci} = B_0/d_\mathrm{i}$. Therefore, with such an approximation, only whistler waves survive at high frequency.

2.5. Complex helicity decomposition

Given the incompressibility constraints (2.13) and (2.16), it is convenient to project the Hall MHD equations in the plane orthogonal to $\mathbf{k}$. We will use the complex helicity decomposition technique which has been shown to be effective in providing

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a compact description of the dynamics of 3D incompressible fluids (Craya 1958; Moffatt 1970; Kraichnan 1973; Cambon and Jacquin 1989; Lesieur 1990; Waleffe 1992; Turner 2000; Galtier 2003; Galtier and Bhattacharjee 2003). The complex helicity basis is also particularly useful since it allows us to diagonalize systems dealing with circularly polarized waves. We introduce the complex helicity decomposition

$$\mathbf{h}^{\mathbf{\Lambda}}(\mathbf{k}) \equiv \mathbf{h}^{\mathbf{\Lambda}}_{\mathbf{k}} = \hat{\mathbf{e}}_\theta + i\Lambda\hat{\mathbf{e}}_\Phi, \tag{2.21}$$

where

$$\hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_\Phi \times \hat{\mathbf{e}}_k, \tag{2.22}$$

$$\hat{\mathbf{e}}_\Phi = \frac{\hat{\mathbf{e}}_\parallel \times \hat{\mathbf{e}}_k}{|\hat{\mathbf{e}}_\parallel \times \hat{\mathbf{e}}_k|}, \tag{2.23}$$

and $|\hat{\mathbf{e}}_\theta(\mathbf{k})| = |\hat{\mathbf{e}}_\Phi(\mathbf{k})| = 1$. We note that $(\hat{\mathbf{e}}_k, h^+_{\mathbf{k}}, h^-_{\mathbf{k}})$ form a complex basis with the following properties:

$$\mathbf{h}^{-\mathbf{\Lambda}}_{\mathbf{k}} = \mathbf{h}^{\mathbf{\Lambda}}_{-\mathbf{k}}, \tag{2.24}$$

$$\hat{\mathbf{e}}_k \times \mathbf{h}^{\mathbf{\Lambda}}_{\mathbf{k}} = -i\Lambda\mathbf{h}^{\mathbf{\Lambda}}_{\mathbf{k}}, \tag{2.25}$$

$$\mathbf{k} \cdot \mathbf{h}^{\mathbf{\Lambda}}_{\mathbf{k}} = 0, \tag{2.26}$$

$$\mathbf{h}^{\mathbf{\Lambda}}_{\mathbf{k}} \cdot \mathbf{h}^{\mathbf{\Lambda}'}_{\mathbf{k}} = 2\delta_{-\Lambda'\Lambda}. \tag{2.27}$$

We project the Fourier transform of the original vectors $\mathbf{v}(\mathbf{x})$ and $\mathbf{b}(\mathbf{x})$ on the helicity basis:

$$\mathbf{v}_{\mathbf{k}} = \sum_\Lambda \mathscr{U}_\Lambda(\mathbf{k})\mathbf{h}^{\mathbf{\Lambda}}_{\mathbf{k}} = \sum_\Lambda \mathscr{U}_\Lambda \mathbf{h}^{\mathbf{\Lambda}}_{\mathbf{k}}, \tag{2.28}$$

$$\mathbf{b}_{\mathbf{k}} = \sum_\Lambda \mathscr{B}_\Lambda(\mathbf{k})\mathbf{h}^{\mathbf{\Lambda}}_{\mathbf{k}} = \sum_\Lambda \mathscr{B}_\Lambda \mathbf{h}^{\mathbf{\Lambda}}_{\mathbf{k}}. \tag{2.29}$$

We introduce expressions of the fields into the Hall MHD equations written in Fourier space and we multiply by vector $\mathbf{h}^{\mathbf{\Lambda}}_{-\mathbf{k}}$. First, we focus on the linear dispersion relation ($\epsilon = 0$) which reads

$$\partial_t \mathscr{Z}^s_\Lambda = -i\omega^s_\Lambda \mathscr{Z}^s_\Lambda, \tag{2.30}$$

with

$$\mathscr{Z}^s_\Lambda \equiv \mathscr{U}_\Lambda + \xi^s_\Lambda \mathscr{B}_\Lambda, \tag{2.31}$$

$$\xi^s_\Lambda(k) = \xi^s_\Lambda = -\frac{s d_i k}{2}\left(s\Lambda + \sqrt{1 + \frac{4}{d_i^2 k^2}}\right). \tag{2.32}$$

Equation (2.30) shows that $\mathscr{Z}^s_\Lambda$ are the 'good' variables for our system. These eigenvectors combine the velocity and the magnetic field in a non-trivial way by a factor ξ^s_Λ (with $\omega^s_\Lambda = -B_0 k_\parallel \xi^s_\Lambda$). In the large-scale limit ($kd_i \to 0$), we see that $\xi^s_\Lambda \to -s$; we recover the Elsässer variables used in standard MHD. In the small-scale limit ($kd_i \to \infty$), we have $\xi^s_\Lambda \to -s\, d_i k$, for $\Lambda = s$ (whistler waves), or $\xi^s_\Lambda \to (-s\, d_i k)^{-1}$, for $\Lambda = -s$. Therefore, $\mathscr{Z}^s_\Lambda$ can be seen as a generalization of the Elsässer variables to Hall MHD. In the rest of the paper, we will use the relation

$$\mathscr{Z}^s_\Lambda = (\xi^s_\Lambda - \xi^{-s}_\Lambda) a^s_\Lambda e^{-i\omega^s_\Lambda t}, \tag{2.33}$$

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where a^s_Λ is the wave amplitude in the interaction representation for which we have, in the linear approximation, $\partial_t a^s_\Lambda = 0$. In particular, that means that weak nonlinearities will modify only slowly in time the Hall MHD wave amplitudes. The coefficient in front of the wave amplitude is introduced in advance to simplify the algebra that we are going to develop.

3. Helical wave turbulence formalism

3.1. Fundamental equations

We decompose the inviscid nonlinear Hall MHD equations (2.14)–(2.15) on the complex helicity decomposition introduced in the previous section. Then we project the equations on vector $\mathbf{h}^{\mathbf{\Lambda}}_{-\mathbf{k}}$. We obtain

$$\begin{aligned}&\partial_t \mathscr{U}_\Lambda - iB_0 k_\parallel \mathscr{B}_\Lambda \\ &\quad = -\frac{i\epsilon}{2}\int \sum_{\Lambda_\mathrm{p},\Lambda_q} (\mathscr{U}_{\Lambda_\mathrm{p}}\mathscr{U}_{\Lambda_q} - \mathscr{B}_{\Lambda_\mathrm{p}}\mathscr{B}_{\Lambda_q})\big(\mathbf{k}\cdot\mathbf{h}^{\mathbf{\Lambda_p}}_{\mathrm{p}}\big)\big(\mathbf{h}^{\mathbf{\Lambda_q}}_{\mathbf{q}}\cdot\mathbf{h}^{-\mathbf{\Lambda}}_{\mathbf{k}}\big)\delta_{pq,k}\,d\mathbf{p}\,d\mathbf{q}, \qquad (3.1)\end{aligned}$$

and

$$\begin{aligned}&\partial_t \mathscr{B}_\Lambda - iB_0 k_\parallel \mathscr{U}_\Lambda + i\Lambda d_\mathrm{i} B_0 k_\parallel k \mathscr{B}_\Lambda \\ &\quad = -\frac{i\epsilon}{2}\int \sum_{\Lambda_\mathrm{p},\Lambda_q} (\mathscr{U}_{\Lambda_\mathrm{p}}\mathscr{B}_{\Lambda_q} - \mathscr{U}_{\Lambda_q}\mathscr{B}_{\Lambda_\mathrm{p}})\big(\mathbf{k}\cdot\mathbf{h}^{\mathbf{\Lambda_p}}_{\mathrm{p}}\big)\big(\mathbf{h}^{\mathbf{\Lambda_q}}_{\mathbf{q}}\cdot\mathbf{h}^{-\mathbf{\Lambda}}_{\mathbf{k}}\big)\delta_{pq,k}\,d\mathbf{p}\,d\mathbf{q} \\ &\qquad + \frac{i\epsilon}{2}\int k\Lambda d_\mathrm{i} \sum_{\Lambda_\mathrm{p},\Lambda_q} \mathscr{B}_{\Lambda_\mathrm{p}}\mathscr{B}_{\Lambda_q}\big[\big(\mathbf{q}\cdot\mathbf{h}^{-\mathbf{\Lambda}}_{\mathbf{k}}\big)\big(\mathbf{h}^{\mathbf{\Lambda_p}}_{\mathrm{p}}\cdot\mathbf{h}^{\mathbf{\Lambda_q}}_{\mathbf{q}}\big) - \big(\mathbf{q}\cdot\mathbf{h}^{\mathbf{\Lambda_p}}_{\mathrm{p}}\big)\big(\mathbf{h}^{\mathbf{\Lambda_q}}_{\mathbf{q}}\cdot\mathbf{h}^{-\mathbf{\Lambda}}_{\mathbf{k}}\big)\big] \\ &\qquad \times \delta_{pq,k}\,d\mathbf{p}\,d\mathbf{q}, \qquad (3.2)\end{aligned}$$

where $\delta_{pq,k} = \delta(\mathbf{p}+\mathbf{q}-\mathbf{k})$. The delta distributions come from the Fourier transforms of the nonlinear terms. We introduce the generalized Elsässer variables a^s_Λ in the interaction representation and we find

$$\partial_t a^s_\Lambda = \frac{i\epsilon}{2}\int \sum_{\substack{\Lambda_\mathrm{p},\Lambda_q \\ s_\mathrm{p},s_q}} L\begin{smallmatrix}\Lambda\Lambda_\mathrm{p}\Lambda_q \\ s\,s_\mathrm{p}\,s_q \\ -k\,p\,q\end{smallmatrix} a^{s_p}_{\Lambda_p} a^{s_q}_{\Lambda_q} e^{-i\Omega_{pq,k}\,t}\delta_{pq,k}\,d\mathbf{p}\,d\mathbf{q}, \qquad (3.3)$$

where

$$\begin{aligned}L\begin{smallmatrix}\Lambda\Lambda_\mathrm{p}\Lambda_q \\ s\,s_\mathrm{p}\,s_q \\ k\,p\,q\end{smallmatrix} &= \frac{1-{\xi^s_\Lambda}^2}{\xi^s_\Lambda - \xi^{-s}_\Lambda}\big(\mathbf{q}\cdot\mathbf{h}^{\mathbf{\Lambda}}_{\mathbf{k}}\big)\big(\mathbf{h}^{\mathbf{\Lambda_p}}_{\mathrm{p}}\cdot\mathbf{h}^{\mathbf{\Lambda_q}}_{\mathbf{q}}\big) \\ &\quad + \frac{{\xi^s_\Lambda}^2 + \xi^s_\Lambda \xi^{-s_p}_{\Lambda_p} - \xi^s_\Lambda \xi^{-s_q}_{\Lambda_q} - \xi^{-s_p}_{\Lambda_p}\xi^{-s_q}_{\Lambda_q}}{\xi^s_\Lambda - \xi^{-s}_\Lambda}(\mathbf{q}\cdot\mathbf{h}^{\mathbf{\Lambda_p}}_{\mathrm{p}})(\mathbf{h}^{\mathbf{\Lambda_q}}_{\mathbf{q}}\cdot\mathbf{h}^{\mathbf{\Lambda}}_{\mathbf{k}}), \qquad (3.4)\end{aligned}$$

and

$$\Omega_{pq,k} = \omega^{s_p}_{\Lambda_p} + \omega^{s_q}_{\Lambda_q} - \omega^s_\Lambda. \qquad (3.5)$$

Equation (3.3) is the wave amplitude equation from which it is possible to extract some information. As expected we see that the nonlinear terms are of order ϵ. This means that weak nonlinearities will modify only slowly in time the Hall MHD wave amplitude. They contain an exponentially oscillating term which is essential for the asymptotic closure. Indeed, wave turbulence deals with variations of spectral

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densities at very large time, i.e. for a nonlinear transfer time much greater than the wave period. As a consequence, most of the nonlinear terms are destroyed by phase mixing and only a few of them, the resonance terms, survive (see, e.g., Newell et al. 2001). The expression obtained for the fundamental equation (3.3) is usual in wave turbulence. The main difference between problems is localized in the matrix L which is interpreted as a complex geometrical coefficient. We will see below that the local decomposition allows us to get a polar form for such a coefficient which is much easier to manipulate. From (3.3) we see eventually that, contrary to incompressible MHD, there are no exact solutions to the nonlinear problem in incompressible Hall MHD. The origin of such a difference is that in MHD the nonlinear term involves Alfvén waves traveling only in opposite directions whereas in Hall MHD this constraint does not exist (we have a summation over Λ and s). In other words, if one type of wave is not present in incompressible MHD then the nonlinear term cancels whereas in incompressible Hall MHD it is not the case. The conclusion reached by Mahajan and Krishan (2005) is therefore not correct: the condition $\mathscr{U}_\Lambda = -\xi^s_\Lambda \mathscr{B}_\Lambda$ does not correspond to a nonlinear solution of incompressible Hall MHD.

3.2. Local decomposition

In order to evaluate the scalar products of complex helical vectors found in the geometrical coefficient (3.4), it is convenient to introduce a vector basis local to each particular triad (Waleffe 1992; Turner 2000; Galtier 2003; Galtier and Bhattacharjee 2003). For example, for a given vector $\mathbf{p}$, we define the orthonormal basis vectors,

$$\begin{aligned} \hat{\mathbf{O}}^{(\mathbf{1})}(\mathbf{p}) &= \hat{\mathbf{n}}, \\ \hat{\mathbf{O}}^{(\mathbf{2})}(\mathbf{p}) &= \hat{\mathbf{e}}_p \times \hat{\mathbf{n}}, \\ \hat{\mathbf{O}}^{(\mathbf{3})}(\mathbf{p}) &= \hat{\mathbf{e}}_p, \end{aligned} \tag{3.6}$$

where $\hat{\mathbf{e}}_p = \mathbf{p}/|\mathbf{p}|$ and

$$\hat{\mathbf{n}} = \frac{\mathbf{p}\times\mathbf{k}}{|\mathbf{p}\times\mathbf{k}|} = \frac{\mathbf{q}\times\mathbf{p}}{|\mathbf{q}\times\mathbf{p}|} = \frac{\mathbf{k}\times\mathbf{q}}{|\mathbf{k}\times\mathbf{q}|}. \tag{3.7}$$

We see that the vector $\hat{\mathbf{n}}$ is normal to any vector of the triad ($\mathbf{k}$,$\mathbf{p}$,$\mathbf{q}$) and changes sign if $\mathbf{p}$ and $\mathbf{q}$ are interchanged, i.e. $\hat{\mathbf{n}}_{(\mathbf{k},\mathbf{q},\mathbf{p})} = -\hat{\mathbf{n}}_{(\mathbf{k},\mathbf{p},\mathbf{q})}$. Note that $\hat{\mathbf{n}}$ does not change by cyclic permutation, i.e. $\hat{\mathbf{n}}_{(\mathbf{k},\mathbf{q},\mathbf{p})} = \hat{\mathbf{n}}_{(\mathbf{q},\mathbf{p},\mathbf{k})} = \hat{\mathbf{n}}_{(\mathbf{p},\mathbf{k},\mathbf{q})}$. A sketch of the local decomposition is given in Fig. 1. We now introduce the vectors

$$\Xi^{\Lambda_\mathrm{p}}(\mathbf{p}) \equiv \Xi^{\Lambda_\mathrm{p}}_{\mathbf{p}} = \hat{\mathbf{O}}^{(\mathbf{1})}(\mathbf{p}) + i\Lambda_\mathrm{p}\hat{\mathbf{O}}^{(\mathbf{2})}(\mathbf{p}), \tag{3.8}$$

and define the rotation angle Φ_p, so that

$$\begin{aligned} \cos\Phi_\mathrm{p} &= \hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_\theta(\mathbf{p}), \\ \sin\Phi_\mathrm{p} &= \hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_{\boldsymbol{\Phi}}(\mathbf{p}). \end{aligned} \tag{3.9}$$

The decomposition of the helicity vector $\mathbf{h}^{\mathbf{\Lambda}_\mathrm{p}}_\mathrm{p}$ in the local basis gives

$$\mathbf{h}^{\mathbf{\Lambda}_\mathrm{p}}_\mathrm{p} = \Xi^{\Lambda_\mathrm{p}}_{\mathbf{p}} e^{i\Lambda_\mathrm{p}\Phi_\mathrm{p}}. \tag{3.10}$$

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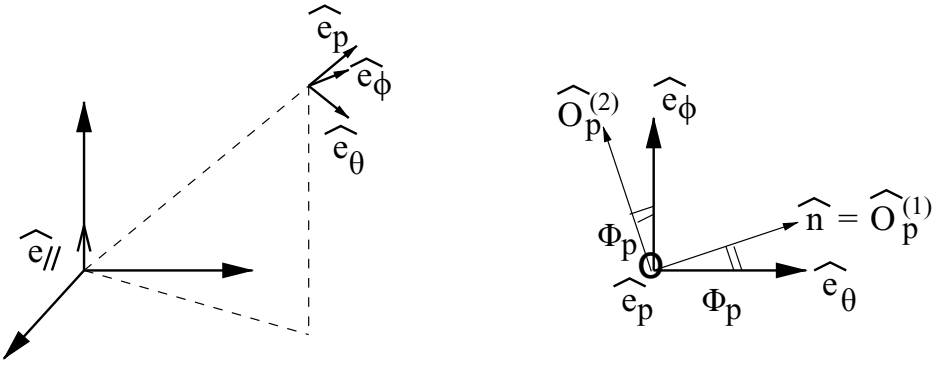


Figure 1. Sketch of the local decomposition for a given wavevector $\mathbf{p}$.

After some algebra we obtain the following polar form for the matrix L,

$$\begin{aligned} L\begin{smallmatrix}\Lambda\Lambda_p\Lambda_q\\ s\,s_p\,s_q\\ k\,p\,q\end{smallmatrix} &= ie^{i(\Lambda\Phi_k+\Lambda_p\Phi_p+\Lambda_q\Phi_q)}\frac{\Lambda\Lambda_p\Lambda_q}{\xi_\Lambda^s-\xi_\Lambda^{-s}}\frac{\sin\psi_k}{k}\\ &\quad\times\big[\big(\xi_\Lambda^{s\,2}-\xi_{\Lambda_p}^{-s_p}\xi_{\Lambda_q}^{-s_q}\big)\Lambda_q q(\Lambda k+\Lambda_p p+\Lambda_q q)\\ &\quad+\xi_\Lambda^s\big(\xi_{\Lambda_p}^{-s_p}-\xi_{\Lambda_q}^{-s_q}\big)kq\big(\Lambda\Lambda_q+\cos\psi_p\big)\big]. \end{aligned} \tag{3.11}$$

The angle ψ_k refers to the angle opposite to $\mathbf{k}$ in the triangle defined by $\mathbf{k}=\mathbf{p}+\mathbf{q}$ ($\sin\psi_k=\hat{\mathbf{n}}\cdot(\mathbf{q}\times\mathbf{p})/|(\mathbf{q}\times\mathbf{p})|$). To obtain (3.11), we have also used the well-known triangle relations

$$\frac{\sin\psi_k}{k}=\frac{\sin\psi_p}{p}=\frac{\sin\psi_q}{q}. \tag{3.12}$$

Further modifications have to be made before applying the spectral formalism. In particular, the fundamental equation has to be invariant under interchange of $\mathbf{p}$ and $\mathbf{q}$. To do so, we introduce the symmetrized and renormalized matrix M

$$M\begin{smallmatrix}\Lambda\Lambda_p\Lambda_q\\ s\,s_p\,s_q\\ k\,p\,q\end{smallmatrix} = id_i\frac{1}{\xi_\Lambda^{s\,2}}\frac{\xi_\Lambda^s-\xi_\Lambda^{-s}}{\xi_{\Lambda_q}^{s_q}-\xi_{\Lambda_p}^{s_p}}\left(L\begin{smallmatrix}\Lambda\Lambda_p\Lambda_q\\ s\,s_p\,s_q\\ k\,p\,q\end{smallmatrix}+L\begin{smallmatrix}\Lambda\Lambda_q\Lambda_p\\ s\,s_q\,s_p\\ k\,q\,p\end{smallmatrix}\right). \tag{3.13}$$

Finally, by using the identities given in Appendix A, we obtain

$$\partial_t a_\Lambda^s=\frac{\epsilon}{4d_i}\int\sum_{\substack{\Lambda_p,\Lambda_q\\ s_p,\,s_q}}\xi_\Lambda^{s\,2}\frac{\xi_{\Lambda_q}^{s_q}-\xi_{\Lambda_p}^{s_p}}{\xi_\Lambda^s-\xi_\Lambda^{-s}}M\begin{smallmatrix}\Lambda\Lambda_p\Lambda_q\\ s\,s_p\,s_q\\ -k\,p\,q\end{smallmatrix}a_{\Lambda_p}^{s_p}a_{\Lambda_q}^{s_q}e^{-i\Omega_{pq,k}t}\delta_{pq,k}\,d\mathbf{p}\,d\mathbf{q}, \tag{3.14}$$

where

$$M\begin{smallmatrix}\Lambda\Lambda_p\Lambda_q\\ s\,s_p\,s_q\\ k\,p\,q\end{smallmatrix}=e^{i(\Lambda\Phi_k+\Lambda_p\Phi_p+\Lambda_q\Phi_q)}\Lambda\Lambda_p\Lambda_q\frac{\sin\psi_k}{k}(\Lambda k+\Lambda_p p+\Lambda_q q)(1-\xi_\Lambda^{-s\,2}\xi_{\Lambda_p}^{-s_p\,2}\xi_{\Lambda_q}^{-s_q\,2}). \tag{3.15}$$

The matrix M possesses the following properties ($*$ denotes the complex conjugate),

$$\left(M\begin{smallmatrix}\Lambda\Lambda_p\Lambda_q\\ s\,s_p\,s_q\\ k\,p\,q\end{smallmatrix}\right)^*=M\begin{smallmatrix}-\Lambda-\Lambda_p-\Lambda_q\\ -s-s_p-s_q\\ k\,p\,q\end{smallmatrix}=M\begin{smallmatrix}\Lambda\Lambda_p\Lambda_q\\ s\,s_p\,s_q\\ -k-p-q\end{smallmatrix}, \tag{3.16}$$

$$M\begin{smallmatrix}\Lambda\Lambda_p\Lambda_q\\ s\,s_p\,s_q\\ k\,p\,q\end{smallmatrix}=-M\begin{smallmatrix}\Lambda\Lambda_q\Lambda_p\\ s\,s_q\,s_p\\ k\,q\,p\end{smallmatrix}, \tag{3.17}$$

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$$M\begin{matrix}\Lambda\Lambda_p\Lambda_q\\ s s_p s_q\\ k\,p\,q\end{matrix} = -M\begin{matrix}\Lambda_q\Lambda_p\Lambda\\ s_q\,s_p\,s\\ q\,p\,k\end{matrix}, \tag{3.18}$$

$$M\begin{matrix}\Lambda\Lambda_p\Lambda_q\\ s\,s_p\,s_q\\ k\,p\,q\end{matrix} = -M\begin{matrix}\Lambda_p\Lambda\Lambda_q\\ s_p\,s\,s_q\\ p\,k\,q\end{matrix}. \tag{3.19}$$

Equation (3.14) is the fundamental equation that describes the slow evolution of the Alfvén wave amplitudes due to the nonlinear terms of the incompressible Hall MHD equations. It is the starting point for deriving the wave kinetic equations. The local decomposition used here allows us to represent concisely complex information in an exponential function. As we will see, it will simplify significantly the derivation of the wave kinetic equations.

From (3.14) we note that the nonlinear coupling between helicity states associated with wavevectors, $\mathbf{p}$ and $\mathbf{q}$, vanishes when the wavevectors are collinear (since then, $\sin\psi_k = 0$). This property is similar to the one found in the limit of EMHD (Galtier and Bhattacharjee 2003). It seems to be a general property for helicity waves (Kraichnan 1973; Waleffe 1992; Turner 2000; Galtier 2003). In addition we note that the nonlinear coupling between helicity states vanishes whenever the wavenumbers p and q are equal if their associated wave and directional polarities, Λ_p, Λ_q and s_p, s_q respectively, are also equal. In the case of whistler (EMHD) waves, for which we have $\Lambda = s$ (right circularly polarized), this property was already observed (Galtier and Bhattacharjee 2003). Here we generalize this finding to right and left circularly polarized waves. In the large-scale limit, i.e. when we tend to pure incompressible MHD, this property tends to disappear. For pure incompressible MHD, where Alfvén waves are linearly polarized, this is no longer observed. As noted before, the nature of the polarization seems to be fundamental.

We are interested by the long-time behavior of the Alfvén wave amplitudes. From the fundamental equation (3.14), we see that the nonlinear wave coupling will come from resonant terms such that,

$$\begin{cases}\mathbf{k} = \mathbf{p} + \mathbf{q},\\ k_\parallel \xi^s_\Lambda = p_\parallel \xi^{s_p}_{\Lambda_p} + q_\parallel \xi^{s_q}_{\Lambda_q}.\end{cases} \tag{3.20}$$

The resonance condition may also be written

$$\frac{\xi^s_\Lambda - \xi^{s_p}_{\Lambda_p}}{q_\parallel} = \frac{\xi^{s_q}_{\Lambda_q} - \xi^s_\Lambda}{p_\parallel} = \frac{\xi^{s_q}_{\Lambda_q} - \xi^{s_p}_{\Lambda_p}}{k_\parallel}. \tag{3.21}$$

As we will see below, the relations (3.21) are useful in simplifying the wave kinetic equations and demonstrating the conservation of ideal invariants. In particular, we note that we recover the resonance conditions for whistler waves by taking the appropriate limit.

3.3. Dynamics and wave kinetic equations

Fully developed wave turbulence is a state of a system composed of many simultaneously excited and interacting nonlinear waves where the energy distribution, far from thermodynamic equilibrium, is characterized by a wide power law spectrum. This range of wavenumbers, the inertial range, is generally localized between large scales at which energy is injected in the system and small dissipative scales. The origin of wave turbulence dates back to the early 1960s and since then many papers have been devoted to the subject (see, e.g., Hasselman 1962; Benney and

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Saffman 1966; Zakharov 1967; Benney and Newell 1969; Sagdeev and Galeev 1969; Kuznetsov 1972; Zakharov et al. 1992; Newell et al. 2001). The essence of weak wave turbulence is the statistical study of large ensembles of weakly interacting dispersive waves via a systematic asymptotic expansion in powers of small nonlinearity. This technique leads finally to the derivation of wave kinetic equations for quantities like the energy and more generally for the quadratic invariants of the system studied. Here, we will follow the standard Eulerian formalism of wave turbulence (see, e.g., Benney and Newell 1969).

We define the density tensor $q^s_\Lambda(\mathbf{k})$ for a homogeneous turbulence, such that:

$$\langle a^s_\Lambda(\mathbf{k}) a^{s'}_{\Lambda'}(\mathbf{k}')\rangle \equiv q^s_\Lambda(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}')\delta_{\Lambda\Lambda'}\delta_{ss'}, \tag{3.22}$$

for which we shall write a 'closure' equation. The presence of the delta $\delta_{\Lambda\Lambda'}$ and $\delta_{ss'}$ means that correlations with opposite wave or directional polarities have no long-time influence in the wave turbulence regime; the third delta distribution $\delta(\mathbf{k}+\mathbf{k}')$ is the consequence of the homogeneity assumption. It is strongly linked to the form of the frequency ω^s_Λ (see the discussion in Sec. 4.5). Details of the derivation of the wave kinetic equations are given in Appendix B. We obtain the following result:

$$\begin{aligned}
&\partial_t q^s_\Lambda(\mathbf{k}) \\
&\quad = \frac{\pi\epsilon^2}{4d_i^2 B_0^2}\int \sum_{\substack{\Lambda_p,\Lambda_q\\ s_p,s_q}} \left(\frac{\sin\psi_k}{k}\right)^2 (\Lambda k + \Lambda_p p + \Lambda_q q)^2 \\
&\qquad \times (1-\xi_\Lambda^{-s^2}\xi_{\Lambda_p}^{-s_p{}^2}\xi_{\Lambda_q}^{-s_q{}^2})^2 \left(\frac{\xi_{\Lambda_q}^{s_q}-\xi_{\Lambda_p}^{s_p}}{k_\parallel}\right)^2 \left(\frac{\omega^s_\Lambda}{1+\xi_\Lambda^{-s^2}}\right) \\
&\qquad \times \left[\left(\frac{\omega^s_\Lambda}{1+\xi_\Lambda^{-s^2}}\right)\frac{1}{q^s_\Lambda(\mathbf{k})} - \left(\frac{\omega^{s_p}_{\Lambda_p}}{1+\xi_{\Lambda_p}^{-s_p{}^2}}\right)\frac{1}{q^{s_p}_{\Lambda_p}(\mathbf{p})} - \left(\frac{\omega^{s_q}_{\Lambda_q}}{1+\xi_{\Lambda_q}^{-s_q{}^2}}\right)\frac{1}{q^{s_q}_{\Lambda_q}(\mathbf{q})}\right] \\
&\qquad \times q^s_\Lambda(\mathbf{k}) q^{s_p}_{\Lambda_p}(\mathbf{p}) q^{s_q}_{\Lambda_q}(\mathbf{q})\delta(\Omega_{k,pq})\delta_{k,pq}\, d\mathbf{p}\, d\mathbf{q}.
\end{aligned} \tag{3.23}$$

Equation (3.23) is the main result of the helical wave turbulence formalism. It describes the statistical properties of incompressible Hall MHD wave turbulence at the lowest order, i.e. for three-wave interactions.

4. General properties of Hall MHD wave turbulence

4.1. Triadic conservation of ideal invariants

In Sec. 2.3, we have introduced the 3D ideal invariants of incompressible Hall MHD. The first test that the wave kinetic equations have to satisfy is the detailed conservation of these invariants, that is to say, the conservation of invariants for each triad ($\mathbf{k}$,$\mathbf{p}$,$\mathbf{q}$). Starting from definitions (2.6)–(2.8), we note that the Fourier spectra of the ideal invariants are

$$E(\mathbf{k}) = \sum_{\Lambda,s}(1+\xi_\Lambda^{-s^2})q^s_\Lambda(\mathbf{k}), \tag{4.1}$$

$$H_m(\mathbf{k}) = \sum_{\Lambda,s}\frac{\Lambda}{k}q^s_\Lambda(\mathbf{k}), \tag{4.2}$$

$$H_G(\mathbf{k}) = \sum_{\Lambda,s}\frac{\Lambda\xi_\Lambda^{-s^4}}{k}q^s_\Lambda(\mathbf{k}). \tag{4.3}$$

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We first check the energy conservation. From expression (3.23), we find

$$\partial_t E(t) \equiv \partial_t \int E(\mathbf{k})\, d\mathbf{k} \equiv \partial_t \int \sum_{\Lambda,s} e^s_\Lambda(\mathbf{k})\, d\mathbf{k}$$
$$= \frac{\pi\epsilon^2}{4d_{\rm i}^2 B_0^2} \int \sum_{\substack{\Lambda,\Lambda_{\rm p},\Lambda_q \\ s,s_{\rm p},s_q}} \left(\frac{\sin\psi_k}{k}\right)^2 (\Lambda k + \Lambda_{\rm p} p + \Lambda_q q)^2 \left(1 - \xi_\Lambda^{-s\,2} \xi_{\Lambda_p}^{-s_p\,2} \xi_{\Lambda_q}^{-s_q\,2}\right)^2$$
$$\times \left(\frac{\xi^{s_q}_{\Lambda_q} - \xi^{s_p}_{\Lambda_p}}{k_\parallel}\right)^2 \omega^s_\Lambda \left[\frac{\omega^s_\Lambda}{e^s_\Lambda(\mathbf{k})} - \frac{\omega^{s_p}_{\Lambda_p}}{e^{s_{\rm p}}_{\Lambda_{\rm p}}(\mathbf{p})} - \frac{\omega^{s_q}_{\Lambda_q}}{e^{s_q}_{\Lambda_q}(\mathbf{q})}\right]$$
$$\times q^s_\Lambda(\mathbf{k}) q^{s_p}_{\Lambda_p}(\mathbf{p}) q^{s_q}_{\Lambda_q}(\mathbf{q}) \delta(\Omega_{k,pq}) \delta_{k,pq}\, d\mathbf{k}\, d\mathbf{p}\, d\mathbf{q}. \tag{4.4}$$

Equation (4.4) is invariant under cyclic permutations of wavevectors. That leads to

$$\partial_t E(t) = \frac{\pi\,\epsilon^2}{12\,d_{\rm i}^2 B_0^2} \int \sum_{\substack{\Lambda,\Lambda_{\rm p},\Lambda_q \\ s,s_{\rm p},s_q}} \left(\frac{\sin\psi_k}{k}\right)^2 (\Lambda k + \Lambda_{\rm p} p + \Lambda_q q)^2 \left(1 - \xi_\Lambda^{-s\,2} \xi_{\Lambda_p}^{-s_p\,2} \xi_{\Lambda_q}^{-s_q\,2}\right)^2$$
$$\times \left(\frac{\xi^{s_q}_{\Lambda_q} - \xi^{s_p}_{\Lambda_p}}{k_\parallel}\right)^2 \Omega_{k,pq} \left[\frac{\omega^s_\Lambda}{e^s_\Lambda(\mathbf{k})} - \frac{\omega^{s_p}_{\Lambda_p}}{e^{s_{\rm p}}_{\Lambda_{\rm p}}(\mathbf{p})} - \frac{\omega^{s_q}_{\Lambda_q}}{e^{s_q}_{\Lambda_q}(\mathbf{q})}\right]$$
$$\times q^s_\Lambda(\mathbf{k}) q^{s_p}_{\Lambda_p}(\mathbf{p}) q^{s_q}_{\Lambda_q}(\mathbf{q}) \delta(\Omega_{k,pq}) \delta_{k,pq}\, d\mathbf{k}\, d\mathbf{p}\, d\mathbf{q}. \tag{4.5}$$

Total energy is conserved exactly on the resonant manifold since then $\Omega_{k,pq} = 0$: we have triadic conservation of total energy.

For the other invariants, it is straightforward to show that the difference between them is conserved by the wave kinetic equations. With relation (A 5), we obtain

$$\partial_t (H_{\rm m}(t) - H_{\rm G}(t)) \equiv \partial_t \int (H_{\rm m}(\mathbf{k}) - H_{\rm G}(\mathbf{k}))\, d\mathbf{k}$$
$$= \frac{\pi\epsilon^2}{4d_{\rm i} B_0^2} \int \sum_{\substack{\Lambda,\Lambda_{\rm p},\Lambda_q \\ s,s_{\rm p},s_q}} \left(\frac{\sin\psi_k}{k}\right)^2 (\Lambda k + \Lambda_{\rm p} p + \Lambda_q q)^2$$
$$\times \left(1 - \xi_\Lambda^{-s\,2} \xi_{\Lambda_p}^{-s_p\,2} \xi_{\Lambda_q}^{-s_q\,2}\right)^2 \left(\frac{\xi^{s_q}_{\Lambda_q} - \xi^{s_p}_{\Lambda_p}}{k_\parallel}\right)^2$$
$$\times k_\parallel \left[\left(\frac{\omega^s_\Lambda}{1 + \xi_\Lambda^{-s\,2}}\right) \frac{1}{q^s_\Lambda(\mathbf{k})} - \left(\frac{\omega^{s_p}_{\Lambda_p}}{1 + \xi_{\Lambda_p}^{-s_p\,2}}\right) \frac{1}{q^{s_p}_{\Lambda_p}(\mathbf{p})} - \left(\frac{\omega^{s_q}_{\Lambda_q}}{1 + \xi_{\Lambda_q}^{-s_q\,2}}\right) \frac{1}{q^{s_q}_{\Lambda_q}(\mathbf{q})}\right]$$
$$\times q^s_\Lambda(\mathbf{k}) q^{s_p}_{\Lambda_p}(\mathbf{p}) q^{s_q}_{\Lambda_q}(\mathbf{q}) \delta(\Omega_{k,pq}) \delta_{k,pq}\, d\mathbf{k}\, d\mathbf{p}\, d\mathbf{q}. \tag{4.6}$$

Equation (4.4) is also invariant under cyclic permutations of wavevectors. Then one is led to

$$\partial_t (H_{\rm m}(t) - H_{\rm G}(t))$$
$$= \frac{\pi\,\epsilon^2}{12\,d_{\rm i} B_0^2} \int \sum_{\substack{\Lambda,\Lambda_{\rm p},\Lambda_q \\ s,s_{\rm p},s_q}} \left(\frac{\sin\psi_k}{k}\right)^2 (\Lambda k + \Lambda_{\rm p} p + \Lambda_q q)^2$$
$$\times \left(1 - \xi_\Lambda^{-s\,2} \xi_{\Lambda_p}^{-s_p\,2} \xi_{\Lambda_q}^{-s_q\,2}\right)^2 \left(\frac{\xi^{s_q}_{\Lambda_q} - \xi^{s_p}_{\Lambda_p}}{k_\parallel}\right)^2 (k_\parallel - p_\parallel - q_\parallel)$$
$$\times \left[\left(\frac{\omega^s_\Lambda}{1 + \xi_\Lambda^{-s\,2}}\right) \frac{1}{q^s_\Lambda(\mathbf{k})} - \left(\frac{\omega^{s_p}_{\Lambda_p}}{1 + \xi_{\Lambda_p}^{-s_p\,2}}\right) \frac{1}{q^{s_p}_{\Lambda_p}(\mathbf{p})} - \left(\frac{\omega^{s_q}_{\Lambda_q}}{1 + \xi_{\Lambda_q}^{-s_q\,2}}\right) \frac{1}{q^{s_q}_{\Lambda_q}(\mathbf{q})}\right]$$
$$\times q^s_\Lambda(\mathbf{k}) q^{s_p}_{\Lambda_p}(\mathbf{p}) q^{s_q}_{\Lambda_q}(\mathbf{q}) \delta(\Omega_{k,pq}) \delta_{k,pq}\, d\mathbf{k}\, d\mathbf{p}\, d\mathbf{q}, \tag{4.7}$$

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which is exactly equal to zero on the resonant manifold: we have triadic conservation of the difference between magnetic and global helicity. The last sufficient step would be to show detailed conservation for one of the two helicities. Unfortunately it is not so trivial and we will not give the proof here. However, we note already that magnetic helicity is conserved for equilateral triangles ($k = p = q$), i.e. for strongly local interactions.

4.2. General properties

From the wave kinetic equations (3.23), we find several general properties. Some of them can be obtained directly from the wave amplitude equation (3.14) as explained in Sec. 3.2. First, we observe that there is no coupling between Hall MHD waves associated with wavevectors, $\mathbf{p}$ and $\mathbf{q}$, when the wavevectors are collinear ($\sin\psi_k = 0$). Second, we note that there is no coupling between helical waves associated with these vectors whenever the magnitudes, p and q, are equal if their associated polarities, s_p and s_q on one hand, and Λ_p and Λ_q on the other hand, are also equal (since then, $\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p} = 0$). These properties hold for the three inviscid invariants and generalize what was found previously in EMHD (Galtier and Bhattacharjee 2003) where we only have right circularly polarized waves ($\Lambda = s$). It seems to be a generic property of helical wave interactions (Kraichnan 1973; Waleffe 1992; Turner 2000; Galtier 2003). As noted before, this property tends to disappear when the large-scale limit is taken, i.e. when we tend to standard MHD. Third, it follows from the previous observations that a strong helical perturbation localized initially in a narrow band of wavenumbers will lead to a weak transfer of energy, magnetic and global helicities. Note that these properties can be inferred from the fundamental equation (3.14) as well.

4.3. High frequency limit of electron MHD

In the present section we shall demonstrate that the small-scale limit ($d_\mathrm{i}k \to +\infty$) of the wave kinetic equations (3.23) tends to the expected equations of electron MHD when only right ($\Lambda = s$) circularly polarized waves are taken into account. We recall that, in the small-scale limit, $\xi_s^s \to -sd_\mathrm{i}k$ and $\xi_s^{-s} \to s/d_\mathrm{i}k$. One obtains

$$\begin{aligned}\partial_t q_s^s(\mathbf{k}) = \frac{\pi d_\mathrm{i}^2\epsilon^2}{4}\int\sum_{s_\mathrm{p},s_q}\left(\frac{\sin\psi_k}{k}\right)^2(sk+s_\mathrm{p}p+s_q q)^2\left(\frac{s_q q - s_\mathrm{p}p}{k_\parallel}\right)^2 skk_\parallel \\ \times\left[skk_\parallel q_{s_\mathrm{p}}^{s_\mathrm{p}}(\mathbf{p})q_{s_q}^{s_q}(\mathbf{q}) - s_\mathrm{p}pp_\parallel q_s^s(\mathbf{k})q_{s_q}^{s_q}(\mathbf{q}) - s_q qq_\parallel q_s^s(\mathbf{k})q_{s_\mathrm{p}}^{s_\mathrm{p}}(\mathbf{p})\right] \\ \times\delta(\Omega_{k,pq})\delta_{k,pq}\,d\mathbf{p}\,d\mathbf{q},\end{aligned} \tag{4.8}$$

where $\Omega_{k,pq} = B_0 d_\mathrm{i}(sk_\parallel k - s_\mathrm{p}p_\parallel p - s_q q_\parallel q)$. The kinetic equations found have exactly the same form as in Galtier and Bhattacharjee (2003) (where by definition $\omega_s^s = s\omega_k$, $B_0 = 1$ and $d_\mathrm{i} = 1$) who derived a wave turbulence theory for electron MHD (see the discussion in Sec. 5). In particular, this means that by recovering the electron MHD description from the present Hall MHD theory, we recover all the properties already found for whistler wave turbulence (anisotropy, scaling laws, direct cascade etc.). We will come back to that point in Sec. 5. Note finally that there is a strong analogy between such a limit and wave turbulence in rapidly rotating flows (Galtier 2003; Bellet et al. 2005; Morize et al. 2005). The physical reason is that in both problems (i) there is a privileged direction, played by the rotating axis or the magnetic field $\mathbf{B}_0$, (ii) there are dispersive helical waves, called inertial waves for

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rotating flows, and (iii) the wave frequencies are not very different, i.e. the inertial wave frequency is proportional to $k_\parallel/k$. In the past, comparisons have been made between incompressible rotating turbulence and magnetized plasmas described by incompressible MHD. The results obtained in the framework of wave turbulence show clearly that the comparison is much more relevant if one considers electron MHD plasmas. Indeed, a small nonlinear transfer is found along the privileged direction for rotating and electron MHD turbulence whereas this kind of transfer is strictly forbidden in MHD turbulence.

4.4. High frequency limit of ion MHD

We may also be interested by the small-scale limit ($d_\mathrm{i}k \to +\infty$) of the wave kinetic equations (3.23) when only left ($\Lambda = -s$) circularly polarized waves are taken into account. We decide to call this limit the ion MHD (IMHD) approximation following the example of the electron MHD. We have, for such a limit, $\xi^{-s}_{-s} \to s d_\mathrm{i}k$ and $\xi^{s}_{-s} \to -s/d_\mathrm{i}k$. One finds

$$\begin{aligned}\partial_t q^s_{-s}(\mathbf{k}) &= \frac{\pi\, d_\mathrm{i}^2\epsilon^2}{4}\int \sum_{s_\mathrm{p},s_q}\left(\frac{\sin\psi_k}{k}\right)^2 (sk + s_\mathrm{p}p + s_q q)^2\left(\frac{s_q q - s_\mathrm{p}p}{pqk_\parallel}\right)^2 sk_\parallel kp^4q^4\delta(\Omega_{k,pq})\delta_{k,pq}\\ &\times\left[\frac{sk_\parallel}{k^3}q^{s_\mathrm{p}}_{-s_\mathrm{p}}(\mathbf{p})q^{s_q}_{-s_q}(\mathbf{q}) - \frac{s_\mathrm{p}p_\parallel}{p^3}q^{s}_{-s}(\mathbf{k})q^{s_q}_{-s_q}(\mathbf{q}) - \frac{s_q q_\parallel}{q^3}q^{s}_{-s}(\mathbf{k})q^{s_\mathrm{p}}_{-s_\mathrm{p}}(\mathbf{p})\right]d\mathbf{p}\,d\mathbf{q}, \quad (4.9)\end{aligned}$$

with $\Omega_{k,pq} = (B_0/d_\mathrm{i})(sk_\parallel/k - s_\mathrm{p}p_\parallel/p - s_q q_\parallel/q)$. No existing theory has been developed for such a limit. We will see in Sec. 5 that it is possible to extract from the master equations of ion cyclotron turbulence some exact properties.

4.5. Low frequency limit of standard MHD

This section is devoted to the low frequency limit, i.e. the large-scale limit, of the wave kinetic equations (3.23) of Hall MHD. The MHD limit is somewhat singular for Hall MHD. Indeed the large-scale limit does not tend to the expected wave kinetic equations that were derived first by Galtier et al. (2000). The subtle point resides in the kinematics: for pseudo-dispersive MHD waves, the definition (3.22) of the density tensor $q^s_\Lambda(\mathbf{k})$ that is used for Hall MHD is no longer valid. The reason is that in the large-scale limit the polarization Λ no longer appears in the frequency, which is $\omega = sk_\parallel B_0$. Then the kinematics tells us that the definition for the density tensor is

$$\langle a^s_\Lambda(\mathbf{k})a^{s'}_{\Lambda'}(\mathbf{k}')\rangle \equiv \tilde{q}^{ss'}_{\Lambda\Lambda'}\delta(\mathbf{k}+\mathbf{k}')\delta_{ss'}, \quad (4.10)$$

where the condition $\Lambda = \Lambda'$ is not necessary satisfied. This means that the large-scale limit of (3.23) leads to MHD wave kinetic equations in the particular case where helicity terms are supposed to be absent and where equality between shear-Alfvén and pseudo-Alfvén wave energy is assumed. Indeed, helicity terms involve quantities for which $\Lambda = -\Lambda'$ and the total energy involves only terms for which $\Lambda = \Lambda'$. However, energies for shear-Alfvén waves and pseudo-Alfvén waves involve terms with different polarities. For shear-Alfvén waves, we have

$$\mathbf{v}_\mathbf{k} - s\mathbf{b}_\mathbf{k} = \mathscr{Z}^s_+\mathbf{h}^+_\mathbf{k} + \mathscr{Z}^s_-\mathbf{h}^-_\mathbf{k} = (\mathscr{Z}^s_+ - \mathscr{Z}^s_-)i\hat{\mathbf{e}}_\Phi, \quad (4.11)$$

and for pseudo-Alfvén waves

$$\mathbf{v}_\mathbf{k} - s\mathbf{b}_\mathbf{k} = \mathscr{Z}^s_+\mathbf{h}^+_\mathbf{k} + \mathscr{Z}^s_-\mathbf{h}^-_\mathbf{k} = (\mathscr{Z}^s_+ + \mathscr{Z}^s_-)\hat{\mathbf{e}}_\theta. \quad (4.12)$$

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Energies associated to shear- and pseudo-Alfvén waves are, respectively,

$$(\mathscr{Z}^s_+ - \mathscr{Z}^s_-)(\mathscr{Z}^s_+ - \mathscr{Z}^s_-)^* = |\mathscr{Z}^s_+|^2 + |\mathscr{Z}^s_-|^2 - \mathscr{Z}^s_+(\mathscr{Z}^s_-)^* - \mathscr{Z}^s_-(\mathscr{Z}^s_+)^*, \tag{4.13}$$

and

$$(\mathscr{Z}^s_+ + \mathscr{Z}^s_-)(\mathscr{Z}^s_+ + \mathscr{Z}^s_-)^* = |\mathscr{Z}^s_+|^2 + |\mathscr{Z}^s_-|^2 + \mathscr{Z}^s_+(\mathscr{Z}^s_-)^* + \mathscr{Z}^s_-(\mathscr{Z}^s_+)^*. \tag{4.14}$$

We see clearly that if, in the wave kinetic equations, we only take into account (quadratic) terms with the same polarizations (Λ) then it is equivalent to assume equality between shear- and pseudo-Alfvén wave energies. We will see below that the large-scale limit of the wave kinetic equations (3.23) of Hall MHD tends to the expected MHD counterpart when the previous assumptions about helicities and energies are satisfied. A derivation of the wave kinetic equations is given in Appendix C. The result is given for the density tensor $q^{ss'}_{\Lambda\Lambda'}$. In contrast to Hall MHD, and actually to any problem in wave turbulence, principal value terms appear for incompressible MHD. The reason for the presence of principal value terms is linked to the nature of Alfvén waves which are pseudo-dispersive.

Further comparisons between the results in Appendix C and the wave kinetic equations obtained by Galtier et al. (2000) are of course possible (see, e.g., the discussion in Sec. 5) but, for simplicity, we prefer to focus our attention to the case where the density tensor is symmetric in Λ. Therefore, we start our analysis with the general kinetic equation (3.23) and take the large-scale limit ($d_i k \to 0$) for which we have, at the leading order, $\xi^s_\Lambda \to -s - \Lambda d_i k/2$. After some simplifications, we arrive at

$$\begin{aligned}\partial_t q^s_\Lambda(\mathbf{k}) = \frac{\pi\,\epsilon^2}{16}\int \sum_{\substack{\Lambda_p,\Lambda_q\\ s_p,s_q}} \left(\frac{\sin\psi_k}{k}\right)^2 (\Lambda k + \Lambda_p p + \Lambda_q q)^2 \\ \times (sk\Lambda + s_p p\Lambda_p + s_q q\Lambda_q)^2 \left(\frac{s_q - s_p}{k_\parallel}\right)^2 \\ \times sk_\parallel (sk_\parallel q^{s_p}_{\Lambda_p}(\mathbf{p}) q^{s_q}_{\Lambda_q}(\mathbf{q}) - s_p p_\parallel q^s_\Lambda(\mathbf{k}) q^{s_q}_{\Lambda_q}(\mathbf{q}) - s_q q_\parallel q^s_\Lambda(\mathbf{k}) q^{s_p}_{\Lambda_p}(\mathbf{p})) \\ \times \delta(\Omega_{k,pq})\delta_{k,pq}\, d\mathbf{p}\, d\mathbf{q},\end{aligned} \tag{4.15}$$

where $\Omega_{k,pq} = B_0(sk_\parallel - s_p p_\parallel - s_q q_\parallel)$. Note that we only have a nonlinear contribution when the wave polarities s_p and s_q are different. We recover here a well-known property of incompressible MHD: in such a limit, we only have nonlinear interactions between Alfvén waves propagating in different directions. One expands the summation over the directional polarities s_p and s_q, and finds

$$\begin{aligned}\partial_t q^s_\Lambda(\mathbf{k}) = \frac{\pi\epsilon^2}{4B_0}\int \sum_{\Lambda_p,\Lambda_q} \left(\frac{\sin\psi_k}{k}\right)^2 (\Lambda k + \Lambda_p p + \Lambda_q q)^2 (\Lambda k + \Lambda_p p - \Lambda_q q)^2 \\ \times q^{-s}_{\Lambda_q}(\mathbf{q})[q^s_{\Lambda_p}(\mathbf{p}) - q^s_\Lambda(\mathbf{k})]\delta(q_\parallel)\delta_{k,pq}\, d\mathbf{p}\, d\mathbf{q}.\end{aligned} \tag{4.16}$$

This result is exactly the same as in Appendix D when MHD wave kinetic equations are considered in the particular case where only terms symmetric in Λ are retained, i.e. terms such as $\tilde{q}^{ss}_{\Lambda\Lambda}$. Therefore, under these assumptions the MHD description does not appear like a singular limit for Hall MHD. (Note that this continuity in the description is *a priori* not so trivial if we remember that as soon as the Hall term is included in the standard MHD equations, whatever its magnitude is, it changes the polarity of the Alfvén waves which become circularly polarized.)

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The comparison with the wave kinetic equations derived by Galtier et al. (2000) (see their equations (26)) is not direct since here the problem has been decomposed at the origin on a complex helicity basis. The main signature of this decomposition is the dependency on the wave polarity Λ. In spite of this difficulty, a common point is clearly seen in the presence of the delta $\delta(q_\parallel)$ which arises because of the three-wave frequency resonance condition. This means that in any triadic resonant interaction, there is always one wave that corresponds to a purely 2D motion ($q_\parallel = 0$) whereas the two others have equal parallel components ($p_\parallel = k_\parallel$). The direct consequence is the absence of nonlinear transfer along $\mathbf{B}_0$, a result predicted earlier by several authors (see, e.g., Montgomery and Turner 1981; Shebalin et al. 1983). In other words, we have a two-dimensionalization of the Alfvén wave turbulence (see also e.g. Ng and Bhattacharjee 1997; Lithwick and Goldreich 2003).

5. Master equations of Hall MHD turbulence

5.1. General case

In order to extract further information about Hall MHD turbulence, we are going to write the expression of the spectral density $q^s_\Lambda(\mathbf{k})$ in terms of the invariants. In practice, we need to add a fourth variable, E_d, which has been chosen to be the difference between the kinetic and magnetic energy, namely

$$E_\mathrm{d}(\mathbf{k}) = \sum_{\Lambda,s} (\xi_\Lambda^{-s\,2} - 1) q^s_\Lambda(\mathbf{k}). \tag{5.1}$$

Note that contrary to the MHD case, wave turbulence in Hall MHD allows us to have a departure from equipartition between kinetic and magnetic energy and therefore a non-trivial value for E_d. We will see, in Sec. 7, that this property is fundamental to get non-trivial non-local interactions between waves. It is possible to inverse the system $q^s_\Lambda(E, E_\mathrm{d}, H_\mathrm{m}, H_\mathrm{G})$; one finds

$$q^s_\Lambda(\mathbf{k}) = \frac{(\xi_\Lambda^{s\,2} + \xi_\Lambda^{-s\,2})[(\xi_\Lambda^{s\,2} - 1)E(\mathbf{k}) - (\xi_\Lambda^{s\,2} + 1)E_\mathrm{d}(\mathbf{k})] + 2\Lambda k(\xi_\Lambda^{s\,4} H_\mathrm{m}(\mathbf{k}) - H_\mathrm{G}(\mathbf{k}))}{4(\xi_\Lambda^{s\,4} - \xi_\Lambda^{-s\,4})}. \tag{5.2}$$

The introduction of expression (5.2) into (3.23) leads to the wave kinetic equations for E, E_d, H_m and H_G. We will not make such a lengthy development and we will rather focus on the master equations which drive the Hall MHD turbulence. Indeed, because of the presence of the factor Λ in expression (5.2), we see that the nonlinear terms with different polarities will not play the same role. In particular, for the wave kinetic equations of energies, only the interactions between nonlinear terms involving either energies or helicities will give a contribution. For the wave kinetic equations of helicities, we only have contributions from nonlinear terms involving energies and helicities. Thus the energy equations are the master equations driving Hall MHD turbulence. In other words, this means that an initial state with zero helicity will not generate any helicity at any scale. However, an initial state of zero helicity does not preclude the development of energy spectra. Assuming $H_\mathrm{m}(\mathbf{k}) = 0$ and $H_\mathrm{G}(\mathbf{k}) = 0$, the master kinetic equations of Hall MHD turbulence, written for

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the kinetic energy E^V and the magnetic energy E^B, are

$$\partial_t \begin{Bmatrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{Bmatrix} = \frac{\pi\epsilon^2}{8 d_\mathrm{i}^2 B_0^2} \int \sum_{\substack{\Lambda,\Lambda_\mathrm{p},\Lambda_q \\ s,s_\mathrm{p},s_q}} \left(\frac{\sin\psi_k}{k}\right)^2 \frac{(\Lambda k + \Lambda_\mathrm{p} p + \Lambda_q q)^2 \left(1 - \xi_\Lambda^{-s2} \xi_{\Lambda_p}^{-s_p 2} \xi_{\Lambda_q}^{-s_q 2}\right)^2}{(1+\xi_\Lambda^{-s2})(1+\xi_{\Lambda_p}^{-s_p 2})(1+\xi_{\Lambda_q}^{-s_q 2})}$$
$$\times \left(\frac{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p}}{k_\parallel}\right)^2 \begin{Bmatrix} \xi_\Lambda^{-s2} \\ 1 \end{Bmatrix} \frac{\omega_\Lambda^s \omega_{\Lambda_p}^{s_p}}{\xi_\Lambda^{-s2} + 1} \left(\frac{\xi_{\Lambda_q}^{-s_q 2} E^V(\mathbf{q}) - E^B(\mathbf{q})}{\xi_{\Lambda_q}^{-s_q 2} - 1}\right)$$
$$\times \left[\left(\frac{\xi_{\Lambda_p}^{-s_p 2} E^V(\mathbf{p}) - E^B(\mathbf{p})}{\xi_{\Lambda_p}^{-s_p 2} - 1}\right) - \left(\frac{\xi_\Lambda^{-s2} E^V(\mathbf{k}) - E^B(\mathbf{k})}{\xi_\Lambda^{-s2} - 1}\right)\right] \delta(\Omega_{k,pq}) \delta_{k,pq}\, d\mathbf{p}\, d\mathbf{q}. \tag{5.3}$$

In Appendix E the equivalent kinetic equations for the variables E and E_d are given. Below we describe the small-scale (whistler and ion cyclotron waves) and large-scale (pure Alfvén waves) limits of such a description. In particular, we will see that the role played by the kinetic and magnetic energies may be very different and that the master equations can be simplified further.

5.2. Master equations in the limit of whistler wave turbulence

In the small-scale limit ($d_\mathrm{i} k \to +\infty$), one can distinguish between the whistler branch and the ion cyclotron branch. For right circularly polarized wave ($\Lambda = s$) we have $\xi_\Lambda^{-s2} \to 1/d_\mathrm{i}^2 k^2$, whereas for left circularly polarized wave ($\Lambda = -s$), $\xi_\Lambda^{-s2} \to d_\mathrm{i}^2 k^2$. Then in the limit of whistler wave turbulence (EMHD limit), we obtain

$$\partial_t \begin{Bmatrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{Bmatrix} = \frac{\pi\epsilon^2 d_\mathrm{i}^2}{8} \int \sum_{s,s_\mathrm{p},s_q} \left(\frac{\sin\psi_k}{k}\right)^2 (sk + s_\mathrm{p} p + s_q q)^2 \left(\frac{s_q q - s_\mathrm{p} p}{k_\parallel}\right)^2$$
$$\times \begin{Bmatrix} 1/d_\mathrm{i}^2 k^2 \\ 1 \end{Bmatrix} s k_\parallel k s_\mathrm{p} p_\parallel p E^B(\mathbf{q}) [E^B(\mathbf{p}) - E^B(\mathbf{k})] \delta(\Omega_{k,pq}) \delta_{k,pq}\, d\mathbf{p}\, d\mathbf{q}, \tag{5.4}$$

with $\Omega_{k,pq} = B_0 d_\mathrm{i} (s k_\parallel k - s_\mathrm{p} p_\parallel p - s_q q_\parallel q)$. We see clearly that the kinetic energy is not a relevant quantity in such a limit since the factor $1/d_\mathrm{i}^2 k^2$ damps the nonlinear contributions. This is not in contradiction with what we know if we remember that the velocity in the original Hall MHD equations is a combination of the ion and electron velocities. In the EMHD limit, i.e. in the small spatio-temporal scale limit, this velocity tends to be small: ions do not have time to follow electrons and they provide a static homogeneous background on which electrons move. Since the magnetic field is frozen into the ideal electron flow (see Sec. 2.3 for the proof) we understand why the magnetic energy is the dominant contribution to the wave kinetic equations. The wave kinetic equations found for the magnetic energy have, at leading order, exactly the same form as in Galtier and Bhattacharjee (2003) (where $\omega_s^s = s\omega_k$ and $B_0 = 1$) but they are more general in the sense that we have at our disposal not only an equation for the magnetic field, the dominant contribution, but also for the velocity that behaves like the magnetic field. The consequence is that we recover all the properties already known. In particular, in

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such a regime the energy spectrum follows the power law $E(k_\perp,k_\parallel)\sim k_\perp^{-5/2}k_\parallel^{-1/2}$ ($k_\parallel$ is assumed positive) which is an exact solution of the wave kinetic equations.

5.3. Master equations in the limit of ion cyclotron wave turbulence

For the ion cyclotron branch ($\Lambda=-s$), the small-scale limit ($d_\mathrm{i}k\to+\infty$) gives

$$\partial_t\begin{Bmatrix}E^V(\mathbf{k})\\E^B(\mathbf{k})\end{Bmatrix}=\frac{\pi\epsilon^2}{8d_\mathrm{i}^2}\int\sum_{s,s_\mathrm{p},s_q}\left(\frac{\sin\psi_k}{k}\right)^2(sk+s_\mathrm{p}p+s_qq)^2\left(\frac{s_\mathrm{p}p-s_qq}{k_\parallel pq}\right)^2 \times\begin{Bmatrix}d_\mathrm{i}^2k^2\\1\end{Bmatrix}\frac{sk_\parallel s_\mathrm{p}p_\parallel pq^2}{k}E^V(\mathbf{q})[E^V(\mathbf{p})-E^V(\mathbf{k})]\delta(\Omega_{k,pq})\delta_{k,pq}\,d\mathbf{p}\,d\mathbf{q}, \tag{5.5}$$

with $\Omega_{k,pq}=(B_0/d_\mathrm{i})(sk_\parallel/k-s_\mathrm{p}p_\parallel/p-s_qq_\parallel/q)$. In such a limit, we are able to describe both the kinetic and the magnetic energies but we see that the dominant quantity is the kinetic energy since the factor $d_\mathrm{i}^2k^2$ amplifies the nonlinear contributions in the corresponding equation (see, e.g., Mahajan and Krishan 2005). The physical reason is that in such a limit, we are mainly dealing with ions whereas the magnetic field is frozen into the ideal electron flow (see Sec. 2.3). As for the EMHD limit, it is possible to derive the exact power law solutions. First of all, we note that for strongly local interactions the resonance condition is written as

$$\left(\frac{s_\mathrm{p}-s_q}{k_\parallel}\right)^2\approx\left(\frac{s_q-s}{p_\parallel}\right)^2\approx\left(\frac{s-s_\mathrm{p}}{q_\parallel}\right)^2, \tag{5.6}$$

which leads, as for the whistler waves case, to anisotropic turbulence. The same analysis for strongly non-local interactions leads to the same conclusion. Thus, we may write the wave kinetic equations in the limit $k_\perp\gg k_\parallel$. For an axisymmetric turbulence $E^V(\mathbf{k}_\perp,k_\parallel)=E^V(k_\perp,k_\parallel)/2\pi k_\perp$; we find

$$\begin{aligned}\partial_tE^V(k_\perp,k_\parallel)=\frac{\epsilon^2}{16}\int\sum_{s,s_\mathrm{p},s_q}&\sin\psi_{q_\perp}(sk_\perp+s_\mathrm{p}p_\perp+s_qq_\perp)^2\left(\frac{s_\mathrm{p}p_\perp-s_qq_\perp}{k_\parallel p_\perp q_\perp}\right)^2sk_\parallel s_\mathrm{p}p_\parallel\\&\times E^V(q_\perp,q_\parallel)[k_\perp E^V(p_\perp,p_\parallel)-p_\perp E^V(k_\perp,k_\parallel)]\\&\times\delta(\Omega_{k,pq})\delta_{k_\parallel,p_\parallel q_\parallel}dp_\perp dq_\perp dp_\parallel dq_\parallel,\end{aligned} \tag{5.7}$$

where $\Omega_{k,pq}=(B_0/d_\mathrm{i})(sk_\parallel/k_\perp-s_\mathrm{p}p_\parallel/p_\perp-s_qq_\parallel/q_\perp)$. Note that in such a limit the well-known triangle relations are: $\sin\psi_{k_\perp}/k_\perp=\sin\psi_{p_\perp}/p_\perp=\sin\psi_{q_\perp}/q_\perp$. The wave kinetic equation (5.7) is now symmetric enough to apply a conformal transformation, called the Kuznetsov–Zakharov transformation. This transformation, a 2D generalization of the Zakharov transformation, has been applied in various anisotropic problems (Kuznetsov 1972, 2001; Balk et al. 1990; Galtier and Bhattacharjee 2003; Galtier 2003). The bihomogeneity of the collision integrals in the wavenumbers $k_\perp$ and $k_\parallel$ allows us to use the transformation

$$\begin{aligned}p_\perp&\to k_\perp^2/p_\perp,\\q_\perp&\to k_\perp q_\perp/p_\perp,\\p_\parallel&\to k_\parallel^2/p_\parallel,\\q_\parallel&\to k_\parallel q_\parallel/p_\parallel.\end{aligned} \tag{5.8}$$

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We search for stationary solutions in the power law form $E(k_\perp, k_\parallel) \sim k_\perp^{-n} k_\parallel^{-m}$. (We will only consider positive parallel wavenumber.) The new form of the collision integral of (5.7), resulting from the summation of the integrand in its primary form and after the Kuznetsov–Zakharov transformation, is

$$\begin{aligned}\partial_t E^V(k_\perp, k_\parallel) = -\frac{\epsilon^2}{32} \int \sum_{s, s_\mathrm{p}, s_q} \sin\psi_{q_\perp} (s k_\perp + s_\mathrm{p} p_\perp + s_q q_\perp)^2 \left(\frac{s_\mathrm{p} p_\perp - s_q q_\perp}{k_\parallel p_\perp q_\perp}\right)^2 \\ \times s k_\parallel s_\mathrm{p} p_\parallel \delta_{k_\parallel, p_\parallel q_\parallel} \delta(\Omega_{k,pq}) k_\perp^{-n} k_\parallel^{-m} p_\perp q_\perp^{-n} q_\parallel^{-m} \\ \times \left(1 - \left(\frac{p_\perp}{k_\perp}\right)^{-n-1} \left(\frac{p_\parallel}{k_\parallel}\right)^{-m}\right) \left(1 - \left(\frac{p_\perp}{k_\perp}\right)^{2n-5} \left(\frac{p_\parallel}{k_\parallel}\right)^{2m-1}\right) \\ \times dp_\perp\, dq_\perp\, dp_\parallel\, dq_\parallel. \qquad (5.9)\end{aligned}$$

The above collision integral vanishes for specific values of n and m. The exact power law solutions correspond to these values. There are two different kinds of solutions. The fluxless solution, also called the thermodynamic equilibrium solution, corresponds to the equipartition state for which the flux of energy is zero. For this case, we have

$$\begin{aligned} n &= -1, \\ m &= 0. \end{aligned} \qquad (5.10)$$

This result can easily be checked by direct substitution in the original wave kinetic equation. The most interesting solution of the wave kinetic equation (5.9) is the one for which the flux is non-zero and finite. The exact solution is called the Kuznetsov–Zakharov–Kolmogorov (KZK) spectrum and corresponds to the values,

$$\begin{aligned} n &= \tfrac{5}{2}, \\ m &= \tfrac{1}{2}. \end{aligned} \qquad (5.11)$$

In other words, the KZK solution scales as

$$E^V(k_\perp, k_\parallel) \sim k_\perp^{-5/2} k_\parallel^{-1/2}. \qquad (5.12)$$

We note that the power law scaling found is the same as the one for whistler wave turbulence. However, here we are dealing with the kinetic energy not the magnetic one. A necessary condition for the realizability of the KZK spectra is that the turbulence be local in the sense that the behavior of the turbulence is determined primarily (but not only) by interaction between wave packets of comparable spatial scale (see, e.g., Balk and Nazarenko 1990). To check *a posteriori* the validity of the solutions, we need to determine the domain of locality, i.e., the domain where the collision integral converges. In practice, we check that the contribution of non-local interactions does not lead to a divergence of the collision integral. Here, the condition of locality is automatically satisfied since the anisotropic limit introduces in the problem a cut-off which prevents any infra-red divergence of the collision integral. This characteristic is also observed for internal gravity waves (Caillol and Zeitlin 2000), inertial waves (Galtier 2003) and whistler waves (Galtier and Bhattacharjee 2003). The other consequence of the existence of such a cut-off is that it is not possible to evaluate precisely the Kolmogorov constant, i.e. the prefactor of the spectra. However, the sign of the energy transfer can be computed for a

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reasonable range of cut-offs. We observe a positive sign which means that we have a direct energy cascade. Note that this information cannot be obtained through a heuristic reasoning but only with a rigorous analysis.

5.4. Master equations in the limit of pure Alfvén wave turbulence

In the large-scale limit ($d_i k \to 0$) for which terms such as $\xi_\Lambda^{-s^2}$ tend to 1, we note that an equipartition between the kinetic and magnetic energies may be obtained since their wave kinetic equations tend to be identical: if initially there is equipartition ($E^V(\mathbf{k}) = E^B(\mathbf{k}) = E(\mathbf{k})/2$) then at any time the equipartition will be conserved. The large-scale limit corresponds to the standard MHD approximation for which the equipartition is in fact automatically satisfied by the kinematics. The large-scale limit of expression (5.3), for a state of equipartition, gives

$$\partial_t E(\mathbf{k}) = \frac{\pi\epsilon^2}{256}\int \sum_{\substack{\Lambda,\Lambda_\mathrm{p},\Lambda_q\\ s,s_\mathrm{p},s_q}} \left(\frac{\sin\psi_k}{k}\right)^2 (\Lambda k + \Lambda_\mathrm{p} p + \Lambda_q q)^2 (s\Lambda k + s_\mathrm{p}\Lambda_\mathrm{p} p + s_q\Lambda_q q)^2$$
$$\times \left(\frac{s_q - s_\mathrm{p}}{k_\parallel}\right)^2 s k_\parallel s_\mathrm{p} p_\parallel E(\mathbf{q})[E(\mathbf{p}) - E(\mathbf{k})]\delta(\Omega_{k,pq})\delta_{k,pq}\, d\mathbf{p}\, d\mathbf{q}, \qquad (5.13)$$

with $\Omega_{k,pq} = B_0(sk_\parallel - s_\mathrm{p} p_\parallel - s_q q_\parallel)$. As expected, only nonlinear interactions between Alfvén waves with different directional polarities $s_\mathrm{p} = -s_q$ will contribute to the dynamics. After such a consideration and an expansion over the polarities, one finds

$$\partial_t E(\mathbf{k}) = \frac{\pi\epsilon^2}{4B_0}\int \sin^2\psi_k (1+\cos^2\psi_q) p^2 E(\mathbf{q})[E(\mathbf{p}) - E(\mathbf{k})]\delta(q_\parallel)\delta_{k,pq}\, d\mathbf{p}\, d\mathbf{q}, \quad (5.14)$$

where we have used the identities

$$\sum_{\Lambda,\Lambda_\mathrm{p},\Lambda_q} (\Lambda k + \Lambda_\mathrm{p} p + \Lambda_q q)^2 (\Lambda k + \Lambda_\mathrm{p} p - \Lambda_q q)^2$$
$$= 4[((k+p)^2 - q^2)^2 + ((k-p)^2 - q^2)^2]$$
$$= 32 k^2 p^2 (1 + \cos^2\psi_q), \qquad (5.15)$$

and the well-known triangle relations. The angle ψ_q refers to the angle opposite to $\mathbf{q}$ in the triangle defined by $\mathbf{k} = \mathbf{p} + \mathbf{q}$. (The same result can actually be found directly from (4.16); the inversion of the system made in Sec. 5.1 is thus compatible with such a limit.) As noted before, the large-scale limit of Hall MHD leads to the particular case where helicities are absent and where shear- and pseudo-Alfvén wave energies are the same. Equation (5.14) can be recovered from the general kinetic equations obtained by Galtier et al. (2000) when the same assumptions are made. The proof is given in Appendix F. Therefore, all the properties previously derived are recovered (scaling laws, direction of the cascade, Kolmogorov constant etc.). In particular, the energy spectrum follows the power law $E(k_\perp, k_\parallel) \sim k_\perp^{-2} f(k_\parallel)$, where f is an arbitrary function that is due to the dynamical decoupling of parallel planes in Fourier space. This is an exact solution of the wave kinetic equations.

6. Anisotropic heuristic description for Hall MHD

The study of wave turbulence in Hall MHD is a difficult task. We have seen above that for three specific limits we are able to find the exact power law energy spectra. Two of them (Alfvén and whistler wave turbulence) were already known

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and a heuristic description was given by Galtier et al. (2000) and Galtier and Bhattacharjee (2003). We shall derive here a generalized heuristic description able to recover the essential physics underlying the KZK spectra including the ion cyclotron wave turbulence. However, note that several phenomenologies may be used to take into account the various regimes involving a non-trivial balance between propagation and nonlinear effects as for standard MHD (see, e.g., Zhou et al. 2004).

We introduce anisotropy in our description by considering that

$$k \approx k_\perp \gg k_\parallel. \tag{6.1}$$

The primary variables used in the formalism are the generalized Elsässer variables $\mathscr{Z}^s_\Lambda$. Thus the nonlinear time built on the generalized Elsässer variables is

$$\tau_{\rm NL} \sim \frac{1}{k_\perp \mathscr{Z}^s_\Lambda}. \tag{6.2}$$

Note here that $\mathscr{Z}^s_\Lambda$ has a dimension of a velocity. In other words, it is *not* taken in Fourier space as it was introduced in Sec. 2.5. In a similar way, we find the following Hall MHD wave period

$$\tau_{\rm w} \sim \frac{1}{\omega^s_\Lambda} = \frac{1}{-B_0 k_\parallel \xi^s_\Lambda}. \tag{6.3}$$

We introduce now the mean rate of energy dissipation per unit mass Π. Contrary to the electron MHD case, we do not have to renormalize this quantity since it is automatically taken into account by the generalized Elsässer variables. Then we have

$$\Pi \sim \frac{E}{\tau_{\rm tr}} \sim \frac{E(k_\perp, k_\parallel) k_\perp k_\parallel}{\tau_{\rm tr}}, \tag{6.4}$$

where the transfer time $\tau_{\rm tr}$ has the usual form given by the wave kinetic equation

$$\tau_{\rm tr} \sim \tau_{\rm NL} \frac{\tau_{\rm NL}}{\tau_{\rm w}}; \tag{6.5}$$

it gives

$$\Pi \sim \frac{E(k_\perp, k_\parallel) k_\perp^3 {\mathscr{Z}^s_\Lambda}^2}{-B_0 \xi^s_\Lambda}. \tag{6.6}$$

To proceed further, we note that

$$(\mathscr{Z}^s_\Lambda)^2 \sim (\xi^s_\Lambda - \xi^{-s}_\Lambda)^2 {a^s_\Lambda}^2, \tag{6.7}$$

and

$$E \sim (1 + \xi^{-s2}_\Lambda) {a^s_\Lambda}^2. \tag{6.8}$$

In relations (6.7) and (6.8), both $\mathscr{Z}^s_\Lambda$ and a^s_Λ are written in the physical space, *not* in Fourier space. By using the relationships given in Appendix A, we finally obtain

$$E(k_\perp, k_\parallel) \sim \sqrt{\Pi B_0} k_\perp^{-2} k_\parallel^{-1/2} (1 + k_\perp^2 d_{\rm i}^2)^{-1/4}. \tag{6.9}$$

The heuristic prediction proposed here is able to describe anisotropic turbulence for the three different limits discussed above. We recover, in the small-scale limit ($k_\perp d_{\rm i} \to \infty$), the expected scaling law for whistler as well as ion cyclotron wave turbulence and, in the large-scale limit ($k_\perp d_{\rm i} \to 0$), the Alfvén wave turbulence scaling law (since then the parallel wavenumber is a mute variable). The prediction is given for the total energy, therefore for the previous small-scale limits one needs to consider only the magnetic or the kinetic energy respectively. One may understand this point by looking at (2.31) where the relation between the generalized Elsässer variables, the velocity and the magnetic field is given. In the small-scale limit, we

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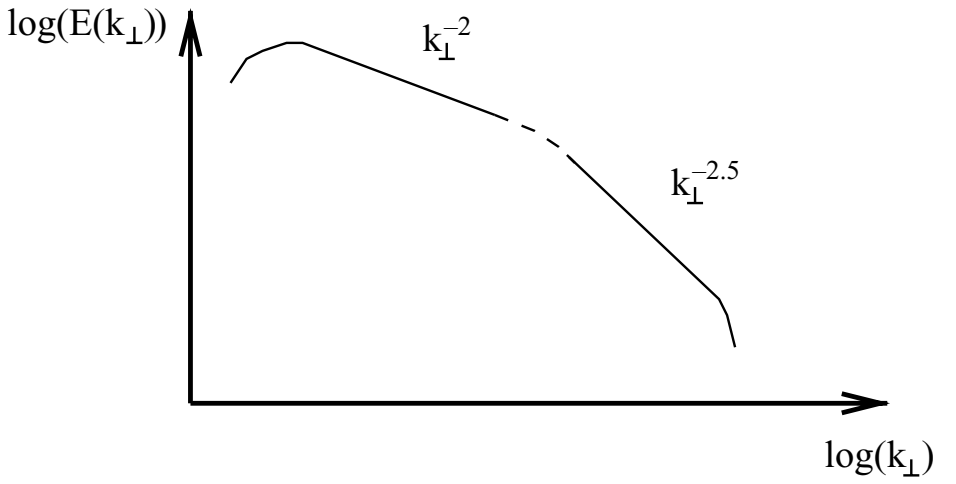


Figure 2. Sketch of the power law spectrum, at a given $k_\parallel$, expected for incompressible Hall MHD in the regime of wave turbulence.

have for whistler waves $\xi^s_\Lambda \to -sd_i k$ and for ion cyclotron waves $\xi^s_\Lambda \to (-sd_i k)^{-1}$ (whereas in the large scale limit $\xi^s_\Lambda \to -s$); thus we see that either the kinetic or the magnetic field will dominate. Note that we ignore anisotropy and assume $k_\perp \sim k_\parallel \sim k$, we arrive at the scaling

$$E(k) \sim \sqrt{\Pi B_0} k^{-3/2} (1 + k^2 d_i^2)^{-1/4} \tag{6.10}$$

for the one-dimensional isotropic spectrum from which one recognizes the Iroshnikov (1963) and Kraichnan (1965) prediction for MHD.

To summarize our finding we propose the picture given in Fig. 2. It is a sketch of the power law energy spectrum predictions for perpendicular wavenumbers at a given $k_\parallel$. The energy spectrum of Hall MHD is characterized by two inertial ranges—the exact power law solutions of the wave kinetic equations—separated by a knee. The position of the knee corresponds to the scale where the Hall term becomes (sub-) dominant, i.e. when $k_\perp d_i \sim 1$. The heuristic prediction tries to make the link continuously (dashed line in Fig. 2) between these power laws but the heuristic spectrum may not be correct at intermediate scales since, in particular, Hall MHD turbulence is not necessarily anisotropic. However, we will show, in Sec. 8, that even at intermediate scales a moderate anisotropy is expected. The presence of a knee in the Solar wind spectrum is well attested by *in situ* measurements of magnetic fluctuations (Coroniti et al. 1982; Denskat et al. 1983, Leamon et al. 1998; Bale et al. 2005). We will discuss this point in Sec. 10; we will also comment on the fact that the comparison with observations is not direct, in particular, for the high frequency part of the spectrum. One reason is that the Taylor hypothesis, usually used at low frequency, is not applicable anymore.

7. Non-local interactions

7.1. Introduction

Non-local interactions may play a significant role in turbulence (see, e.g., Laval et al. 2001). By non-local interactions, we mean interactions between well-separated scales. For triadic processes, it corresponds to the interaction of two highly

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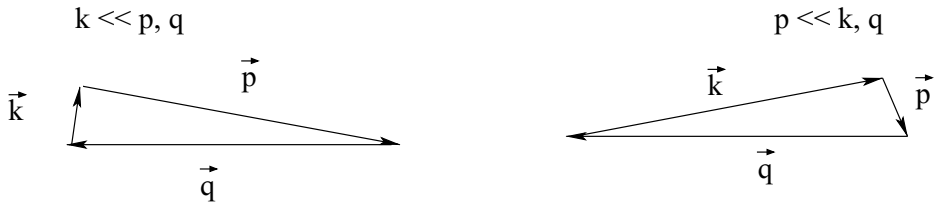


Figure 3. Non-local triadic interactions.

elongated wavevectors with a third small wavevector. In Hall MHD the study of non-local interactions and non-local transfers is particularly interesting since the turbulent flow behaves very differently at different scales (see Mininni et al. 2005). At the larger scales it follows the MHD dynamics whereas at the smallest scales the flow may follow, for example, the EMHD dynamics. For simplicity, we will focus our analysis on the master equations (5.3). Therefore, we will neglect the helicity effects and keep only the kinetic and magnetic energy terms. According to the situation, we will consider two different kinds of non-local interactions, namely $k \ll p, q$ or $p \ll k, q$ (see Fig. 3). In practice, small-scale and large-scale effects will be effective when, respectively, the limits $kd_\mathrm{i} \to +\infty$ and $kd_\mathrm{i} \to 0$ will be taken. Therefore, we see that large scale effects (small wavenumbers) will be driven by Alfvén waves (A) whereas small scale effects (large wavenumbers) will be driven by whistler (W) or ion cyclotron (C) waves. Note that we will not study non-local interactions between the same type of waves (AAA, WWW or CCC interactions) which are somewhat more restrictive. Under such a condition, three different families of interactions may happen, namely WWA, CCA and WCA. We describe below such interactions.

7.2. WWA interactions

First we shall consider the strongly non-local interactions of two whistler waves on one Alfvén wave. In other words, it means that the Alfvén wave will be supported by the wavevector $\mathbf{k}$, with $k \ll p, q$. For such a limit the master equations (5.3) reduce to

$$\partial_t \begin{Bmatrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{Bmatrix} = \frac{\pi\epsilon^2}{8d_\mathrm{i}^2 B_0^2} \int \sum_{\substack{\Lambda \\ s,s_\mathrm{p},s_q}} \left(\frac{\sin\psi_k}{k}\right)^2 \frac{(\Lambda k + s_\mathrm{p} p + s_q q)^2}{2} \left(\frac{s_q q d_\mathrm{i} - s_\mathrm{p} p d_\mathrm{i}}{k_\parallel}\right)^2 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\times \frac{B_0^2 s k_\parallel s_\mathrm{p} p_\parallel p d_\mathrm{i}}{2} E^B(\mathbf{q}) \left[E^B(\mathbf{p}) - \left(\frac{E^V(\mathbf{k}) - E^B(\mathbf{k})}{-s\Lambda d_\mathrm{i} k}\right)\right]$$

$$\times \delta(\Omega_{k,pq})\delta_{k,pq}\, d\mathbf{p}\, d\mathbf{q}. \tag{7.1}$$

Note that for whistler waves the polarizations are fixed, i.e. $\Lambda_\mathrm{p} = s_\mathrm{p}$ and $\Lambda_q = s_q$. Further simplifications are possible and one obtains finally

$$\partial_t \begin{Bmatrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{Bmatrix} = \frac{\pi\epsilon^2}{8} \int \sum_{s,s_\mathrm{p},s_q} \left(\frac{\sin\psi_k}{k}\right)^2 \left(\frac{s_q q - s_\mathrm{p} p}{k_\parallel}\right)^2 (s_\mathrm{p} p + s_q q) k_\parallel s_\mathrm{p} p_\parallel p \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\times E^B(\mathbf{q})[E^V(\mathbf{k}) - E^B(\mathbf{k})]\delta(\Omega_{k,pq})\delta_{k,pq}\, d\mathbf{p}\, d\mathbf{q}. \tag{7.2}$$

A very interesting result is obtained: at leading order, the Alfvén wave is only affected by small-scale whistler waves when the local (i.e. at a given wavevector $\mathbf{k}$) equipartition between the Alfvénic kinetic and magnetic energies is broken, i.e. when $E^V(\mathbf{k}) \neq E^B(\mathbf{k})$. In addition we see that these non-local effects are the same

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for the Alfvénic kinetic and the magnetic energies. This is not so trivial since whistler waves are described only by the magnetic energy. We also note that the collisional integral is linearly dependent on the difference between the large-scale kinetic and magnetic energies which can be extracted from the integral. These comments may be summarized by the relation

$$\partial_t \begin{Bmatrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{Bmatrix} = I_1(\mathbf{k})[E^V(\mathbf{k}) - E^B(\mathbf{k})] \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \tag{7.3}$$

where $I_1(\mathbf{k})$ is a real function that measures the effects of whistler waves.

The second case that we shall consider is the one where a whistler wave is affected by the interaction between an Alfvén wave and another whistler wave. In other words, it means that the Alfvén wave will be supported by the wavevector $\mathbf{p}$ with $p \ll k, q$. The same kind of manipulations as before leads finally to the following wave kinetic equations

$$\partial_t \begin{Bmatrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{Bmatrix} = \frac{\pi\epsilon^2}{4} \int \sum_{s,s_\mathrm{p},s_q} \left(\frac{\sin\psi_k}{k}\right)^2 \left(\frac{sk - s_q q}{p_\parallel}\right)^2 (sk + s_q q) s k_\parallel k p_\parallel \begin{Bmatrix} \frac{1}{d_\mathrm{i}^2 k^2} \\ 1 \end{Bmatrix}$$

$$\times E^B(\mathbf{q})[E^B(\mathbf{p}) - E^V(\mathbf{p})]\delta(\Omega_{k,pq})\delta_{k,pq}\, d\mathbf{p}\, d\mathbf{q}. \tag{7.4}$$

As expected, we note that at leading order the kinetic energy of whistler waves is negligible and only the magnetic energy is relevant. We see that the Alfvén wave affects the whistler wave only when the local equipartition is broken. Thus for any kind of WWA non-local interactions, one needs a discrepancy from the Alfvénic equipartition to have a non-trivial dynamics.

7.3. CCA interactions

We study here the strongly non-local interactions of two ion cyclotron waves on one Alfvén wave: in other words, it means that the Alfvén wave will be supported by the wavevector $\mathbf{k}$, with $k \ll p, q$. For such a limit the wave kinetic equations (5.3) are written as

$$\partial_t \begin{Bmatrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{Bmatrix}$$

$$= \frac{\pi\epsilon^2}{8\, d_\mathrm{i}^2 B_0^2} \int \sum_{\substack{\Lambda \\ s,s_\mathrm{p},s_q}} \left(\frac{\sin\psi_k}{k}\right)^2 \frac{(\Lambda k - s_\mathrm{p} p - s_q q)^2 d_\mathrm{i}^8 p^4 q^4}{2 d_\mathrm{i}^4 p^2 q^2} \left(\frac{s_q/q d_\mathrm{i} - s_\mathrm{p}/p d_\mathrm{i}}{k_\parallel}\right)^2 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\times \frac{B_0^2 s k_\parallel s_\mathrm{p} p_\parallel}{2 p d_\mathrm{i}} E^V(\mathbf{q}) \left[E^V(\mathbf{p}) - \left(\frac{E^V(\mathbf{k}) - E^B(\mathbf{k})}{-s\Lambda d_\mathrm{i} k}\right)\right] \delta(\Omega_{k,pq})\delta_{k,pq}\, d\mathbf{p}\, d\mathbf{q}. \tag{7.5}$$

For ion cyclotron waves the polarizations are $\Lambda_\mathrm{p} = -s_\mathrm{p}$ and $\Lambda_q = -s_q$. Further simplifications lead to

$$\partial_t \begin{Bmatrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{Bmatrix} = \frac{\pi\epsilon^2}{8 d_\mathrm{i}^2} \int \sum_{s,s_\mathrm{p},s_q} \left(\frac{\sin\psi_k}{k}\right)^2 \left(\frac{s_q/q - s_\mathrm{p}/p}{k_\parallel}\right)^2 (s_\mathrm{p} p + s_q q) p q^2 k_\parallel s_\mathrm{p} p_\parallel \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\times E^V(\mathbf{q})[E^B(\mathbf{k}) - E^V(\mathbf{k})]\delta(\Omega_{k,pq})\delta_{k,pq}\, d\mathbf{p}\, d\mathbf{q}. \tag{7.6}$$

As before, we see that at leading order the Alfvén wave is only affected by small scale ion cyclotron waves when the local equipartition between the Alfvénic kinetic

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and magnetic energies is broken. In addition we see that these non-local effects are the same for the Alfvénic kinetic and magnetic energies. This is not so trivial since ion cyclotron waves are described only by the kinetic energy. As before, we note that the collisional integral is linearly dependent on the difference between the large scale kinetic and magnetic energies which can be extracted from the integral. These comments may be summarized by the relation

$$\partial_t \begin{Bmatrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{Bmatrix} = I_2(\mathbf{k})[E^B(\mathbf{k}) - E^V(\mathbf{k})] \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \tag{7.7}$$

where $I_2(\mathbf{k})$ is a real function that measures the effects of ion cyclotron waves.

The second case that we shall consider is the one where an ion cyclotron wave is affected by the interaction between an Alfvén wave and another ion cyclotron wave. In other words, it means that the Alfvén wave will be supported by the wavevector $\mathbf{p}$ with $p \ll k, q$. The same kind of manipulations as before leads finally to

$$\partial_t \begin{Bmatrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{Bmatrix} = \frac{\pi\epsilon^2}{4d_\mathrm{i}^4} \int \sum_{s,s_\mathrm{p},s_q} \left(\frac{\sin\psi_k}{k}\right)^2 \left(\frac{s_q/q - s/k}{p_\parallel}\right)^2 (sk + s_q q) \frac{sk_\parallel p_\parallel q^2}{k} \begin{Bmatrix} d_\mathrm{i}^2 k^2 \\ 1 \end{Bmatrix}$$

$$\times E^V(\mathbf{q})[E^V(\mathbf{p}) - E^B(\mathbf{p})]\delta(\Omega_{k,pq})\delta_{k,pq}\, d\mathbf{p}\, d\mathbf{q}. \tag{7.8}$$

As expected, we note that at leading order the magnetic energy of ion cyclotron waves is negligible and only the kinetic energy is relevant. We see that the Alfvén wave affects the ion cyclotron wave only when its local equipartition is broken. Thus for any kind of CCA non-local interactions, one needs a discrepancy from the Alfvénic equipartition to obtain a non-trivial dynamics.

7.4. WCA interactions

First, we investigate the strongly non-local interactions of a whistler wave and an ion cyclotron wave on an Alfvén wave: in this case the Alfvén wave will be supported by the wavevector $\mathbf{k}$; the whistler and ion cyclotron waves will be associated respectively to $\mathbf{p}$ and $\mathbf{q}$, with $k \ll p, q$. For such a limit the wave kinetic equations (5.3) are written as

$$\partial_t \begin{Bmatrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{Bmatrix} = \frac{\pi\epsilon^2}{8d_\mathrm{i}^2 B_0^2} \int \sum_{\substack{\Lambda \\ s,s_\mathrm{p},s_q}} \left(\frac{\sin\psi_k}{k}\right)^2 \frac{(\Lambda k + s_\mathrm{p} p - s_q q)^2 (1 - q^2/p^2)^2}{2d_\mathrm{i}^2 q^2}$$

$$\times \left(\frac{s_\mathrm{p} p d_\mathrm{i} - s_q/q d_\mathrm{i}}{k_\parallel}\right)^2 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \frac{B_0^2 s k_\parallel s_\mathrm{p} p_\parallel p d_\mathrm{i}}{2}$$

$$\times E^V(\mathbf{q}) \left[E^B(\mathbf{p}) - \left(\frac{E^V(\mathbf{k}) - E^B(\mathbf{k})}{-s\Lambda d_\mathrm{i} k}\right)\right] \delta(\Omega_{k,pq})\delta_{k,pq}\, d\mathbf{p}\, d\mathbf{q}. \tag{7.9}$$

Further simplifications give finally

$$\partial_t \begin{Bmatrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{Bmatrix} = \frac{\pi\epsilon^2}{8d_\mathrm{i}^4} \int \sum_{s,s_\mathrm{p},s_q} \left(\frac{\sin\psi_k}{k}\right)^2 \left(\frac{s_\mathrm{p} p d_\mathrm{i} - s_q/q d_\mathrm{i}}{k_\parallel}\right)^2 (1 - q^2/p^2)^2 (s_\mathrm{p} p - s_q q)$$

$$\times \frac{k_\parallel s_\mathrm{p} p_\parallel p}{q^2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} E^V(\mathbf{q})[E^V(\mathbf{k}) - E^B(\mathbf{k})]\delta(\Omega_{k,pq})\delta_{k,pq}\, d\mathbf{p}\, d\mathbf{q}. \tag{7.10}$$

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As before, at leading order the Alfvén wave is only affected by small-scale effects when the local Alfvénic equipartition is broken. These small-scale effects are only due to ion cyclotron waves. The reason for this apparent asymmetry is actually due to the initial association between wavevectors and waves (see below). We see that these non-local effects are the same for the Alfvénic kinetic and the magnetic energies. Once again, we note that the collisional integral is linearly dependent on the difference between the large-scale kinetic and magnetic energies which can be extracted from the integral. Then one can write

$$\partial_t \begin{Bmatrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{Bmatrix} = I_3(\mathbf{k})[E^V(\mathbf{k}) - E^B(\mathbf{k})] \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \tag{7.11}$$

where $I_3(\mathbf{k})$ is a real function that measures the small-scale effects due to the ion cyclotron wave.

Because of the apparent asymmetry mentioned above, we will also consider the situation where the whistler wave is associated to the wavevector $\mathbf{q}$ and the ion cyclotron wave to the wavevector $\mathbf{p}$. For such a limit, we obtain in the same way as before the following wave kinetic equations

$$\partial_t \begin{Bmatrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{Bmatrix} = \frac{\pi\epsilon^2}{8d_\mathrm{i}^6} \int \sum_{s,s_\mathrm{p},s_q} \left(\frac{\sin\psi_k}{k}\right)^2 \left(\frac{s_q q d_\mathrm{i} - s_\mathrm{p}/p d_\mathrm{i}}{k_\parallel}\right)^2 (1-p^2/q^2)^2 (s_\mathrm{p} p - s_q q)$$

$$\times \frac{k_\parallel s_\mathrm{p} p_\parallel}{p^3} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} E^B(\mathbf{q})[E^B(\mathbf{k}) - E^V(\mathbf{k})] \delta(\Omega_{k,pq}) \delta_{k,pq}\, d\mathbf{p}\, d\mathbf{q}. \tag{7.12}$$

The same comments as before hold: at leading order the Alfvén wave is only affected by small-scale effects when the local Alfvénic equipartition is broken. Here these small-scale effects are only due to whistler waves. We also see that these non-local effects are the same for the Alfvénic kinetic and the magnetic energies. Once again, we note that the collisional integral is linearly dependent on the difference between the large-scale kinetic and magnetic energies. Then we can write

$$\partial_t \begin{Bmatrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{Bmatrix} = I_4(\mathbf{k})[E^B(\mathbf{k}) - E^V(\mathbf{k})] \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \tag{7.13}$$

where $I_4(\mathbf{k})$ is a real function that measures the small-scale effects due to the whistler wave.

The next situation that we are going to investigate is the one where an ion cyclotron wave is affected by the interaction between an Alfvén wave and a whistler wave. In other words, it means that the Alfvén wave will be supported by the wavevector $\mathbf{p}$, the whistler and ion cyclotron waves will be associated respectively to $\mathbf{q}$ and $\mathbf{k}$, with $p \ll k, q$. The same type of manipulations as before leads to

$$\partial_t \begin{Bmatrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{Bmatrix} = \frac{\pi\epsilon^2}{4d_\mathrm{i}^8} \int \sum_{s,s_\mathrm{p},s_q} \left(\frac{\sin\psi_k}{k}\right)^2 \left(\frac{s_q q d_\mathrm{i} - s/k d_\mathrm{i}}{p_\parallel}\right)^2 (1-k^2/q^2)^2 (sk - s_q q)$$

$$\times \frac{s k_\parallel p_\parallel}{k^5} \begin{Bmatrix} d_\mathrm{i}^2 k^2 \\ 1 \end{Bmatrix} E^B(\mathbf{q})[E^V(\mathbf{p}) - E^B(\mathbf{p})] \delta(\Omega_{k,pq}) \delta_{k,pq}\, d\mathbf{p}\, d\mathbf{q}. \tag{7.14}$$

As expected, at leading order the magnetic energy of ion cyclotron waves is negligible and only the kinetic energy is relevant. Once again, the Alfvén wave affects the ion cyclotron wave only when the local equipartition is broken.

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The last case is the one where a whistler wave is affected by the interaction between an Alfvén wave and an ion cyclotron wave. In other words, it means that the Alfvén wave will be supported by the wavevector $\mathbf{p}$, the whistler and ion cyclotron waves will be associated, respectively, with $\mathbf{k}$ and $\mathbf{q}$, with $p \ll k, q$. Under these conditions and after some simplifications the master equations are written as

$$\partial_t \begin{Bmatrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{Bmatrix} = \frac{\pi\epsilon^2}{4d_\mathrm{i}^4} \int \sum_{s,s_\mathrm{p},s_q} \left(\frac{\sin\psi_k}{k}\right)^2 \left(\frac{s_q/qd_\mathrm{i} - skd_\mathrm{i}}{p_\parallel}\right)^2 (1-q^2/k^2)^2 (sk - s_q q) \times \frac{sk_\parallel kp_\parallel}{q^2} \begin{Bmatrix} 1/d_\mathrm{i}^2k^2 \\ 1 \end{Bmatrix} E^V(\mathbf{q})[E^B(\mathbf{p}) - E^V(\mathbf{p})]\delta(\Omega_{k,pq})\delta_{k,pq}\, d\mathbf{p}\, d\mathbf{q}. \tag{7.15}$$

As expected, at leading order the kinetic energy of whistler waves is negligible and only the magnetic energy is relevant. Once again, the Alfvén wave affects the whistler wave only when the local equipartition is broken. In conclusion, for any type of WCA non-local interactions, one needs a discrepancy from the Alfvénic equipartition to obtain a non-trivial dynamics.

7.5. Discussion

We have seen that for any kind of non-local interaction (WWA, CCA or WCA) a non-trivial dynamics is present as long as the large-scale kinetic and magnetic energies are different. This result revealed from the above detailed analysis is new and may explain some Solar wind features. Indeed at first sight, the Solar wind may be seen as a MHD flow where the kinetic and magnetic energies are balanced. However, fluctuations around this equipartition state exist. The present work may explain how these fluctuations are maintained by small-scale processes. Small scales may also be affected by what happens at the largest scales of the system. In this case Alfvén waves may be seen as a potential source of energy for small scales. We report in Appendix G the result of the non-local analysis made on WCC interactions.

8. Source of anisotropy

8.1. Local interactions

In previous sections, we have seen three different limits for which anisotropic turbulence develops more or less strongly, i.e. the small-scale limit, with right or left polarization, and the large scale limit of Hall MHD. For intermediate wavenumbers the situation is more difficult to analyze. Indeed, the wave kinetic equations are in general too complex to give a simple answer. In this section, we shall see what happens when only strongly local interactions are taken into account. It is a situation where wavevectors have approximately the same amplitude, i.e. $k \approx p \approx q$ (we consider triangles, $\mathbf{k} = \mathbf{p} + \mathbf{q}$, approximately equilaterals). We are particularly interested by what happens at intermediate scales.

Our analysis may start from the master equations (5.3). The hypothesis of strongly local interactions does not lead to drastic simplifications; it seems therefore more interesting to look at the resonance condition (3.21) that appears in (5.3). The first case that we may investigate is the one for which waves have the same polarization (left or right). After simplifications, one finds the approximate relation

$$\left(\frac{s_\mathrm{p} - s_q}{k_\parallel}\right)^2 \approx \left(\frac{s_q - s}{p_\parallel}\right)^2 \approx \left(\frac{s - s_\mathrm{p}}{q_\parallel}\right)^2. \tag{8.1}$$

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Necessarily two directional wave polarities will be equal which implies that one of the parallel wavenumbers will be (approximately) equal to zero. In other words, local interactions between the same kind of waves lead to anisotropic turbulence where small scales are preferentially generated perpendicular to the external magnetic field.

The second case that we shall analyze is the one where we have local interactions between waves with different polarizations (left and right). Then the resonance condition simplifies to

$$\left(\frac{s_{\rm p}(s_{\rm p}\Lambda_{\rm p}+\sqrt{1+4/d_{\rm i}^2p^2})-s_q(s_q\Lambda_q+\sqrt{1+4/d_{\rm i}^2q^2})}{k_\parallel}\right)^2$$
$$\approx\left(\frac{s_q(s_q\Lambda_q+\sqrt{1+4/d_{\rm i}^2q^2})-s(s\Lambda+\sqrt{1+4/d_{\rm i}^2k^2})}{p_\parallel}\right)^2$$
$$\approx\left(\frac{s(s\Lambda+\sqrt{1+4/d_{\rm i}^2k^2})-s_{\rm p}(s_{\rm p}\Lambda_{\rm p}+\sqrt{1+4/d_{\rm i}^2p^2})}{q_\parallel}\right)^2. \qquad (8.2)$$

Several combinations are possible. We note that for the particular case where two waves have both the same polarization (left or right) and the same directional polarity, one parallel wavenumber will be (approximately) equal to zero. Thus, such a situation leads to the same type of anisotropic turbulence as before. A similar analysis applied to all the other cases does not lead to such a conclusion: an isotropic turbulence may be generated by the nonlinear dynamics.

The wave kinetic equations (5.3) are composed of a sum over the different polarities (Λ- and s-type). In other words, all the cases that we have mentioned above have to be taken into account to understand what happens for strongly local interactions. Both the mechanisms leading to anisotropy and isotropy are in competition and, therefore, it is likely that globally a moderate anisotropy (less strong than what happens in the limits of large-scale or small-scale Hall MHD) is generated at intermediate wavenumbers.

8.2. Discussion

The local interaction analysis gives some hints to understand globally the nonlinear dynamics of wave turbulence in Hall MHD. It is also instructive to analyze the opposite limit, namely the limit of strongly non-local interactions. In Sec. 7, we have introduced the concept of non-local interactions and discussed some properties. Information about anisotropy can be extracted from the different cases analyzed. More precisely, one needs to study the solution of the condition $\Omega_{k,pq}=0$ that has to be satisfied at the resonance. The general tendency that we observe (we will not give a detailed description of every case) is that most non-local interactions lead to anisotropic turbulence where small scales are preferentially generated perpendicular to the external magnetic field.

Both local and non-local interactions contribute to develop anisotropy more or less strongly. In addition we know that a direct energy cascade is predicted in the large-scale limit (pure Alfvén wave turbulence) as well as in the small-scale limit (whistler and ion cyclotron wave turbulence) of Hall MHD. Thus it is expected that a turbulent anisotropic spectrum generated, or developed, at large scale will spread out over wavenumbers, from $kd_{\rm i}<1$ to $kd_{\rm i}>1$, extending anisotropy to smaller scales. This scenario may be applicable to the Solar wind for which various indirect

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lines of evidence, at low and high frequencies, show that waves propagate at large angles to the background magnetic field and that the power in fluctuations parallel to the background magnetic field is much less than the perpendicular case (Coroniti et al. 1982; Leamon et al. 1998).

9. Wave versus strong turbulence

9.1. Domain of validity of wave turbulence

Wave turbulence deals with asymptotic developments which are based on a time scale separation, with a transfer time assumed to be much larger than the wave period. A consequence of this assumption is that the theory is not valid uniformly in all of **k**-space. In this section, we are going to evaluate the condition of validity of the present theory. Generally, one needs to start with the wave kinetic equations for which exact power law solutions exist. In our case, different solutions have been found for different limits but in general—for intermediate wavenumbers—no solution has been found. For that reason we are going to use a dimensional analysis and, in particular, the heuristic spectrum derived in Sec. 6. This spectrum is based on anisotropic arguments, therefore the condition that we are going to write is only valid for anisotropic turbulence and might not be applicable at intermediate wavenumbers.

We remind the reader that the nonlinear time and the wave period are, respectively,

$$\tau_{\rm NL} \sim \frac{1}{\epsilon k_\perp \mathscr{Z}^s_\Lambda} \tag{9.1}$$

and

$$\tau_{\rm w} \sim \frac{1}{\omega^s_\Lambda} = \frac{1}{-B_0 k_\parallel \xi^s_\Lambda}. \tag{9.2}$$

Note that we have included explicitly the small parameter ϵ in the nonlinear time. The transfer time $\tau_{\rm tr}$ is built on these time scales and is written

$$\tau_{\rm tr} \sim \tau_{\rm NL} \frac{\tau_{\rm NL}}{\tau_{\rm w}}. \tag{9.3}$$

Then the asymptotic condition $\tau_{\rm tr} \gg \tau_{\rm w}$ is

$$\frac{-B_0 k_\parallel \xi^s_\Lambda}{\epsilon^2 k_\perp^2 \mathscr{Z}^s_\Lambda} \gg \frac{1}{-B_0 k_\parallel \xi^s_\Lambda}, \tag{9.4}$$

where $\mathscr{Z}^s_\Lambda$ has a dimension of a velocity (i.e. it is not taken in Fourier space as it was introduced in Sec. 2.5). To proceed further we use relations (6.7)–(6.8) and we take into account the heuristic spectrum (6.9); one obtains (with $B_0 = 1$)

$$(\mathscr{Z}^s_\Lambda)^2 \sim \frac{(\xi^s_\Lambda - \xi^{-s}_\Lambda)^2}{1 + \xi^{-s\,2}_\Lambda} k_\perp^{-1} k_\parallel^{1/2} (1 + k_\perp^2 d_{\rm i}^2)^{-1/4}. \tag{9.5}$$

Then we insert relation (9.5) into (9.4). By using relation (A 3), we finally obtain the inequality

$$k_\parallel \gg \epsilon^{4/3} k_\perp^{2/3} (1 + k_\perp^2 d_{\rm i}^2)^{1/2} (1 + \xi^{s\,2}_\Lambda)^{-2/3}. \tag{9.6}$$

Equation (9.6) is the condition of validity of wave turbulence for Hall MHD at the level of three-wave interactions. This condition is *not* in contradiction with the anisotropic assumption ($k_\perp \gg k_\parallel$) since the small parameter ϵ is present. Thus we

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can always find a domain of the **k**-space where the theory is valid. We note, in particular, that for whistler and ion cyclotron wave turbulence the condition of validity reduces, respectively, to $k_\parallel \gg \epsilon^{4/3} k_\perp^{1/3}$ and $k_\parallel \gg \epsilon^{4/3} k_\perp^{5/3}$. We remind the reader that it is an evaluation based on an anisotropic assumption and therefore it might not be valid at intermediate wavenumbers. In the prohibited region (small $k_\parallel$) other higher-order processes such as four-wave interaction have to be taken into account. Basically it is the domain where strong turbulence occurs.

9.2. The 2D state

The wave turbulence theory developed in this paper describes the Hall MHD dynamics of 3D incompressible waves. Indeed, the condition of validity found previously shows that the 2D state, i.e. modes with $k_\parallel = 0$, is not described by the wave kinetic equations. Actually, if we look at (3.23), we see that the nonlinear transfers decrease (linearly) with $k_\parallel$. For the forbidden value $k_\parallel = 0$ the transfer is exactly null. We see therefore that the 2D—slow—modes decouple from the 3D Hall MHD waves. Such decoupling is not new; it is found in a variety of problems such as for internal gravity waves (see, e.g., Phillips 1968), rotating stratified flows (see, e.g., Bartello 1995) or pure rotating flows (see, e.g., Smith and Waleffe 1999).

As we have already noticed in Sec. 4.5 the MHD limit is somewhat singular for Hall MHD. A signature of this singular behavior is the appearance of principal value terms in the wave kinetic equations. The consequence is that there is an important difference between Hall MHD and MHD: in the latter case the 2D modes drive the 3D dynamics. Therefore, in this particular case, we need information from the forbidden region to describe the dynamics in the domain of applicability of wave turbulence. Note that even in this case the wave turbulence theory at the level of three-wave interactions has been shown to be applicable (Galtier et al. 2000; Nazarenko et al. 2001; Bhattacharjee and Ng 2001). For Hall MHD in general, higher-order processes like four-wave interactions may lead to a coupling between the 2D state and 3D modes.

9.3. Strong Hall MHD turbulence

We shall derive now the equivalent heuristic prediction as in Sec. 6 but for strong turbulence. We would like to compare, at least at the level of the phenomenology, the wave and strong turbulence regimes.

We will assume that a moderate external magnetic field B_0 is applied and that it still affects the Hall MHD dynamics. In other words we will distinguish the wavenumber $k_\perp$ from $k_\parallel$, with $k_\perp \gg k_\parallel$, and we suppose that the nonlinear time is of the order of the wave period. Then one has

$$\Pi \sim \frac{E}{\tau_{\rm tr}} \sim \frac{E(k_\perp, k_\parallel) k_\perp k_\parallel}{\tau_{\rm NL}}. \tag{9.7}$$

The equivalent manipulations as in Sec. 6 lead to the prediction

$$E(k_\perp, k_\parallel) \sim \Pi^{2/3} k_\perp^{-5/3} k_\parallel^{-1} (1 + \delta_{\Lambda s} k_\perp d_{\rm i})^{-2/3}, \tag{9.8}$$

where $\delta_{\Lambda s}$ is introduced to distinguish the electron MHD case ($\delta_{ss} = 1$) from the ion MHD case ($\delta_{-ss} = 0$). The heuristic prediction gives the energy spectrum scaling laws for the three different limits of standard, EMHD and IMHD as well as for intermediate wavenumbers. Contrary to the wave turbulence regime the prediction shows that EMHD and IMHD follow different power laws. If the external magnetic

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field is small enough one may ignore anisotropy and write $k_\perp \sim k_\parallel \sim k$; it gives

$$E(k) \sim \Pi^{2/3} k^{-5/3} (1 + \delta_{\Lambda s} k d_\mathrm{i})^{-2/3}, \tag{9.9}$$

for the one-dimensional isotropic spectrum. It is interesting to note that the EMHD prediction is the one found and observed in direct numerical simulations by Biskamp et al. (1999). Note that the Kolmogorov scaling found at large scales is an unavoidable result since we have used in fact the same phenomenology as Kolmogorov. Note finally that the scaling law predictions made for strong and wave turbulence in the presence of anisotropy may be described in the context of the generalized critical balance proposed by Galtier et al. (2005). In that context the solutions found here would be two particular cases of a family of solutions. The result obtained here has to be contrasted with those obtained by Krishan and Mahajan (2004, 2005) for strong isotropic turbulence. In these papers, the authors give several predictions separately for the magnetic and kinetic energy spectra. In particular, they conclude that the whistler-type prediction is very different from the observations and evoke a possible stronger damping for whistler waves. In the present paper, it is thought that the damping (e.g. due to the resonance) affects mainly the ion cyclotron waves (see also Goldstein et al. 1994).

10. Conclusion

10.1. Summary

In this paper we have investigated the steepening of the magnetic fluctuation power law spectra observed in the Solar wind for frequencies higher than 0.5 Hz. We have shown that the high-frequency part of the spectrum may be attributed to dispersive nonlinear processes present in the incompressible Hall MHD equations rather than pure dissipation. In that context, we have introduced the generalized Elsässer variables adapted to Hall MHD and developed a wave turbulence formalism based on a complex helical decomposition. Then, we have derived the wave kinetic equations for 3D incompressible Hall MHD turbulence at the level of three-wave interactions, in the presence of a strong external magnetic field, and in the most general case, i.e. with global and magnetic helicities. In such a regime, Hall MHD turbulence is mainly anisotropic with small scales preferentially generated perpendicular to the external magnetic field. In particular, strong anisotropy is developed at small scales (ion cyclotron/IMHD and whistler/EMHD wave turbulence) and large scales (pure Alfvén/MHD wave turbulence). For intermediate wavenumbers, i.e. $k d_\mathrm{i} \approx 1$, only a moderate anisotropy may happen. We have found two different power law energy spectra that are exact solutions of the wave kinetic equations, with a steeper power law at smaller scales. To illustrate this finding, we have developed an anisotropic phenomenology that describes continuously the different scaling laws for the energy spectrum: the existence of a double inertial range is recovered with a knee at intermediate scales where the Hall term becomes (sub-)dominant. We have also shown that the large-scale limit of Hall MHD, i.e. the standard MHD, is singular and leads to the appearance of principal value terms in the wave kinetic equations. We have analyzed the non-local interactions between Alfvén, whistler and ion cyclotron waves and we have shown that, at leading order, a non-trivial dynamics is present only when a discrepancy from the equipartition between the large-scale kinetic and magnetic energies of Alfvén waves happens. Finally, we have

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given the conditions of validity of wave turbulence in Hall MHD and compared the wave and strong turbulence scaling laws.

10.2. Discussion

Solar wind magnetic fluctuation power law spectra are commonly observed to have an index around $\frac{-5}{3}$ at frequencies lower than 0.5 Hz and a steeper power law behavior at higher frequencies with a spectral index on average around -3 (Coroniti et al. 1982; Denskat et al. 1983; Leamon et al. 1998). The latest analysis made from Cluster spacecraft data has, however, reported a spectral index about -2.12 (Bale et al. 2005). These small scale fluctuations are also characterized by a bias of the polarization suggesting that they are likely to be right-hand polarized, outward propagating waves (Goldstein et al. 1994). For these reasons, it is claimed that Alfvén—left circularly polarized—fluctuations may be suppressed by proton cyclotron damping and that the high frequency power law spectra are likely to consist of whistler waves (see, e.g., Stawicki et al. 2001). In addition it is often observed that the high frequency fluctuations of the magnetic field are much smaller than the background magnetic field (see, e.g., Stawicki et al. 2001). Therefore, a wave turbulence approach seems to be well adapted to this problem. It is well known that whistler waves are a solution to the Hall MHD equations. We have seen in the regime of wave turbulence that when the Hall term is added to the MHD equations a non-trivial dynamics happens at small scales (i.e. when $kd_{\mathrm{i}} > 1$) with, in particular, a steepening of the power law energy spectra. This behavior is similar to what is observed in the Solar wind and may be seen as an indication that the steepening of the Solar wind magnetic fluctuation power law spectra is mainly due to nonlinear processes rather than pure dissipation. Under this new interpretation, the resistive dissipation range of frequencies may be moved to frequencies higher than the electron cyclotron frequency.

The Taylor hypothesis, that allows us to connect directly a frequency to a wavenumber, is widely used to interpret the single spacecraft Solar wind data. It is thought that this approximation is well adapted to analyze, in particular, the low frequency part of the magnetic and velocity fluctuations. The subsequent interpretation is mainly relevant for isotropic media. However, there is evidence that anisotropy is present in the Solar wind at high and low frequencies (see, e.g., Belcher and Davis 1971). From a theoretical point of view, it is well known that the presence of a strong magnetic field influences the MHD turbulent flows (see, e.g., Pouquet 1978; Montgomery and Turner 1981; Shebalin et al. 1983; Goldreich and Sridhar 1995; Ng and Bhattacharjee 1996, 1997; Galtier et al. 2000, 2005). The main effect is that MHD turbulence becomes mainly bidimensional with a nonlinear transfer essentially perpendicular to its direction. Direct numerical simulations of $2\frac{1}{2}$D compressible Hall MHD for high and low β plasma (Ghosh and Goldstein 1997) have also displayed such an anisotropic property when a strong magnetic field is present. As we have seen, the model that we propose here is also able to exhibit such an anisotropy. The predictions that we have made are for anisotropic turbulence, i.e. a situation where we distinguish the wavenumber $k_\perp$ from $k_\parallel$. Therefore, a proper comparison with observational data will be possible only when a 3D energy spectrum will be accessible. It is important to note that average effects may alter significantly the power law scaling. An illustration of such effects may be given from the recent heuristic MHD predictions made by Galtier et al. (2005) who have generalized the concept of critical balance. For a medium where a strong

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or moderate magnetic field $B_0\hat{\mathbf{e}}_\parallel$ is present, they predict the anisotropic energy spectrum $E(k_\perp, k_\parallel) \sim k_\perp^{-\alpha} k_\parallel^{-\beta}$, with $3\alpha + 2\beta = 7$. This model is able to describe both the strong and wave turbulence regimes as well as the transition between them; it also satisfies the critical balance relationship $k_\parallel \propto k_\perp^{2/3}$. It is very interesting to note that if one averages the heuristic spectrum to obtain the one-dimensional counterpart $E(k_\perp)$, one obtains systematically a scaling law in $k_\perp^{-5/3}$ for any family of solutions (α, β). Actually, this remark might explain the apparent contradiction between, on the one hand, the presence of anisotropy in the Solar wind and, on the other hand, the Kolmogorov $(-5/3)$ scaling law for the low frequency energy spectrum. Of course, other explanations of the 5/3-spectrum are possible; an example is given in Oughton and Matthaeus (2005). The direct comparison between theoretical predictions and *in situ* measurements of 3D energy spectra is particularly crucial for the high frequency part of the power spectrum for which the usual Taylor hypothesis that allows us to connect directly the frequency to the wavenumber is no longer applicable. Efforts are currently being made with Cluster spacecraft data from which it is possible to extract the 3D magnetic turbulent spectra of the magnetosheath thanks to multipoint measurements and a k-filtering technique (see, e.g., Sahraoui et al. 2004). As explained in Bale et al. (2005) Cluster spacecraft may exit from the terrestrial magnetosphere to make Solar wind measurements. The application of the k-filtering technique to the high frequency part of the Solar wind magnetic fluctuations seems then possible. It may lead, for the first time, to a direct and rigorous comparison with a model prediction of Solar wind. Note that using multi-spacecraft analysis, Matthaeus et al. (2005) have recently investigated some spatial correlations from two-point measurements.

The model proposed in this paper is able to recover some Solar wind properties but several aspects have not been discussed. For example, it would be interesting to investigate the effects of asymmetry, i.e. the fact that outward propagating waves dominate in the Solar wind. In the framework of incompressible MHD, we know that asymmetry changes the index of the power law spectra. Similarly, the indices found here may be affected by asymmetry; this question will be tackled in the future. In this paper, we have found that a turbulent state made of ion cyclotron waves may exist around a fraction of the ion cyclotron frequency ω_{ci}, namely for a frequency around $\omega_{\mathrm{R}} = (k_\parallel/k)\omega_{\mathrm{ci}}$. The presence of a resonance at a frequency ω_{R} lower than ω_{ci} is rarely discussed in the literature (see White et al. 2002) since the analysis is generally focused on parallel propagations ($k_\parallel = k$) for which the resonance frequency is exactly the ion cyclotron frequency. As we have seen, in the presence of a strong external magnetic field, Hall MHD turbulence becomes mainly bidimensional with an energy spectrum mainly spread out over the wavenumbers $k_\perp$. In this case, the resonance frequency may appear at a small fraction of ω_{ci} (this fraction being even smaller for heavier mass ions). These remarks may be crucial to understand the Solar coronal heating problem in which the coronal temperature is far beyond what one can predict by the resistive MHD approximation: although the Hall MHD model is a fluid model that does not describe the resonance between waves and particles and therefore the particle heating, it offers the possibility to evaluate the rate of particle heating by assuming that the turbulent energy of ion cyclotron waves is mainly transfer into heating. This point is currently being investigated and will be reported elsewhere. In the context of the Solar wind, the measure of the position of the knee in the magnetic fluctuation power law spectrum may be seen as a proxy to measure the Solar wind anisotropy.

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The Hall effect is relevant in many astrophysical problems to understand, for example, the presence of instabilities in protostellar disks (Balbus and Terquem 2001), the magnetic field evolution in neutron star crusts (Goldreich and Reisenegger 1992; Cumming et al. 2004) or impulsive magnetic reconnection (see, e.g., Bhattacharjee 2004). Small-scale turbulence is also a key issue in a number of problems from the interstellar medium (see the review by Elmegreen and Scalo (2004)), to Solar physics (see, e.g., Petrosyan et al. 2005), magnetospheric physics (see, e.g., the recent paper by Goldstein (2005)) and laboratory devices such as tokamaks (see, e.g., Wild et al. 1981; Taylor, 1993). Therefore, it is likely that the present model will interest several other (astrophysical) problems.

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Appendix A. Some useful relationships

It is convenient to note the following identities:

$$1-{\xi_\Lambda^s}^2 = k\Lambda d_{\rm i}\xi_\Lambda^s, \tag{A 1}$$

$$\xi_\Lambda^s\xi_\Lambda^{-s} = -1, \tag{A 2}$$

$$\xi_\Lambda^s - \xi_\Lambda^{-s} = -s\sqrt{4+k^2d_{\rm i}^2}, \tag{A 3}$$

$$\xi_\Lambda^s + \xi_\Lambda^{-s} = -kd_{\rm i}\Lambda, \tag{A 4}$$

$$\Lambda\left(\frac{1-{\xi_\Lambda^{-s}}^4}{k}\right)\left(\frac{\omega_\Lambda^s}{1+{\xi_\Lambda^{-s}}^2}\right) = d_{\rm i}k_\parallel. \tag{A 5}$$

Appendix B. Derivation of the wave kinetic equations

The starting point of the derivation of the wave kinetic equations for incompressible Hall MHD is the fundamental equation (3.14). We write successively equations for the second- and third-order moments,

$$\begin{aligned}\partial_t\langle a_\Lambda^s a_{\Lambda'}^{s'}\rangle &= \frac{\epsilon}{4d_{\rm i}}\int\sum_{\substack{\Lambda_{\rm p},\Lambda_q\\ s_{\rm p},s_q}} {\xi_\Lambda^s}^2\frac{\xi_{\Lambda_q}^{s_q}-\xi_{\Lambda_p}^{s_p}}{\xi_\Lambda^s-\xi_\Lambda^{-s}}M\begin{matrix}\Lambda\Lambda_{\rm p}\Lambda_q\\ ss_{\rm p}s_q\\ -k\,p\,q\end{matrix}\langle a_{\Lambda_p}^{s_p}a_{\Lambda_q}^{s_q}a_{\Lambda'}^{s'}\rangle e^{-i\Omega_{pq,k}t}\delta_{pq,k}\,d\mathbf{p}\,d\mathbf{q}\\ &+\frac{\epsilon}{4d_{\rm i}}\int\sum_{\substack{\Lambda_{\rm p},\Lambda_q\\ s_{\rm p},s_q}} {\xi_{\Lambda'}^{s'}}^2\frac{\xi_{\Lambda_q}^{s_q}-\xi_{\Lambda_p}^{s_p}}{\xi_{\Lambda'}^{s'}-\xi_{\Lambda'}^{-s'}}M\begin{matrix}\Lambda'\Lambda_{\rm p}\Lambda_q\\ s's_{\rm p}s_q\\ -k'\,p\,q\end{matrix}\langle a_{\Lambda_p}^{s_p}a_{\Lambda_q}^{s_q}a_{\Lambda}^{s}\rangle e^{-i\Omega_{pq,k'}t}\delta_{pq,k'}\,d\mathbf{p}\,d\mathbf{q},\end{aligned} \tag{B 1}$$

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and

$$\begin{aligned}
&\partial_t \langle a^s_\Lambda a^{s'}_{\Lambda'} a^{s''}_{\Lambda''}\rangle \\
&\quad = \frac{\epsilon}{4d_{\rm i}} \int \sum_{\substack{\Lambda_{\rm p},\Lambda_q \\ s_{\rm p},s_q}} \xi^s_\Lambda{}^2 \frac{\xi^{s_q}_{\Lambda_q} - \xi^{s_p}_{\Lambda_p}}{\xi^s_\Lambda - \xi^{-s}_\Lambda} M \begin{matrix} \Lambda\Lambda_{\rm p}\Lambda_q \\ s s_{\rm p} s_q \\ -k\, p\, q \end{matrix} \langle a^{s_p}_{\Lambda_p} a^{s_q}_{\Lambda_q} a^{s'}_{\Lambda'} a^{s''}_{\Lambda''}\rangle e^{-i\Omega_{pq,k}t} \delta_{pq,k}\, d\mathbf{p}\, d\mathbf{q} \\
&\qquad + \frac{\epsilon}{4d_{\rm i}} \int \sum_{\substack{\Lambda_{\rm p},\Lambda_q \\ s_{\rm p},s_q}} \xi^{s'}_{\Lambda'}{}^2 \frac{\xi^{s_q}_{\Lambda_q} - \xi^{s_p}_{\Lambda_p}}{\xi^{s'}_{\Lambda'} - \xi^{-s'}_{\Lambda'}} M \begin{matrix} \Lambda'\Lambda_{\rm p}\Lambda_q \\ s' s_{\rm p} s_q \\ -k'\, p\, q \end{matrix} \langle a^{s_p}_{\Lambda_p} a^{s_q}_{\Lambda_q} a^{s}_{\Lambda} a^{s''}_{\Lambda''}\rangle e^{-i\Omega_{pq,k'}t} \delta_{pq,k'}\, d\mathbf{p}\, d\mathbf{q} \\
&\qquad + \frac{\epsilon}{4d_{\rm i}} \int \sum_{\substack{\Lambda_{\rm p},\Lambda_q \\ s_{\rm p},s_q}} \xi^{s''}_{\Lambda''}{}^2 \frac{\xi^{s_q}_{\Lambda_q} - \xi^{s_p}_{\Lambda_p}}{\xi^{s''}_{\Lambda''} - \xi^{-s''}_{\Lambda''}} M \begin{matrix} \Lambda''\Lambda_{\rm p}\Lambda_q \\ s'' s_{\rm p} s_q \\ -k''\, p\, q \end{matrix} \langle a^{s_p}_{\Lambda_p} a^{s_q}_{\Lambda_q} a^{s}_{\Lambda} a^{s'}_{\Lambda'}\rangle e^{-i\Omega_{pq,k''}t} \delta_{pq,k''}\, d\mathbf{p}\, d\mathbf{q}.
\end{aligned} \tag{B 2}$$

We shall write asymptotic closure (Newell et al. 2001) for our system. For that, we basically need to write the fourth-order moment in terms of a sum of the fourth-order cumulant plus products of second-order ones. The asymptotic closure depends on two ingredients: the first is the degree to which the linear waves interact to randomize phases; the second relies on the fact that the nonlinear regeneration of the third-order moment by the fourth-order moment in (B 2) depends more on the product of the second-order moments than it does on the fourth-order cumulant. The fourth-order moment decomposes into the sum of three products of second-order moments, and a fourth-order cumulant. The latter does not contribute to secular behavior, and among the other products one is absent because of the homogeneity assumption. If we use the symmetric relations (3.16)–(3.19) and perform wavevector integrations, summations over polarities and time integration, then (B 2) becomes

$$\begin{aligned}
&\langle a^s_\Lambda a^{s'}_{\Lambda'} a^{s''}_{\Lambda''}\rangle \\
&\quad = \frac{\epsilon}{4d_{\rm i}} \Delta(\Omega_{kk'k''}) \delta_{kk'k''} \\
&\qquad \times \left\{ \xi^s_\Lambda{}^2 \left[\frac{\xi^{s''}_{\Lambda''} - \xi^{s'}_{\Lambda'}}{\xi^s_\Lambda - \xi^{-s}_\Lambda} \left(M \begin{matrix} \Lambda\Lambda'\Lambda'' \\ s s' s'' \\ k k' k'' \end{matrix} \right)^* + \frac{\xi^{s'}_{\Lambda'} - \xi^{s''}_{\Lambda''}}{\xi^s_\Lambda - \xi^{-s}_\Lambda} \left(M \begin{matrix} \Lambda\Lambda''\Lambda' \\ s s'' s' \\ k k'' k' \end{matrix} \right)^* \right] q^{s'}_{\Lambda'} q^{s''}_{\Lambda''} \right. \\
&\qquad + \xi^{s'}_{\Lambda'}{}^2 \left[\frac{\xi^{s''}_{\Lambda''} - \xi^{s}_{\Lambda}}{\xi^{s'}_{\Lambda'} - \xi^{-s'}_{\Lambda'}} \left(M \begin{matrix} \Lambda'\Lambda\Lambda'' \\ s' s s'' \\ k' k k'' \end{matrix} \right)^* + \frac{\xi^{s}_{\Lambda} - \xi^{s''}_{\Lambda''}}{\xi^{s'}_{\Lambda'} - \xi^{-s'}_{\Lambda'}} \left(M \begin{matrix} \Lambda'\Lambda''\Lambda \\ s' s'' s \\ k' k'' k \end{matrix} \right)^* \right] q^{s}_{\Lambda} q^{s''}_{\Lambda''} \\
&\qquad \left. + \xi^{s''}_{\Lambda''}{}^2 \left[\frac{\xi^{s}_{\Lambda} - \xi^{s'}_{\Lambda'}}{\xi^{s''}_{\Lambda''} - \xi^{-s''}_{\Lambda''}} \left(M \begin{matrix} \Lambda''\Lambda'\Lambda \\ s'' s' s \\ k'' k' k \end{matrix} \right)^* + \frac{\xi^{s'}_{\Lambda'} - \xi^{s}_{\Lambda}}{\xi^{s''}_{\Lambda''} - \xi^{-s''}_{\Lambda''}} \left(M \begin{matrix} \Lambda''\Lambda\Lambda' \\ s'' s s' \\ k'' k k' \end{matrix} \right)^* \right] q^{s'}_{\Lambda'} q^{s}_{\Lambda} \right\},
\end{aligned} \tag{B 3}$$

where

$$\Delta(\Omega_{kk'k''}) = \int_0^t e^{i\Omega_{kk'k''}t'} dt' = \frac{e^{i\Omega_{kk'k''}t} - 1}{i\Omega_{kk'k''}}. \tag{B 4}$$

The introduction of symmetric relations (3.16)–(3.19) into (B 3) allows us to simplify

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further the previous equation; one obtains

$$\langle a^s_\Lambda a^{s'}_{\Lambda'} a^{s''}_{\Lambda''}\rangle$$

$$= \frac{\epsilon}{2d_{\rm i}}\Delta(\Omega_{kk'k''})\delta_{kk'k''}\begin{pmatrix} M\ {}^{\Lambda\Lambda'\Lambda''}_{ss's''}\\ {}_{kk'k''}\end{pmatrix}^*$$

$$\times\left[\xi^s_\Lambda{}^2\frac{\xi^{s''}_{\Lambda''}-\xi^{s'}_{\Lambda'}}{\xi^s_\Lambda-\xi^{-s}_\Lambda}q^{s'}_{\Lambda'}q^{s''}_{\Lambda''}+\xi^{s'}_{\Lambda'}{}^2\frac{\xi^s_\Lambda-\xi^{s''}_{\Lambda''}}{\xi^{s'}_{\Lambda'}-\xi^{-s'}_{\Lambda'}}q^s_\Lambda q^{s''}_{\Lambda''}+\xi^{s''}_{\Lambda''}{}^2\frac{\xi^{s'}_{\Lambda'}-\xi^s_\Lambda}{\xi^{s''}_{\Lambda''}-\xi^{-s''}_{\Lambda''}}q^s_\Lambda q^{s'}_{\Lambda'}\right]. \quad (B\,5)$$

We insert expression (B 5) into (B 1); it leads to

$$\partial_t q^s_\Lambda(\mathbf{k})$$

$$= \frac{\epsilon^2}{8d_{\rm i}^2}\int\sum_{\substack{\Lambda_{\rm p},\Lambda_q\\ s_{\rm p},s_q}}\xi^s_\Lambda{}^2\frac{\xi^{s_q}_{\Lambda_q}-\xi^{s_p}_{\Lambda_p}}{\xi^s_\Lambda-\xi^{-s}_\Lambda}\left|M\ {}^{\Lambda\Lambda_{\rm p}\Lambda_q}_{ss_{\rm p}s_q}_{\ -kpq}\right|^2\Delta(\Omega_{pq,k})e^{-i\Omega_{pq,k}t}\delta_{pq,k}$$

$$\times\left[\xi^{s_p}_{\Lambda_p}{}^2\frac{\xi^s_\Lambda-\xi^{s_q}_{\Lambda_q}}{\xi^{s_p}_{\Lambda_p}-\xi^{-s_p}_{\Lambda_p}}q^s_\Lambda q^{s_q}_{\Lambda_q}+\xi^{s_q}_{\Lambda_q}{}^2\frac{\xi^{s_p}_{\Lambda_p}-\xi^s_\Lambda}{\xi^{s_q}_{\Lambda_q}-\xi^{-s_q}_{\Lambda_q}}q^s_\Lambda q^{s_p}_{\Lambda_p}+\xi^s_\Lambda{}^2\frac{\xi^{s_q}_{\Lambda_q}-\xi^{s_p}_{\Lambda_p}}{\xi^s_\Lambda-\xi^{-s}_\Lambda}q^{s_p}_{\Lambda_p}q^{s_q}_{\Lambda_q}\right]d\mathbf{p}\,d\mathbf{q}$$

$$+\frac{\epsilon^2}{8d_{\rm i}^2}\int\sum_{\substack{\Lambda_{\rm p},\Lambda_q\\ s_{\rm p},s_q}}\xi^{s'}_{\Lambda'}{}^2\frac{\xi^{s_q}_{\Lambda_q}-\xi^{s_p}_{\Lambda_p}}{\xi^{s'}_{\Lambda'}-\xi^{-s'}_{\Lambda'}}\left|M\ {}^{\Lambda'\Lambda_{\rm p}\Lambda_q}_{s's_{\rm p}s_q}_{\ -k'pq}\right|^2\Delta(\Omega_{pq,k'})e^{-i\Omega_{pq,k'}t}\delta_{pq,k'}$$

$$\times\left[\xi^{s_p}_{\Lambda_p}{}^2\frac{\xi^{s'}_{\Lambda'}-\xi^{s_q}_{\Lambda_q}}{\xi^{s_p}_{\Lambda_p}-\xi^{-s_p}_{\Lambda_p}}q^{s'}_{\Lambda'}q^{s_q}_{\Lambda_q}+\xi^{s_q}_{\Lambda_q}{}^2\frac{\xi^{s_p}_{\Lambda_p}-\xi^{s'}_{\Lambda'}}{\xi^{s_q}_{\Lambda_q}-\xi^{-s_q}_{\Lambda_q}}q^{s'}_{\Lambda'}q^{s_p}_{\Lambda_p}\right.$$

$$\left.+\xi^{s'}_{\Lambda'}{}^2\frac{\xi^{s_q}_{\Lambda_q}-\xi^{s_p}_{\Lambda_p}}{\xi^{s'}_{\Lambda'}-\xi^{-s'}_{\Lambda'}}q^{s_p}_{\Lambda_p}q^{s_q}_{\Lambda_q}\right]d\mathbf{p}\,d\mathbf{q}. \quad (B\,6)$$

The long-time behavior of the wave kinetic equation (B 6) is given by the Riemann–Lebesgue lemma which tells us that, for $t\to+\infty$, we have

$$e^{-ixt}\Delta(x)=\Delta(-x)\to\pi\delta(x)-i\mathscr{P}(1/x), \quad (B\,7)$$

where $\mathscr{P}$ is the principal value of the integral. The two terms of (B 6) are complex conjugated therefore if in the second term we replace the dummy integration variables $\mathbf{p}$, $\mathbf{q}$, by $-\mathbf{p}$, $-\mathbf{q}$, we can simplify further (B 6) since, in particular, principal value terms compensate exactly. Finally, we obtain the wave kinetic equations for incompressible Hall MHD

$$\partial_t q^s_\Lambda(\mathbf{k})=\frac{\pi\epsilon^2}{4d_{\rm i}^2}\int\sum_{\substack{\Lambda_{\rm p},\Lambda_q\\ s_{\rm p},s_q}}\xi^s_\Lambda\frac{\xi^{s_q}_{\Lambda_q}-\xi^{s_p}_{\Lambda_p}}{1-\xi^{-s^2}_\Lambda}\left|M\ {}^{\Lambda\Lambda_{\rm p}\Lambda_q}_{ss_{\rm p}s_q}_{\ -kpq}\right|^2\delta(\Omega_{k,pq})\delta_{k,pq}$$

$$\times\left[\xi^{s_p}_{\Lambda_p}\frac{\xi^s_\Lambda-\xi^{s_q}_{\Lambda_q}}{1-\xi^{-s_p}_{\Lambda_p}{}^2}q^s_\Lambda q^{s_q}_{\Lambda_q}+\xi^{s_q}_{\Lambda_q}\frac{\xi^{s_p}_{\Lambda_p}-\xi^s_\Lambda}{1-\xi^{-s_q}_{\Lambda_q}{}^2}q^s_\Lambda q^{s_p}_{\Lambda_p}\right.$$

$$\left.+\xi^s_\Lambda\frac{\xi^{s_q}_{\Lambda_q}-\xi^{s_p}_{\Lambda_p}}{1-\xi^{-s^2}_\Lambda}q^{s_p}_{\Lambda_p}q^{s_q}_{\Lambda_q}\right]d\mathbf{p}\,d\mathbf{q}, \quad (B\,8)$$

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where

$$\left| M \begin{smallmatrix} \Lambda\Lambda_{\rm p}\Lambda_q \\ s s_{\rm p} s_q \\ -kpq \end{smallmatrix} \right|^2 = \left(\frac{\sin\psi_k}{k}\right)^2 (\Lambda k + \Lambda_{\rm p} p + \Lambda_q q)^2 (1 - \xi_\Lambda^{-s^2} \xi_{\Lambda_p}^{-s_p{}^2} \xi_{\Lambda_q}^{-s_q{}^2})^2.$$

The last step that we have to follow to obtain the same expression as (3.23) is to include the resonance relations (3.21) into the previous equations.

Appendix C. Pseudo-dispersive MHD waves and complex helical decomposition

This section is devoted to the derivation of the wave kinetic equations for pure Alfvénic turbulence ($kd_{\rm i} = 0$). These equations were already derived by Galtier et al. (2000) but here we use the complex helicity decomposition. We will see that some differences appear in the kinematics that render the MHD description somewhat singular (principal value terms appear) by opposition to the dispersive Hall MHD description. We start from the standard incompressible and inviscid MHD equations,

$$\nabla \cdot \mathbf{V} = 0, \tag{C 1}$$

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla P_* + \mathbf{B} \cdot \nabla \mathbf{B}, \tag{C 2}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{V}, \tag{C 3}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{C 4}$$

The same notation as before is used. We introduce the fluctuating fields $\mathbf{B}(\mathbf{x}) = B_0 \hat{\mathbf{e}}_\parallel + \epsilon \mathbf{b}(\mathbf{x})$, $\mathbf{V}(\mathbf{x}) = \epsilon \mathbf{v}(\mathbf{x})$ and we Fourier transform the MHD equations. We obtain

$$\partial_t \mathbf{z}^{\mathbf{s}}_{\mathbf{k}} - i s k_\parallel B_0 \mathbf{z}^{\mathbf{s}}_{\mathbf{k}} = -\epsilon \{\mathbf{z}^{-\mathbf{s}} \cdot \nabla \mathbf{z}^{\mathbf{s}} - \nabla P_*\}_{\mathbf{k}}, \tag{C 5}$$

$$\mathbf{k} \cdot \mathbf{z}^{\mathbf{s}}{}_{\mathbf{k}} = 0, \tag{C 6}$$

where we use the standard Elsässer variables $\mathbf{z}^{\mathbf{s}} = \mathbf{v} + s\mathbf{b}$. The linear solution ($\epsilon = 0$) corresponds to linearly polarized Alfvén waves for which $\omega_k = B_0 k_\parallel$. We introduce the complex helicity basis and define

$$\mathbf{z}^s_{\mathbf{k}} = \sum_\Lambda \mathscr{Z}^s_\Lambda(\mathbf{k}) e^{i s \omega_k t} \mathbf{h}^{\mathbf{\Lambda}}_{\mathbf{k}} = \sum_\Lambda \mathscr{Z}^s_\Lambda e^{i s \omega_k t} \mathbf{h}^{\mathbf{\Lambda}}_{\mathbf{k}}. \tag{C 7}$$

We see that this decomposition is not natural for linear polarized Alfvén waves since for a given direction of propagation (a given s) we have two contributions for each value of Λ. We nevertheless use this decomposition to show the compatibility with the Hall MHD turbulence description. After the substitution of (C 7) into (C 5) and (C 6), we obtain

$$\partial_t \mathscr{Z}^s_\Lambda = -\frac{i\epsilon}{2} \int \sum_{\Lambda_{\rm p}, \Lambda_q} \mathscr{Z}^{-s}_{\Lambda_{\rm p}} \mathscr{Z}^s_{\Lambda_q} (\mathbf{k} \cdot \mathbf{h}^{\mathbf{\Lambda}_{\rm p}}_{\rm p})(\mathbf{h}^{\mathbf{\Lambda}_{\mathbf{q}}}_{\mathbf{q}} \cdot \mathbf{h}^{-\mathbf{\Lambda}}_{\mathbf{k}}) e^{-i s(\omega_k + \omega_{\rm p} - \omega_q) t} \delta_{pq,k} \, d\mathbf{p} \, d\mathbf{q}. \tag{C 8}$$

Note that we do not have a summation over the directional polarity $s_{\rm p}$ and s_q. The physical reason is that the nonlinear coupling in incompressible MHD involves only Alfvén waves propagating in opposite directions. Thus the information about the

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direction of propagation is already taken into account in (C 5). We use the local decomposition and find after some algebra

$$\partial_t \mathscr{Z}^s_\Lambda = \frac{\epsilon}{4}\int \sum_{\Lambda_\mathrm{p},\Lambda_q} (k\Lambda + q\Lambda_q - p\Lambda_\mathrm{p}) N^{\Lambda\Lambda_\mathrm{p}\Lambda_q}_{-k\,p\,q}\, \mathscr{Z}^{-s}_{\Lambda_\mathrm{p}} \mathscr{Z}^{s}_{\Lambda_q} e^{-2isB_0 p_\parallel t}\delta_{pq,k}\, d\mathbf{p}\, d\mathbf{q}, \tag{C 9}$$

where

$$N^{\Lambda\Lambda_\mathrm{p}\Lambda_q}_{k\,p\,q} = e^{i(\Lambda\Phi_k+\Lambda_\mathrm{p}\Phi_\mathrm{p}+\Lambda_q\Phi_q)}\Lambda\Lambda_\mathrm{p}\Lambda_q \frac{\sin\psi_k}{k}(\Lambda k + \Lambda_\mathrm{p} p + \Lambda_q q). \tag{C 10}$$

We note that the matrix N possesses the following properties ($*$ denotes the complex conjugate),

$$\left(N^{\Lambda\Lambda_\mathrm{p}\Lambda_q}_{k\,p\,q}\right)^* = N^{-\Lambda-\Lambda_\mathrm{p}-\Lambda_q}_{k\,p\,q} = N^{\Lambda\Lambda_\mathrm{p}\Lambda_q}_{-k-p-q}, \tag{C 11}$$

$$N^{\Lambda\Lambda_\mathrm{p}\Lambda_q}_{k\,p\,q} = -N^{\Lambda\Lambda_q\Lambda_\mathrm{p}}_{k\,q\,p}, \tag{C 12}$$

$$N^{\Lambda\Lambda_\mathrm{p}\Lambda_q}_{k\,p\,q} = -N^{\Lambda_q\Lambda_\mathrm{p}\Lambda}_{q\,p\,k}, \tag{C 13}$$

$$N^{\Lambda\Lambda_\mathrm{p}\Lambda_q}_{k\,p\,q} = -N^{\Lambda_\mathrm{p}\Lambda\Lambda_q}_{p\,k\,q}. \tag{C 14}$$

Equation (C 9) is the fundamental equation that describes the slow evolution of the Alfvén wave amplitudes due to the nonlinear terms of the incompressible MHD equations. Note that (i) we have already used the resonance condition to simplify the coefficient in the complex exponential function; (ii) a comparison with the large-scale limit of (3.14) is possible if we sum over the directional polarities. We follow the same steps as in Appendix B: we write successively equations for the second- and third-order moments,

$$\begin{aligned}
&\partial_t \langle \mathscr{Z}^s_\Lambda \mathscr{Z}^{s'}_{\Lambda'}\rangle \\
&\quad= \frac{\epsilon}{4}\int \sum_{\Lambda_\mathrm{p},\Lambda_q} (k\Lambda + q\Lambda_q - p\Lambda_\mathrm{p}) N^{\Lambda\Lambda_\mathrm{p}\Lambda_q}_{-k\,p\,q} \langle \mathscr{Z}^{-s}_{\Lambda_\mathrm{p}} \mathscr{Z}^{s}_{\Lambda_q} \mathscr{Z}^{s'}_{\Lambda'}\rangle e^{-2isB_0 p_\parallel t}\delta_{pq,k}\, d\mathbf{p}\, d\mathbf{q}, \\
&\qquad+ \frac{\epsilon}{4}\int \sum_{\Lambda_\mathrm{p},\Lambda_q} (k'\Lambda' + q\Lambda_q - p\Lambda_\mathrm{p}) N^{\Lambda'\Lambda_\mathrm{p}\Lambda_q}_{-k'\,p\,q} \langle \mathscr{Z}^{-s'}_{\Lambda_\mathrm{p}} \mathscr{Z}^{s'}_{\Lambda_q} \mathscr{Z}^{s}_{\Lambda}\rangle e^{-2is'B_0 p_\parallel t}\delta_{pq,k'}\, d\mathbf{p}\, d\mathbf{q},
\end{aligned} \tag{C 15}$$

and

$$\begin{aligned}
&\partial_t \langle \mathscr{Z}^s_\Lambda \mathscr{Z}^{s'}_{\Lambda'} \mathscr{Z}^{s''}_{\Lambda''}\rangle \\
&\quad= \frac{\epsilon}{4}\int \sum_{\Lambda_\mathrm{p},\Lambda_q} (k\Lambda + q\Lambda_q - p\Lambda_\mathrm{p}) N^{\Lambda\Lambda_\mathrm{p}\Lambda_q}_{-k\,p\,q} \langle \mathscr{Z}^{-s}_{\Lambda_\mathrm{p}} \mathscr{Z}^{s}_{\Lambda_q} \mathscr{Z}^{s'}_{\Lambda'} \mathscr{Z}^{s''}_{\Lambda''}\rangle e^{-2isB_0 p_\parallel t}\delta_{pq,k}\, d\mathbf{p}\, d\mathbf{q}, \\
&\qquad+ \frac{\epsilon}{4}\int \sum_{\Lambda_\mathrm{p},\Lambda_q} (k'\Lambda' + q\Lambda_q - p\Lambda_\mathrm{p}) N^{\Lambda'\Lambda_\mathrm{p}\Lambda_q}_{-k'\,p\,q} \langle \mathscr{Z}^{-s'}_{\Lambda_\mathrm{p}} \mathscr{Z}^{s'}_{\Lambda_q} \mathscr{Z}^{s}_{\Lambda} \mathscr{Z}^{s''}_{\Lambda''}\rangle \\
&\qquad\times e^{-2is'B_0 p_\parallel t}\delta_{pq,k'}\, d\mathbf{p}\, d\mathbf{q},
\end{aligned}$$

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$$+\frac{\epsilon}{4}\int\sum_{\Lambda_{\rm p},\Lambda_q}(k''\Lambda''+q\Lambda_q-p\Lambda_{\rm p})N^{\Lambda''\Lambda_{\rm p}\Lambda_q}_{-k''\,p\,q}\langle\mathscr{Z}^{-s''}_{\Lambda_{\rm p}}\mathscr{Z}^{s''}_{\Lambda_q}\mathscr{Z}^{s}_{\Lambda}\mathscr{Z}^{s'}_{\Lambda'}\rangle$$

$$\times\, e^{-2is''B_0p_\parallel t}\delta_{pq,k''}\,d\mathbf{p}\,d\mathbf{q}. \tag{C 16}$$

We define the density tensor $q^{ss'}_{\Lambda\Lambda'}(\mathbf{k})$ for a homogeneous turbulence,

$$\langle\mathscr{Z}^{s}_{\Lambda}(\mathbf{k})\mathscr{Z}^{s'}_{\Lambda'}(\mathbf{k}')\rangle\equiv q^{ss'}_{\Lambda\Lambda'}(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}')\delta_{ss'}. \tag{C 17}$$

The presence of the delta $\delta_{ss'}$ means that correlations with opposite polarities have no long-time influence in the wave turbulence regime; the second delta distribution $\delta(\mathbf{k}+\mathbf{k}')$ is the consequence of the homogeneity assumption. We note that the kinematics does not impose any condition about the polarization Λ. The reason is that Λ does not appear in the Alfvén wave frequency. This remark shows a fundamental difference with the Hall MHD case. As we will see it is the reason why principal value terms appear in MHD but not in Hall MHD. After the same kind of manipulations as in Appendix B, the third-order moment equation (C 16) becomes

$$\begin{aligned}
&\langle\mathscr{Z}^{s}_{\Lambda}\mathscr{Z}^{s'}_{\Lambda'}\mathscr{Z}^{s''}_{\Lambda''}\rangle\\
&\quad=\frac{\epsilon}{4}\delta_{k\,k'\,k''}\Bigg\{\sum_{\Lambda_{\rm p},\Lambda_q}(k\Lambda+k''\Lambda_q-k'\Lambda_{\rm p})\begin{pmatrix}N^{\Lambda\Lambda_{\rm p}\Lambda_q}\\ k\,k'\,k''\end{pmatrix}^{*}\Delta(2sB_0k'_\parallel)q^{-s-s}_{\Lambda_{\rm p}\Lambda'}q^{ss}_{\Lambda_q\Lambda''}\\
&\qquad+\sum_{\Lambda_{\rm p},\Lambda_q}(k\Lambda+k'\Lambda_q-k''\Lambda_{\rm p})\begin{pmatrix}N^{\Lambda\Lambda_{\rm p}\Lambda_q}\\ k\,k''\,k'\end{pmatrix}^{*}\Delta(2sB_0k''_\parallel)q^{-s-s}_{\Lambda_{\rm p}\Lambda''}q^{ss}_{\Lambda_q\Lambda'}\\
&\qquad+\sum_{\Lambda_{\rm p},\Lambda_q}(k'\Lambda'+k''\Lambda_q-k\Lambda_{\rm p})\begin{pmatrix}N^{\Lambda'\Lambda_{\rm p}\Lambda_q}\\ k'\,k\,k''\end{pmatrix}^{*}\Delta(2s'B_0k_\parallel)q^{-s'-s'}_{\Lambda_{\rm p}\Lambda}q^{s's'}_{\Lambda_q\Lambda''}\\
&\qquad+\sum_{\Lambda_{\rm p},\Lambda_q}(k'\Lambda'+k\Lambda_q-k''\Lambda_{\rm p})\begin{pmatrix}N^{\Lambda'\Lambda_{\rm p}\Lambda_q}\\ k'\,k''\,k\end{pmatrix}^{*}\Delta(2s'B_0k''_\parallel)q^{-s'-s'}_{\Lambda_{\rm p}\Lambda''}q^{s's'}_{\Lambda_q\Lambda}\\
&\qquad+\sum_{\Lambda_{\rm p},\Lambda_q}(k''\Lambda''+k'\Lambda_q-k\Lambda_{\rm p})\begin{pmatrix}N^{\Lambda''\Lambda_{\rm p}\Lambda_q}\\ k''\,k\,k'\end{pmatrix}^{*}\Delta(2s''B_0k_\parallel)q^{-s''-s''}_{\Lambda_{\rm p}\Lambda}q^{s''s''}_{\Lambda_q\Lambda'}\\
&\qquad+\sum_{\Lambda_{\rm p},\Lambda_q}(k''\Lambda''+k\Lambda_q-k'\Lambda_{\rm p})\begin{pmatrix}N^{\Lambda''\Lambda_{\rm p}\Lambda_q}\\ k''\,k'\,k\end{pmatrix}^{*}\Delta(2s''B_0k'_\parallel)q^{-s''-s''}_{\Lambda_{\rm p}\Lambda'}q^{s''s''}_{\Lambda_q\Lambda}\Bigg\}.
\end{aligned}\tag{C 18}$$

We insert expression (C 18) into (C 15). We note that only the third and the fifth terms of expression (C 18) will contribute. We obtain

$$\begin{aligned}
&\partial_t q^{ss}_{\Lambda\Lambda'}(\mathbf{k})\\
&\quad=\frac{\epsilon}{16}\Bigg\{\int\sum_{\substack{\Lambda_{\rm p},\Lambda_q\\ \bar\Lambda_{\rm p},\bar\Lambda_q}}(k\Lambda+q\Lambda_q-p\Lambda_{\rm p})(q\Lambda_q+k\bar\Lambda_q-p\bar\Lambda_{\rm p})N^{\Lambda\Lambda_{\rm p}\Lambda_q}_{-k\,p\,q}\begin{pmatrix}-N^{\bar\Lambda_q\bar\Lambda_{\rm p}\Lambda_q}\\ -k\,p\,q\end{pmatrix}^{*}\\
&\qquad\times\Delta(-2sB_0p_\parallel)q^{-s-s}_{\bar\Lambda_{\rm p}\Lambda_{\rm p}}q^{ss}_{\bar\Lambda_q\Lambda'}\delta_{pq,k}\,d\mathbf{p}\,d\mathbf{q}
\end{aligned}$$

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$$+\int \sum_{\substack{\Lambda_\mathrm{p},\Lambda_q\\ \bar\Lambda_\mathrm{p},\bar\Lambda_q}} (k\Lambda+q\Lambda_q-p\Lambda_\mathrm{p})(k\Lambda'+q\bar\Lambda_q-p\bar\Lambda_\mathrm{p}) N^{\Lambda\Lambda_\mathrm{p}\Lambda_q}_{-k\,p\,q} \begin{pmatrix} N^{\Lambda\bar\Lambda_\mathrm{p}\bar\Lambda_q}_{-k\,p\,q}\end{pmatrix}^*$$

$$\times \Delta(-2sB_0p_\parallel) q^{-s-s}_{\bar\Lambda_\mathrm{p}\Lambda_\mathrm{p}} q^{ss}_{\bar\Lambda_q\Lambda_q}\delta_{pq,k}\,d\mathbf{p}\,d\mathbf{q}$$

$$+\int \sum_{\substack{\Lambda_\mathrm{p},\Lambda_q\\ \bar\Lambda_\mathrm{p},\bar\Lambda_q}} (k\Lambda'+q\Lambda_q-p\Lambda_\mathrm{p})(q\Lambda_q+k\bar\Lambda_q-p\bar\Lambda_\mathrm{p}) N^{\Lambda'\Lambda_\mathrm{p}\Lambda_q}_{k\,p\,q} \begin{pmatrix} -N^{\bar\Lambda_q\bar\Lambda_\mathrm{p}\Lambda_q}_{k\,p\,q}\end{pmatrix}^*$$

$$\times \Delta(-2sB_0p_\parallel) q^{-s-s}_{\bar\Lambda_\mathrm{p}\Lambda_\mathrm{p}} q^{ss}_{\bar\Lambda_q\Lambda}\delta_{k\,p\,q}\,d\mathbf{p}\,d\mathbf{q}$$

$$+\int \sum_{\substack{\Lambda_\mathrm{p},\Lambda_q\\ \bar\Lambda_\mathrm{p},\bar\Lambda_q}} (k\Lambda'+q\Lambda_q-p\Lambda_\mathrm{p})(k\Lambda+q\bar\Lambda_q-p\bar\Lambda_\mathrm{p}) N^{\Lambda'\Lambda_\mathrm{p}\Lambda_q}_{k\,p\,q} \begin{pmatrix} N^{\Lambda\bar\Lambda_\mathrm{p}\bar\Lambda_q}_{k\,p\,q}\end{pmatrix}^*$$

$$\times \Delta(-2sB_0p_\parallel) q^{-s-s}_{\bar\Lambda_\mathrm{p}\Lambda_\mathrm{p}} q^{ss}_{\bar\Lambda_q\Lambda_q}\delta_{k\,p\,q}\,d\mathbf{p}\,d\mathbf{q}\Big\}. \tag{C 19}$$

We change signs for wavevectors $\mathbf{p}$ and $\mathbf{q}$ in the last two terms and obtain

$$\begin{aligned}\partial_t q^{ss}_{\Lambda\Lambda'}(\mathbf{k}) &= \frac{\epsilon}{16}\int \sum_{\substack{\Lambda_\mathrm{p},\Lambda_q\\ \bar\Lambda_\mathrm{p},\bar\Lambda_q}} \Big\{(k\Lambda+q\Lambda_q-p\Lambda_\mathrm{p})(q\Lambda_q+k\bar\Lambda_q-p\bar\Lambda_\mathrm{p})\\ &\times N^{\Lambda\Lambda_\mathrm{p}\Lambda_q}_{-k\,p\,q}\begin{pmatrix} -N^{\bar\Lambda_q\bar\Lambda_\mathrm{p}\Lambda_q}_{-k\,p\,q}\end{pmatrix}^* \Delta(-2sB_0p_\parallel) q^{-s-s}_{\bar\Lambda_\mathrm{p}\Lambda_\mathrm{p}} q^{ss}_{\bar\Lambda_q\Lambda'}\\ &+(k\Lambda+q\Lambda_q-p\Lambda_\mathrm{p})(k\Lambda'+q\bar\Lambda_q-p\bar\Lambda_\mathrm{p}) N^{\Lambda\Lambda_\mathrm{p}\Lambda_q}_{-k\,p\,q}\begin{pmatrix} N^{\Lambda\bar\Lambda_\mathrm{p}\bar\Lambda_q}_{-k\,p\,q}\end{pmatrix}^*\\ &\times \Delta(-2sB_0p_\parallel) q^{-s-s}_{\bar\Lambda_\mathrm{p}\Lambda_\mathrm{p}} q^{ss}_{\bar\Lambda_q\Lambda_q} + (k\Lambda'+q\Lambda_q-p\Lambda_\mathrm{p})(q\Lambda_q+k\bar\Lambda_q-p\bar\Lambda_\mathrm{p})\\ &\times N^{\bar\Lambda_q\bar\Lambda_\mathrm{p}\Lambda_q}_{-k\,p\,q}\begin{pmatrix} -N^{\Lambda'\Lambda_\mathrm{p}\Lambda_q}_{-k\,p\,q}\end{pmatrix}^* \Delta(2sB_0p_\parallel) q^{-s-s}_{\bar\Lambda_\mathrm{p}\Lambda_\mathrm{p}} q^{ss}_{\bar\Lambda_q\Lambda}\\ &+(k\Lambda'+q\Lambda_q-p\Lambda_\mathrm{p})(k\Lambda+q\bar\Lambda_q-p\bar\Lambda_\mathrm{p}) N^{\Lambda\bar\Lambda_\mathrm{p}\bar\Lambda_q}_{-k\,p\,q}\begin{pmatrix} N^{\Lambda'\Lambda_\mathrm{p}\Lambda_q}_{-k\,p\,q}\end{pmatrix}^*\\ &\times \Delta(2sB_0p_\parallel) q^{-s-s}_{\bar\Lambda_\mathrm{p}\Lambda_\mathrm{p}} q^{ss}_{\bar\Lambda_q\Lambda_q}\Big\}\delta_{pq,k}\,d\mathbf{p}\,d\mathbf{q}.\end{aligned} \tag{C 20}$$

The long-time behavior is then given by the Riemann–Lebesgue lemma

$$\Delta(\pm 2sB_0p_\parallel)\to \pi\delta(2sB_0p_\parallel)\mp i\mathscr{P}(1/2sB_0p_\parallel). \tag{C 21}$$

We see that principal value terms will appear in the long-time limit. The reason is that the polarizations Λ and Λ' in the density tensor $q^{ss}_{\Lambda\Lambda'}(\mathbf{k})$ are not the same in general. Therefore we lose the symmetry between terms that we had in the Hall MHD case (see Appendix B).

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Appendix D. Simplified MHD wave kinetic equations

In this section, we continue the analysis made in Appendix C when only terms symmetric in Λ are retained, i.e. terms such as $q^{ss}_{\Lambda\Lambda}$. Then expression (C 20) simplifies to

$$\begin{aligned}\partial_t q^{ss}_{\Lambda\Lambda}(\mathbf{k}) = \frac{\epsilon}{16}\int \sum_{\Lambda_\mathrm{p},\Lambda_q}\Bigg\{&(k\Lambda+q\Lambda_q-p\Lambda_\mathrm{p})^2 N\begin{matrix}\Lambda\Lambda_\mathrm{p}\Lambda_q\\ -k\,p\,q\end{matrix}\left(-N\begin{matrix}\Lambda\Lambda_\mathrm{p}\Lambda_q\\ -k\,p\,q\end{matrix}\right)^*\\ &\times\Delta(-2sB_0p_\parallel)q^{-s-s}_{\Lambda_\mathrm{p}\Lambda_\mathrm{p}}q^{ss}_{\Lambda\Lambda}+(k\Lambda+q\Lambda_q-p\Lambda_\mathrm{p})^2 N\begin{matrix}\Lambda\Lambda_\mathrm{p}\Lambda_q\\ -k\,p\,q\end{matrix}\left(N\begin{matrix}\Lambda\Lambda_\mathrm{p}\Lambda_q\\ -k\,p\,q\end{matrix}\right)^*\\ &\times\Delta(-2sB_0p_\parallel)q^{-s-s}_{\Lambda_\mathrm{p}\Lambda_\mathrm{p}}q^{ss}_{\Lambda_q\Lambda_q}+(k\Lambda+q\Lambda_q-p\Lambda_\mathrm{p})^2 N\begin{matrix}\Lambda\Lambda_\mathrm{p}\Lambda_q\\ -k\,p\,q\end{matrix}\\ &\times\left(-N\begin{matrix}\Lambda\Lambda_\mathrm{p}\Lambda_q\\ -k\,p\,q\end{matrix}\right)^*\Delta(2sB_0p_\parallel)q^{-s-s}_{\Lambda_\mathrm{p}\Lambda_\mathrm{p}}q^{ss}_{\Lambda\Lambda}+(k\Lambda+q\Lambda_q-p\Lambda_\mathrm{p})^2\\ &\times N\begin{matrix}\Lambda\Lambda_\mathrm{p}\Lambda_q\\ -k\,p\,q\end{matrix}\left(N\begin{matrix}\Lambda\Lambda_\mathrm{p}\Lambda_q\\ -k\,p\,q\end{matrix}\right)^*\Delta(2sB_0p_\parallel)q^{-s-s}_{\Lambda_\mathrm{p}\Lambda_\mathrm{p}}q^{ss}_{\Lambda_q\Lambda_q}\Bigg\}\delta_{pq,k}\,d\mathbf{p}\,d\mathbf{q}.\end{aligned}\tag{D 1}$$

The symmetry between terms is recovered; further simplifications lead to

$$\begin{aligned}\partial_t q^{ss}_{\Lambda\Lambda}(\mathbf{k}) = \frac{\epsilon}{16}\int \sum_{\Lambda_\mathrm{p},\Lambda_q}&\left|N\begin{matrix}\Lambda\Lambda_\mathrm{p}\Lambda_q\\ -k\,p\,q\end{matrix}\right|^2(k\Lambda+q\Lambda_q-p\Lambda_\mathrm{p})^2\\ &\times(\Delta(-2sB_0p_\parallel)+\Delta(2sB_0p_\parallel))q^{-s-s}_{\Lambda_\mathrm{p}\Lambda_\mathrm{p}}(q^{ss}_{\Lambda_q\Lambda_q}-q^{ss}_{\Lambda\Lambda})\delta_{pq,k}\,d\mathbf{p}\,d\mathbf{q}.\end{aligned}\tag{D 2}$$

The long-time behavior is given by the Riemann–Lebesgue lemma; one finds

$$\begin{aligned}\partial_t q^{ss}_{\Lambda\Lambda}(\mathbf{k}) = \frac{\pi\epsilon}{16B_0}\int \sum_{\Lambda_\mathrm{p},\Lambda_q}&\left(\frac{\sin\psi_k}{k}\right)^2(k\Lambda+p\Lambda_\mathrm{p}+q\Lambda_q)^2(k\Lambda+q\Lambda_q-p\Lambda_\mathrm{p})^2\\ &\times q^{-s-s}_{\Lambda_\mathrm{p}\Lambda_\mathrm{p}}(q^{ss}_{\Lambda_q\Lambda_q}-q^{ss}_{\Lambda\Lambda})\delta(p_\parallel)\delta_{k,pq}\,d\mathbf{p}\,d\mathbf{q}.\end{aligned}\tag{D 3}$$

Equations (D 3) are the wave kinetic equations for incompressible MHD turbulence at the level of three-wave interactions when helicities are absent and when equality between shear- and pseudo-Alfvén wave energies is assumed (see Sec. 4.5). We find the same equations as (4.16) since we have the relation $q^{ss}_{\Lambda\Lambda}(\mathbf{k}) = 4q^s_\Lambda(\mathbf{k})$.

Appendix E. Kinetic equations for the energies E and E_d

We introduce the expression (5.2) into (3.23) and we consider a state of zero helicities ($H_\mathrm{m}(\mathbf{k}) = 0$ and $H_\mathrm{G}(\mathbf{k}) = 0$). The kinetic equations for the energies E and E_d are

$$\begin{aligned}\partial_t&\begin{Bmatrix}E(\mathbf{k})\\ E_\mathrm{d}(\mathbf{k})\end{Bmatrix}\\ &=\frac{\pi\epsilon^2}{32d_\mathrm{i}^2B_0^2}\int\sum_{\substack{\Lambda,\Lambda_\mathrm{p},\Lambda_q\\ s,s_\mathrm{p},s_q}}\left(\frac{\sin\psi_k}{k}\right)^2\frac{(\Lambda k+\Lambda_\mathrm{p}p+\Lambda_q q)^2(1-\xi_\Lambda^{-s\,2}\xi_{\Lambda_p}^{-s_p\,2}\xi_{\Lambda_q}^{-s_q\,2})^2}{(1+\xi_\Lambda^{-s\,2})(1+\xi_{\Lambda_p}^{-s_p\,2})(1+\xi_{\Lambda_q}^{-s_q\,2})}\\ &\times\left(\frac{\xi_{\Lambda_q}^{s_q}-\xi_{\Lambda_p}^{s_p}}{k_\parallel}\right)^2\begin{Bmatrix}\xi_\Lambda^{-s}+1\\ \xi_\Lambda^{-s}-1\end{Bmatrix}\frac{\omega_\Lambda^s\omega_{\Lambda_p}^{s_p}}{\xi_\Lambda^{-s\,2}+1}\left[E(\mathbf{q})+\left(\frac{\xi_{\Lambda_q}^{-s_q\,2}+1}{\xi_{\Lambda_q}^{-s_q\,2}-1}\right)E_\mathrm{d}(\mathbf{q})\right]\end{aligned}$$

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$$\times \left[E(\mathbf{p}) - E(\mathbf{k}) + \left(\frac{\xi_{\Lambda_p}^{-s_p\,2}+1}{\xi_{\Lambda_p}^{-s_p\,2}-1}\right)E_{\rm d}(\mathbf{p}) - \left(\frac{\xi_{\Lambda}^{-s\,2}+1}{\xi_{\Lambda}^{-s\,2}-1}\right)E_{\rm d}(\mathbf{k})\right]$$

$$\times \delta(\Omega_{k,pq})\delta_{k,pq}\, d\mathbf{p}\, d\mathbf{q}. \tag{E 1}$$

Appendix F. Compatibility with Galtier et al. (2000)

We start from equation (26) of Galtier et al. (2000). We assume that helicities are absent ($I^s(\mathbf{k}) = 0$) and that shear- and pseudo-Alfvén wave energies are identical. We introduce the following notation:

$$k_\perp^2 \Psi^s(\mathbf{k}) = e^s(\mathbf{k})/2, \tag{F 1}$$

and

$$k_\perp^2 k^2 \Phi^s(\mathbf{k}) = e^s(\mathbf{k})/2. \tag{F 2}$$

Then equation (26) becomes

$$\partial_t e^s(\mathbf{k}) = \frac{\pi\epsilon^2}{4B_0}\int\Bigg\{\left(1 - \frac{(\mathbf{k}\times\kappa)^2}{L_\perp^2 k^2} + \frac{k_\parallel^2\kappa_\perp^2}{L_\perp^2 k^2}\right)e^s(\mathbf{L}) - \left(1 - \frac{(\mathbf{k}\times\kappa)^2}{k_\perp^2 L^2} + \frac{k_\parallel^2\kappa_\perp^2}{k_\perp^2 L^2}\right)e^s(\mathbf{k})$$

$$+ \left(1 - \frac{k_\parallel^2\kappa^2\cos^2\psi_k}{L_\perp^2 k^2}\right)e^s(\mathbf{L}) - \left(1 - \frac{k_\parallel^2\kappa^2\cos^2\psi_L}{k_\perp^2 L^2}\right)e^s(\mathbf{k})\Bigg\}$$

$$\times \left((\mathbf{k}\times\kappa)^2 - k_\parallel^2\kappa_\perp^2\right)\frac{e^{-s}(\kappa)}{\kappa_\perp^2} + k_\parallel^2 e^{-s}(\kappa)\Big\}\delta(\kappa_\parallel)\delta_{k,\kappa L}\, d\kappa\, d\mathbf{L}. \tag{F 3}$$

We remind the reader that the angle ψ_k refers to the angle opposite to the 3D vector $\mathbf{k}$ in the triangle defined by $\mathbf{k} = \mathbf{L} + \kappa$. (In Galtier et al. (2000), angles are introduced in reference to 2D wavevectors.) Some manipulations lead to

$$\partial_t e^s(\mathbf{k}) = \frac{\pi\epsilon^2}{4B_0}\int\Bigg\{\left(2 - \frac{\kappa_\perp^2\sin^2\psi_L}{L_\perp^2} + \frac{k_\parallel^2\kappa_\perp^2}{L_\perp^2 k^2} - \frac{k_\parallel^2\kappa_\perp^2\cos^2\psi_k}{L_\perp^2 k^2}\right)e^s(\mathbf{L})$$

$$\times \left(2 - \frac{k^2\kappa_\perp^2\sin^2\psi_L}{k_\perp^2 L^2} + \frac{k_\parallel^2\kappa_\perp^2}{k_\perp^2 L^2} - \frac{k_\parallel^2\kappa_\perp^2\cos^2\psi_L}{k_\perp^2 L^2}\right)e^s(\mathbf{k})\Bigg\}$$

$$\times k^2\sin^2\psi_L e^{-s}(\kappa)\delta(\kappa_\parallel)\delta_{k,\kappa L} d\kappa d\mathbf{L}, \tag{F 4}$$

which can be written as

$$\partial_t e^s(\mathbf{k}) = \frac{\pi\epsilon^2}{4B_0}\int(1+\cos^2\psi_\kappa)L^2\sin^2\psi_k e^{-s}(\kappa)(e^s(\mathbf{L}) - e^s(\mathbf{k}))\delta(\kappa_\parallel)\delta_{k,\kappa L}\, d\kappa\, d\mathbf{L}. \tag{F 5}$$

Equations (F 5) are the wave kinetic equations for incompressible MHD turbulence when helicities are absent and when equality between shear- and pseudo-Alfvén wave energies is assumed. The addition over the index s will give the wave kinetic equations for the total energy.

Appendix G. Non-local WCC interactions

We shall consider the strongly non-local interactions of two ion cyclotron waves at small scales on one whistler wave at large scales. In other words, it means that the

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whistler wave will be supported by the wavevector $\mathbf{k}$, with $k \ll p, q$. This type of interaction is the most interesting since smaller scales may be reached more easily by ion cyclotron waves. For such a limit, the master equations (5.3) reduce to

$$\partial_t \begin{Bmatrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{Bmatrix} = \frac{\pi\epsilon^2}{8d_{\mathrm{i}}^4} \int \sum_{s,s_{\mathrm{p}},s_q} \left(\frac{\sin\psi_k}{k}\right)^2 \left(\frac{s_q/q - s_{\mathrm{p}}/p}{k_\parallel}\right)^2 (s_{\mathrm{p}}p + s_q q)^2$$

$$\times \frac{pq^2 k_\parallel p_\parallel}{k^3} \begin{Bmatrix} \frac{1}{d_{\mathrm{i}}^2 k^2} \\ 1 \end{Bmatrix} E^V(\mathbf{q})[E^V(\mathbf{p}) - E^B(\mathbf{k})]\delta(\Omega_{k,pq})\delta_{k,pq}\, d\mathbf{p}\, d\mathbf{q}. \quad \text{(G 1)}$$

As expected, at leading order the whistler wave is described by the magnetic energy. Contrary to the other cases studied before, a non-trivial dynamics happens as long as a discrepancy exists between the (kinetic) energy of the ion cyclotron waves and the (magnetic) energy of the whistler wave.

References

Armstrong, J. W., Coles, W. A., Rickett, B. J. and Kojima, M. 1990 Observations of field-aligned density fluctuations in the inner Solar wind. *Astrophys. J.* **358**, 685–692.

Balbus, S. A. and Terquem, C. 2001 Linear analysis of the Hall effect in protostellar disks. *Astrophys. J.* **552**, 235–247.

Bale, S. D., Kellogg, P. J., Mozer, F. S., Horbury, T. S. and Reme, H. 2005 Measurement of the electric field fluctuation spectrum of MHD turbulence. *Phys. Rev. Lett.* **94**, 215 002.

Balk, A. M. and Nazarenko, S. V. 1990 Physical realizability of anisotropic weak-turbulence Kolmogorov spectra. *Sov. Phys. J. Exp. Theor. Phys.* **70**, 1031–1041.

Balk, A. M., Zakharov, V. E. and Nazarenko, S. V. 1990 Non-local turbulence of drift waves. *Sov. Phys. J. Exp. Theor. Phys.* **71**, 249–260.

Barnes, A. 1981 Interplanetary Alfvénic fluctuations: a stochastic model. *J. Geophys. Res.* **86**, 7498–7506.

Bartello, P. 1995 Geostrophic adjustment and inverse cascades in rotating stratified turbulence. *J. Atmos. Sci.* **52**, 4410–4428.

Bavassano, B., Woo, R. and Bruno, R. 1997 Heliospheric plasma sheet and coronal streamers. *Geophys. Res. Lett.* **24**, 1655–1658.

Belcher, J. W. and Davis, L. 1971 Large-amplitude Alfvén waves in the interplanetary medium. *J. Geophys. Res.* **76**, 3534–3563.

Bellet, F., Godeferd, F. S., Scott, J. F. and Cambon, C. 2005 Wave-turbulence in rapidly rotating flows. *J. Fluid Mech.*, in press.

Benney, J. and Saffman, P. G. 1966 Nonlinear interactions of random waves in a dispersive medium. *Proc. Roy. Soc.* A **289**, 301–320.

Benney, J. and Newell, A. C. 1969 Random wave closures. *Stud. Applied Math.* **48**, 29–53.

Bhattacharjee, A. 2004 Impulsive magnetic reconnection in the Earth's magnetotail and the Solar corona. *Ann. Rev. Astron. Astrophys.* **42**, 365–384.

Bhattacharjee, A. and Ng, C. S. 2001 Random scattering and anisotropic turbulence of shear-Alfvén wave packets. *Astrophys. J.* **548**, 318–322.

Bieber, J. W., Wanner, W. and Matthaeus, W. H. 1996 Dominant two-dimensional Solar wind turbulence with implications for cosmic ray transport. *J. Geophys. Res.* **101**, 2511–2522.

Biskamp, D. 1997 Collision and collisionless magnetic reconnection. *Phys. Plasmas* **4**, 1964–1968.

Biskamp, D., Schwarz, E., Zeiler, A., Celani, A. and Drake, J. F. 1999 Electron magnetohydrodynamic turbulence. *Phys. Plasmas* **6**, 751–758.

Burlaga, L. F. 1991 Intermittent turbulence in the Solar wind. *J. Geophys. Res.* **96**, 5847–5851.

https://doi.org/10.1017/S0022377806004521 Published online by Cambridge University Press

Burlaga, L. F. and Mish, W. H. 1987 Large-scale fluctuations in the interplanetary medium. *J. Geophys. Res.* **92**, 1261–1266.

Cai, C. L., Cao, J. B., Zhou, G. C. and Wang, D. J. 2001 Whistler turbulence at the magnetopause: A nonlinear generation mechanism. *Phys. Plasmas* **8**, 272–276.

Caillol, P. and Zeitlin, V. 2000 Kinetic equations and stationary energy spectra of weakly nonlinear internal gravity waves. *Dyn. Atmos. Oceans* **32**, 81–112.

Cambon, C. and Jacquin, L. 1989 Spectral approach to non-isotropic turbulence subjected to rotation. *J. Fluid Mech.* **202**, 295–317.

Cho, J. and Vishniac, E. T. 2000 The anisotropy of magnetohydrodynamic Alfvénic turbulence. *Astrophys. J.* **539**, 273–282.

Coleman, P. J. 1968 Turbulence, viscosity, and dissipation in the Solar-wind plasma. *Astrophys. J.* **153**, 371–388.

Coroniti, F. V., Kennel, C. F. and Scarf, F. L. 1982 Whistler mode turbulence in the disturbed Solar wind. *J. Geophys. Res.* **87**, 6029–6044.

Craya, A. 1958 Contribution à l'analyse de la turbulence associée à des vitesses moyennes. *P.S.T. Ministère de l'Air* **345**.

Cumming, A., Arras, P. and Zweibel, E. 2004 Magnetic field evolution in neutron star crusts due to the Hall effect and ohmic decay. *Astrophys. J.* **609**, 999–1017.

Davidson, P. A. 2001 *An Introduction to Magnetohydrodynamics*. Cambridge texts in Applied Mathematics. Cambridge: Cambridge University Press.

Denskat, K. U., Beinroth, H. J. and Neubauer, F. M. 1983 Interplanetary magnetic field power spectra with frequencies from 2.4×10^{-5} Hz to 470 Hz from HELIOS-observations during Solar minimum conditions. *J. Geophys.* **54**, 60–67.

Dmitruk, P., Matthaeus, W. H. and Oughton, S. 2005 Direct comparisons of compressible MHD and reduced MHD turbulence. *Phys. Plasmas* **12**, 112 304.

Dreher, J., Laveder, D., Grauer, R., Passot, T. and Sulem, P.-L. 2005 Formation and disruption of Alfvénic filaments in Hall MHD. *Phys. Plasmas* **12**, 523 191–523 197.

Elmegreen, B. G. and Scalo, J. 2004 Interstellar turbulence I: Observations and processes. *Annu. Rev. Astron. Astrophys.* **42**, 211–273.

Forsyth, R. J., Balogh, A., Horbury, T. S., Erdös, G., Smith, E. J. and Burton, M. E. 1996a The heliospheric magnetic field at Solar minimum: Ulysses observations from pole to pole. *Astron. Astrophys.* **316**, 287–295.

Forsyth, R. J., Horbury, T. S., Balogh, A. and Smith, E. J. 1996b Hourly variances of fluctuations in the heliospheric magnetic field out of the ecliptic plane. *Geophys. Res. Lett.* **23**, 595–598.

Galtier, S. 2003 A weak inertial wave turbulence theory. *Phys. Rev.* E **68**, 015301.

Galtier, S. and Bhattacharjee, A. 2003 Anisotropic weak whistler wave turbulence in electron magnetohydrodynamics. *Phys. Plasmas* **10**, 3065–3076.

Galtier, S., Nazarenko, S., Newell, A. C. and Pouquet, A. 2000 A weak turbulence theory for incompressible magnetohydrodynamics. *J. Plasma Physics* **63**, 447–488.

Galtier, S., Nazarenko, S., Newell, A. C. and Pouquet, A. 2002 Anisotropic turbulence of shear-Alfvén waves. *Astrophys. J.* **564**, L49–L52.

Galtier, S., Pouquet, A. and Mangeney, A. 2005 On spectral scaling laws for incompressible anisotropic MHD turbulence. *Phys. Plasmas* **12**, 092 310.

Ghosh, S. and Goldstein, M. L. 1997 Anisotropy in Hall MHD turbulence due to a mean magnetic field. *J. Plasma Physics* **57**, 129–154.

Ghosh, S., Siregar, E., Roberts, D. A. and Goldstein, M. L. 1996 Simulation of high-frequency Solar wind power spectra using Hall magnetohydrodynamic. *J. Geophys. Res.* **101**, 2493–2504.

Goldreich, P. and Reisenegger, A. 1992 Magnetic field decay in isolated neutron stars *Astrophys. J.* **395**, 250–258.

Goldreich, P. and Sridhar, S. 1995 Toward a theory of interstellar turbulence. II. Strong Alfvénic turbulence. *Astrophys. J.* **438**, 763–775.

https://doi.org/10.1017/S0022377806004521 Published online by Cambridge University Press

Goldstein, M. L. 2005 Magnetospheric physics: Turbulence on a small scale. *Nature* **436**, 782–783.

Goldstein, M. L. and Roberts, D. A. 1999 Magnetohydrodynamics turbulence in the Solar wind. *Phys. Plasmas* **6**, 4154–4160.

Goldstein, M. L., Roberts, D. A. and Fitch, C. A. 1994 Properties of the fluctuating magnetic helicity in the inertial and dissipative ranges of Solar wind turbulence. *J. Geophys. Res.* **99**, 11 519–11 538.

Grappin, R., Mangeney, A. and Marsch, E. 1990 On the origin of Solar wind MHD turbulence: Helios data revisited. *J. Geophys. Res.* **95**, 8197–8209.

Hameiri, E., Ishizawa, A. and Ishida, A. 2005 Waves in the Hall-magnetohydrodynamics model. *Phys. Plasmas* **12**, 072 109.

Hasselman, K. 1962 On the nonlinear energy transfer in a gravity-wave spectrum. *J. Fluid Mech.* **12**, 481–500.

Horbury, T. S. 1999 Waves and turbulence in the solar wind – An overview. In: *Proc. Int. Conf. on Plasma Turbulence and Energetic Particles in Astrophysics* (ed. M. Ostrowski and R. Schlickeiser). Universytet Jagiellonski Cracow (Poland), pp. 115–134.

Huba, J. D. 1995 Hall magnetohydrodynamics in space and laboratory plasmas. *Phys. Plasmas* **2**(6), 2504–2513.

Iroshnikov, P. 1963 Turbulence of a conducting fluid in a strong magnetic field. *Sov. Astron.* **7**, 566–571.

Kingsep, A. S., Chukbar, K. V. and Yankov, V. V. 1990 In: *Reviews of Plasma Physics*, vol. 16. New York: Consultant Bureau. "Election MHD", pp. 243–291.

Kinney, R. M. and McWilliams, J. C. 1998 Turbulent cascades in anisotropic magnetohydrodynamic. *Phys. Rev. E* **57**, 7111–7121.

Klein, L., Bruno, R., Bavassano, B. and Rosenbauer, H. 1993 Anisotropy and minimum variance of magnetohydrodynamic fluctuations in the inner heliosphere. *J. Geophys. Res.* **98**, 17 461–17 466.

Kolmogorov, A. N. 1941 The local structure of turbulence in incompressible viscous fluid for very high Reynolds numbers. *Dokl. Akad. Nauk. SSS* **30**, 301–305.

Kraichnan, R. H. 1965 Inertial-range spectrum of hydromagnetic turbulence. *Phys. Fluids* **8**, 1385–1387.

Kraichnan, R. H. 1973 Helical turbulence and absolute equilibrium. *J. Fluid Mech.* **59**, 745–752.

Krishan, V. and Mahajan, S. M. 2004 Magnetic fluctuations and Hall MHD turbulence in the Solar wind. *J. Geophys. Res.* **109**, A111051.

Krishan, V. and Mahajan, S. M. 2005 Modeling of short scale turbulence in the Solar wind. *Nonlin. Proc. Geophys.* **12**, 75–81.

Kuznetsov, E. A. 1972 Turbulence of ion sound in a plasma located in a magnetic field. *Sov. Phys. J. Exp. Theor. Phys.* **35**, 310–314.

Kuznetsov, E. A. 2001 Weak magnetohydrodynamic turbulence of a magnetized plasma. *Sov. Phys. J. Exp. Theor. Phys.* **93**, 1052–1064.

Laval, J.-P., Dubrulle, B. and Nazarenko, S. 2001 Nonlocality and intermittency in three-dimensional turbulence. *Phys. Fluids* **13**, 1995–2012.

Leamon, R. J., Smith, C. W., Ness, N. F. and Matthaeus, W. H. 1998 Observational constraints on the dynamics of the interplanetary magnetic field dissipation range. *J. Geophys. Res.* **103**, 4775–4787.

Lesieur, M. 1990 *Turbulence in Fluids*, 2nd edn. Dordrecht: Kluwer.

Lithwick, Y. and Goldreich, P. 2003 Imbalanced weak magnetohydrodynamic turbulence. *Astrophys. J.* **582**, 1220–1240.

Mahajan, S. M. and Krishan, V. 2005 Exact solution of the incompressible Hall MHD. *Mon. Not. R. Astron. Soc.* **359**, L27–L29.

Matthaeus, W. H. and Goldstein, M. L. 1982 Measurement of the rugged invariants of magnetohydrodynamic turbulence in the Solar wind. *J. Geoph. Res.* **87A**, 6011–6028.

https://doi.org/10.1017/S0022377806004521 Published online by Cambridge University Press

Matthaeus, W. H., Ghosh, S., Oughton, S. and Roberts, D. A. 1996 Anisotropic three dimensional MHD turbulence. *J. Geophys. Res.* **101**, 7619–7629.

Matthaeus, W. H., Dasso, S., Weygand, J. M., Milano, L. J., Smith, C. W. and Kivelson, M. G. 2005 Spatial correlation of Solar-wind turbulence from two-point measurements. *Phys. Rev. Lett.* **95**, 231 101.

Mininni, P. D., Gómez, D. O. and Mahajan, S. M. 2003a Role of the Hall current in magnetohydrodynamic dynamos. *Astrophys. J.* **584**, 1120–1126.

Mininni, P. D., Gómez, D. O. and Mahajan, S. M. 2003b Dynamo action in magnetohydrodynamics and Hall-magnetohydrodynamics. *Astrophys. J.* **587**, 472–481.

Mininni, P. D., Alexakis, A. and Pouquet, A. 2005 Energy transfer in Hall-MHD turbulence: Cascades, backscatter, and dynamo action. *J. Plasma Phys.* submitted.

Moffatt, H. K. 1970 Dynamo action associated with random inertial waves in a rotating conducting fluid. *J. Fluid Mech.* **44**, 705–719.

Montgomery, D. and Turner, L. 1981 Anisotropic magnetohydrodynamic turbulence in a strong external magnetic field. *Phys. Fluids* **24**, 825–831.

Morize, C., Moisy, F. and Rabaud, M. 2005 Decaying grid-generated turbulence in a rotating tank. *Phys. Fluids* **17**, 095 105.

Nazarenko, S. V., Newell, A. C. and Galtier, S. 2001 Non-local MHD turbulence. *Physica D* **152–153**, 646–652.

Newell, A. C., Nazarenko, S. and Biven, L. 2001 Wave turbulence and intermittency. *Physica D* **152–153**, 520–550.

Ng, C. S. and Bhattacharjee, A. 1996 Interaction of shear-Alfvén wave packets: Implication for weak MHD turbulence in astrophysical plasmas. *Astrophys. J.* **465**, 845–854.

Ng, C. S. and Bhattacharjee, A. 1997 Scaling of anisotropic spectra due to the weak interaction of shear-Alfvén wave packets. *Phys. Plasmas* **4**, 605–610.

Ohsaki, S. 2005 Hall effect on relaxation process of flowing plasmas. *Phys. Plasmas* **12**, 032 306.

Oughton, S. and Matthaeus, W. H. 2005 Parallel and perpendicular cascades in Solar wind turbulence. *Nonlin. Proc. Geophys.* **12**, 299–310.

Oughton, S., Priest, E. R. and Matthaeus, W. H. 1994 The influence of a mean magnetic field on three-dimensional magnetohydrodynamic turbulence. *J. Fluid Mech.* **280**, 95–117.

Passot, T. and Sulem, P.-L. 2003 Filamentation instability of long Alfvén waves in warm collisionless plasmas. *Phys. Plasmas* **10**, 3914–3921.

Petrosyan, A., Balogh, A., et al. 2005 Turbulence in the Solar atmosphere and Solar wind. *Solar Phys.*, in preparation.

Phillips, O. M. 1968 The interaction trapping of internal gravity waves. *J. Fluid Mech.* **34**, 407–416.

Pouquet, A. 1978 On two-dimensional MHD turbulence. *J. Fluid Mech.* **88**, 1–16.

Pouquet, A. 1993 Magnetohydrodynamic turbulence. In: *Astrophysical Fluid Dynamics* (ed. J.-P. Zahn and J. Zinn-Justin). Amsterdam: Elsevier, pp. 139–227.

Roberts, D. A. and Goldstein, M. L. 1987 Spectral signatures of jumps and turbulence in interplanetary speed and magnetic field data. *J. Geoph. Res.* **92**, 10 105–10 110.

Roberts, D. A., Goldstein, M. L., Klein, L. W. and Matthaeus, W. H. 1987 Origin and evolution of fluctuations in the Solar wind: Helios observations and Helios-Voyager comparison. *J. Geoph. Res.* **92**, 12 023–12 035.

Sagdeev, R. and Galeev, A. 1969 *Nonlinear Plasma Theory*. New York: Benjamin.

Sahraoui, F., Belmont, G. and Rezeau, L. 2003 Hamiltonian canonical formulation of Hall MHD: Toward an application to weak turbulence. *Phys. Plasmas* **10**, 1325–1337.

Sahraoui, F., Belmont, G., Pinçon, J. L., Rezeau, L., Balogh, A., Robert, P. and Cornilleau-Wehrlin, N. 2004 Magnetic turbulent spectra in the magnetosheath: new insights. *Ann. Geophys.* **22**, 2283–2288.

Sahraoui, F., Galtier, S. and Belmont, G. 2006 On waves in incompressible Hall MHD waves, submitted to J. Plasma Physics.

https://doi.org/10.1017/S0022377806004521 Published online by Cambridge University Press

Scalo, J. and Elmegreen, B. G. 2004 Interstellar turbulence II: Implications and effects. *Ann. Rev. Astron. Astrophys.* **42**, 275–316.

Shebalin, J. V., Matthaeus, W. H. and Montgomery, D. 1983 Anisotropy in MHD turbulence due to a mean magnetic field. *J. Plasma Phys.* **29**, 525–547.

Shukla, P. K. and Stenflo, L. 1999 Electron magnetohydrodynamic of inhomogeneous plasmas. *Phys. Lett. A* **259**, 49–52.

Smith, L. M. and Waleffe, F. 1999 Transfer of energy to 2D large scales in forced, rotating 3D turbulence. *Phys. Fluids* **11**, 1608–1622.

Spangler, S. R. 2002 The amplitude of magnetohydrodynamic turbulence in the Solar wind. *Astrophys. J.* **576**, 997–1004.

Stawicki, O., Gary, P. S. and Li, H. 2001 Solar wind magnetic fluctuations spectra: Dispersion versus damping. *J. Geophys. Res.* **106**, 8273–8281.

Taylor, J. B. 1993 Filamentation, current profiles, and transport in a tokamak. *Phys. Fluids B* **5**, 4378–4383.

Turner, L. 2000 Using helicity to characterize homogeneous and inhomogeneous turbulent dynamics. *J. Fluid Mech.* **408**, 205–237.

Velli, M., Grappin, R. and Mangeney, A. 1989 Turbulent cascade of incompressible unidirectional Alfvén waves in the interplanetary medium. *Phys. Rev. Lett.* **63**, 1807–1810.

Waleffe, F. 1992 The nature of triad interactions in homogeneous turbulence. *Phys. Fluids A* **4**, 350–363.

White, R., Chen, L. and Lin, Z. 2002 Resonant plasma heating below the cyclotron frequency. *Phys. Plasmas* **9**(5), 1890–1897.

Wild, N., Gekelman, W. and Stenzel, R. L. 1981 Resistivity and energy flow in a plasma undergoing magnetic field line reconnection. *Phys. Rev. Lett.* **46**, 339–342.

Woltjer, L. 1958 On hydromagnetic equilibrium. *Proc. Natl. Acad. Sci. U.S.A.* **44**, 833–841.

Zakharov, V. E. 1967 On the spectrum of turbulence in plasma without magnetic field. *J. Exp. Theor. Phys.* **24**, 455–459.

Zakharov, V. E., L'vov, V. and Falkovich, G. E. 1992 *Kolmogorov Spectra of Turbulence I: Wave Turbulence*. Berlin: Springer.

Zhou, Y., Matthaeus, W. H. and Dmitruk, P. 2004 Magnetohydrodynamic turbulence and time scales in astrophysical and space plasmas. *Rev. Mod. Phys.* **76**, 1015–1035.

https://doi.org/10.1017/S0022377806004521 Published online by Cambridge University Press