Nagoya Math. J., **247** (2022), 615–623
DOI 10.1017/nmj.2021.14

# COMPACT ORBITS OF PARABOLIC SUBGROUPS

LEONARDO BILIOTTI AND OLUWAGBENGA JOSHUA WINDARE

**Abstract.** We study the action of a real reductive group $G$ on a real submanifold $X$ of a Kähler manifold $Z$. We suppose that the action of a compact connected Lie group $U$ with Lie algebra $\mathfrak{u}$ extends holomorphically to an action of the complexified group $U^{\mathbb{C}}$ and that the $U$-action on $Z$ is Hamiltonian. If $G \subset U^{\mathbb{C}}$ is compatible, there exists a gradient map $\mu_{\mathfrak{p}} : X \longrightarrow \mathfrak{p}$ where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. In this paper, we describe compact orbits of parabolic subgroups of $G$ in terms of the gradient map $\mu_{\mathfrak{p}}$.

## §1. Introduction

In this paper, we study the actions of real reductive groups on real submanifolds of Kähler manifolds.

Let $U$ be a compact connected Lie group with Lie algebra $\mathfrak{u}$, and let $U^{\mathbb{C}}$ be its complexification. We say that a subgroup $G$ of $U^{\mathbb{C}}$ is compatible if $G$ is closed and the map $K \times \mathfrak{p} \to G, (k, \beta) \mapsto k\exp(\beta)$ is a diffeomorphism where $K := G \cap U$ and $\mathfrak{p} := \mathfrak{g} \cap \mathrm{i}\mathfrak{u}; \mathfrak{g}$ is the Lie algebra of $G$. The Lie algebra $\mathfrak{u}^{\mathbb{C}}$ of $U^{\mathbb{C}}$ is the direct sum $\mathfrak{u} \oplus i\mathfrak{u}$. It follows that $G$ is compatible with the Cartan decomposition $U^{\mathbb{C}} = U\exp(\mathrm{i}\mathfrak{u})$, $K$ is a maximal compact subgroup of $G$ with Lie algebra $\mathfrak{k}$, and that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Note that $G$ has finitely number of connected components. In the sequel, we always assume that $G$ is connected.

Let $(Z, \omega)$ be a Kähler manifold with a holomorphic action of the complex reductive group $U^{\mathbb{C}}$. We also assume $\omega$ is $U$-invariant, and that there is a $U$-equivariant momentum map $\mu : Z \to \mathfrak{u}^*$. By definition, for any $\xi \in \mathfrak{u}$ and $z \in Z, d\mu^{\xi} = i_{\xi_Z}\omega$, where $\mu^{\xi}(z) := \langle \mu(z), \xi \rangle$, and $\xi_Z$ denotes the fundamental vector field induced on $Z$ by the action of $U$,

$$\xi_Z(z) := \frac{d}{dt}|_{t=0}\exp(t\xi) \cdot z.$$

The inclusion $\mathrm{i}\mathfrak{p} \hookrightarrow \mathfrak{u}$ induces, by restriction, a $K$-equivariant map $\mu_{\mathrm{i}\mathfrak{p}} : Z \to (\mathrm{i}\mathfrak{p})^*$. Using an $\mathrm{Ad}(U)$-invariant inner product on $u$ to identify $(\mathrm{i}\mathfrak{p})^*$ and $\mathfrak{p}, \mu_{\mathrm{i}\mathfrak{p}}$ can be viewed as a map $\mu_{\mathfrak{p}} : Z \to \mathfrak{p}$. For $\beta \in \mathfrak{p}$, let $\mu^{\beta}_{\mathfrak{p}}$ denote $\mu^{-\mathrm{i}\beta}$, that is, $\mu^{\beta}_{\mathfrak{p}}(z) := -\langle \mu(z), i\beta \rangle$. Then, the $\mathrm{grad}\mu^{\beta}_{\mathfrak{p}} = \beta_Z$ where grad is computed with respect to the Riemannian metric induced by the Kähler structure. The map $\mu_{\mathfrak{p}}$ is called the gradient map associated with $\mu$. For a $G$-stable locally closed real submanifold $X$ of $Z$, we consider $\mu_{\mathfrak{p}}$ as a mapping $\mu_{\mathfrak{p}} : X \to \mathfrak{p}$ (see [9] for more details).

Let $\mathfrak{a} \subset \mathfrak{p}$ be an abelian subalgebra, and let $\pi : \mathfrak{p} \longrightarrow \mathfrak{a}$ be the orthogonal projection onto $\mathfrak{a}$. Then, $\mu_{\mathfrak{a}} = \pi_{\mathfrak{a}} \circ \mu_{\mathfrak{p}}$ is the gradient map associated with $A = \exp(\mathfrak{a})$.

Received May 13, 2021. Accepted November 2, 2021.
2020 Mathematics subject classification: 57S20, 32M05
Biliotti was partially supported by the Project PRIN 2015, *Real and Complex Manifolds: Geometry, Topology and Harmonic Analysis*, by the Project PRIN 2017, *Real and Complex Manifolds: Topology, Geometry and Holomorphic Dynamics*, and by the GNSAGA INdAM.

© *(2021) The Authors. Copyright in the Journal, as distinct from the individual articles, is owned by Foundation Nagoya Mathematical Journal.*

https://doi.org/10.1017/nmj.2021.14 Published online by Cambridge University Press

If $\beta \in \mathfrak{p}$, set

$$G^{\beta+} := \{g \in G : \lim_{t\to-\infty} \exp(t\beta)g\exp(-t\beta) \text{ exists}\},$$
$$R^{\beta+} := \{g \in G : \lim_{t\to-\infty} \exp(t\beta)g\exp(-t\beta) = e\}, \qquad \mathfrak{r}^{\beta+} := \bigoplus_{\lambda>0} V_\lambda(\mathrm{ad}\beta),$$
$$G^{\beta} = \{g \in G : \mathrm{Ad}(g)(\beta) = \beta\}.$$

Note that $\mathfrak{g}^{\beta+} = \mathfrak{g}^{\beta} \oplus \mathfrak{r}^{\beta+}$. It is well known that $G^{\beta+}$ is a parabolic subgroup of $G$ with Lie algebra $\mathfrak{g}^{\beta+}$, and every parabolic subgroup of $G$ arises as $G^{\beta+}$ for some $\beta \in \mathfrak{p}$. $R^{\beta+}$ is connected, and it is the unipotent radical of $G^{\beta+}$. $G^{\beta}$ is a Levi factor of $G^{\beta+}$ (see [5, Lem. 9], [6] and [11] for more details). Our first main result is the following.

THEOREM 1. *Let $\beta \in \mathfrak{p}$. Then:*

- *If $G^{\beta+}\cdot x$ is compact, then $\mathcal{O} = G\cdot x$ is compact and $G^{\beta+}\cdot x$ is a finite union of connected components of* $\max_{\mathcal{O}}(\beta) = \left\{p \in \mathcal{O} : \max_{z\in\mathcal{O}} \mu^{\beta}_{\mathfrak{p}} = \mu^{\beta}_{\mathfrak{p}}(p)\right\}$.
- *If $\mathcal{O}$ is a compact $G$-orbit, then* $\max_{\mathcal{O}}(\beta)$ *is a finite union of compact $G^{\beta+}$-orbits.*

*In particular, the number of compact $G^{\beta+}$-orbits is equal or bigger than the number of compact $G$-orbits.*

Observe that $\mu_{\mathfrak{p}}(\mathcal{O})$ is a $K$-orbit, but it is not true, in general, that $\mu_{\mathfrak{p}}$ defines a diffeomporhism between $\mathcal{O}$ and $\mu_{\mathfrak{p}}(\mathcal{O})$, without the assumption that $G$ is a complex reductive group. Therefore, Theorem 1.2 in [5, p. 582] does not apply in our context.

Let $\xi \in \mathfrak{u}$. The standard notation for parabolic subgroups of complex reductive groups (see, e.g., [12]) is given by

$$U^{\mathbb{C}}(\xi) = \{g \in U^{\mathbb{C}} : \lim_{t\to-\infty} \exp(it\xi)g\exp(-it\xi)\text{exists}\}.$$

It is well known that $U^{\mathbb{C}}(\xi)$ is connected, and it contains a Borel subgroup, that is, a maximal solvable subgroup of $U^{\mathbb{C}}$ (see [1]). Hence, if $\beta \in i\mathfrak{u}$, then $U^{\mathbb{C}}(-i\beta)$ corresponds to $(U^{\mathbb{C}})^{\beta+}$ in our notation. If $\tilde{\mathcal{O}}$ is a compact orbit of $U^{\mathbb{C}}$, then it is a complex $U$-orbit (see [9]), and so it is a flag manifold (see [8]). Since

$$\max_{\tilde{\mathcal{O}}}(\beta) = \left\{p \in \tilde{\mathcal{O}} : \max_{z\in\tilde{\mathcal{O}}} \mu^{-i\beta} = \mu^{-i\beta}(p)\right\},$$

it follows that $\max_{\tilde{\mathcal{O}}}(\beta)$ is connected (see [2], [7]). Hence, the following result (see also [5]) holds.

COROLLARY 1.0.1. *The number of compact $(U^{\mathbb{C}})^{\beta+}$-orbits is equal to the number of compact $U^{\mathbb{C}}$-orbits. Moreover, any closed $(U^{\mathbb{C}})^{\beta+}$-orbit arises as* $\max_{\tilde{\mathcal{O}}}(\beta)$*, where $\tilde{\mathcal{O}}$ is a compact $U^{\mathbb{C}}$-orbit.*

Assume that $G$ is a real form of $U^{\mathbb{C}}$. Assume there exists $p \in Z$ such that $X = U^{\mathbb{C}}\cdot p$ is compact. If $Z$ is compact, then the $U^{\mathbb{C}}$-orbit throughout the maximum of the norm square function $\| \mu \|^2$ is a compact orbit, and so it is a flag manifold (see [9]). It is well known that $G$ has a unique closed orbit $\mathcal{O}$ in $X$. This is an old result of Wolf [16] (see also [9]). In this setting, we prove the following result.

https://doi.org/10.1017/nmj.2021.14 Published online by Cambridge University Press

THEOREM 2. *The set* $\max_{\mathcal{O}}(\beta)$ *is the unique closed orbit of* $G^{\beta+}$ *acting on* $X$. *This orbit is connected, and it is a* $(K^{\beta})^{o}$*-orbit.*

As a consequence of the proof, we obtain the following result.

PROPOSITION 1.1. *Let* $\mathfrak{a} \subset \mathfrak{p}$ *be an abelian subalgebra. Let* $\mu_{\mathfrak{a}} : X \longrightarrow \mathfrak{a}$ *be the corresponding* $A = \exp(\mathfrak{a})$*-gradient map. Then,* $\mu_{\mathfrak{a}}(X) = \mu_{\mathfrak{a}}(\mathcal{O})$.

It is well known that both $\mu_{\mathfrak{a}}(X)$ and $\mu_{\mathfrak{a}}(\mathcal{O})$ are polytopes (see [9]). The above result tells us that $\mathcal{O}$ captures much of the information of $\mu_{\mathfrak{a}}$. Note that if $\mathfrak{a} \subset \mathfrak{p}$ is a maximal abelian subalgebra, then by a beautiful theorem of Kostant [14], keeping in mind that $\mu_{\mathfrak{a}} = \pi_{\mathfrak{a}} \circ \mu_{\mathfrak{p}}$ and $\mu_{\mathfrak{p}}(\mathcal{O})$ is a $K$-orbit in $\mathfrak{p}$, the set $\mu_{\mathfrak{a}}(X)$ is the convex hull of an orbit of the Weyl group $W(\mathfrak{g},\mathfrak{a}) = \{\mathrm{Ad}(k) : k \in K, \mathrm{Ad}(k)(\mathfrak{a}) = \mathfrak{a}\}$ (see [13] for more details on the Weyl group).

## §2. Preliminaries

### 2.1 Convex geometry

In this section, some definitions and results in convex geometry are recalled. The reader can see, for example, [5], [15] for further details on the topic.

Let $V$ be a real vector space with a scalar product $\langle , \rangle$, and let $E \subset V$ be a compact convex subset. The *relative interior* of $E$, denoted by $\operatorname{relint} E$, is the interior of $E$ in its affine hull. For $a, b \in E$, denote the close segment joining $a$ and $b$ by $[a,b]$. Then, a face of $E$ is a convex subset $F$ of $E$ such that if $a, b \in E$ and $\operatorname{relint}[a,b] \cap F \neq \emptyset$, then $[a,b] \subset F$. The *extreme points* of $E$, denoted by $\operatorname{ext} E$, are the points $a \in E$ such that $\{a\}$ is a face. Since $E$ is compact, the faces are closed (see [15, p. 62]). The empty set and $E$ are faces of $E$: the other faces are called *proper*.

DEFINITION 2.1. The support function of $E$ is defined by the function $h_E : V \to \mathbb{R}$, $h_E(u) = \max\{\langle x, u\rangle : x \in E\}$. If $u \neq 0$, the hyperplane $H(E,u) := \{x \in E : \langle x, u\rangle = h_E(u)\}$ is called the supporting hyperplane of $E$ for $u$. The set

$$F_u(E) := E \cap H(E,u) \tag{1}$$

is a face, and it is called the *exposed face* of $E$ defined by $u$.

In general, not all faces of a convex subset are exposed. For instance, consider the convex hull of a closed disc and a point outside the disc: the resulting convex set is the union of the disc and a triangle. The two vertices of the triangle that lie on the boundary of the disc are nonexposed 0-faces.

The following result is probably well known. A proof is given in [4]. For the sake of completeness, we give a proof.

PROPOSITION 2.1. *Let* $C_1 \subseteq C_2$ *be two compact convex set of* $V$. *Assume that for any* $\beta \in V$, *we have*

$$\max_{y \in C_1} \langle y, \beta\rangle = \max_{y \in C_2} \langle y, \beta\rangle.$$

*Then,* $C_1 = C_2$.

*Proof.* We may assume without loss of generality that the affine hull of $C_2$ is $V$. Assume by contradiction that $C_1 \subsetneq C_2$. Since $C_1$ and $C_2$ are both compact, it follows that there exists $p \in \partial C_1$ such that $p \in \overset{o}{C}_2$. Since every face of a convex compact set is contained in an

https://doi.org/10.1017/nmj.2021.14 Published online by Cambridge University Press

exposed face (see [15]), there exists $\beta \in V$ such that

$$\max_{y \in C_1} \langle y, \beta \rangle = \langle p, \beta \rangle.$$

This means the linear function $x \mapsto \langle x, \beta \rangle$ restricted on $C_2$ achieves its maximum at an interior point which is a contradiction. □

### 2.2 Gradient map

Let $(Z,\omega)$ be a Kähler manifold. Let $U^{\mathbb{C}}$ acts holomorphically on $Z$, that is, the action $U^{\mathbb{C}} \times Z \to Z$ is holomorphic. Assume that $U$ preserves $\omega$ and that there is a $U$-equivariant momentum map $\mu : Z \to \mathfrak{u}$. If $\xi \in \mathfrak{u}$, we denote by $\xi_Z$ the induced vector field on $Z$, and we let $\mu^\xi \in C^\infty(Z)$ be the function $\mu^\xi(z) := \langle \mu(z), \xi \rangle$. By definition, we have

$$d\mu^\xi = i_{\xi_Z}\omega.$$

Let $G \subset U^{\mathbb{C}}$ be a compatible subgroup of $U^{\mathbb{C}}$. For $x \in Z$, let $\mu_{\mathfrak{p}}^\beta(x)$ denote $-i$ times the component of $\mu(x)$ along $\beta$ in the direction of $i\mathfrak{p}$, that is,

$$\mu_{\mathfrak{p}}^\beta(x) := \langle \mu_{\mathfrak{p}}(x), \beta \rangle = -\langle \mu(x), i\beta \rangle = \mu^{-i\beta}, \tag{2}$$

for any $\beta \in \mathfrak{p}$. Here, $\langle \cdot, \cdot \rangle$ denotes $K$-invariant inner product on $\mathfrak{p} \subset i\mathfrak{u}$. Then, the map defined by $\mu_{\mathfrak{p}} : Z \to \mathfrak{p}$ is called the *gradient map*. Let $\mu_{\mathfrak{p}}^\beta \in C^\infty(Z)$ be the function $\mu_{\mathfrak{p}}^\beta(z) = \langle \mu_{\mathfrak{p}}(z), \beta \rangle = \mu^{-i\beta}(z)$. Let $(\,,\,)$ be the Kähler metric associated with $\omega$, that is, $(v,w) = \omega(v, Jw)$, for all $z \in Z$ and $v, w \in T_z Z$, where $J$ denotes the complex structure on $TZ$. Then, $\beta_Z$ is the gradient of $\mu_{\mathfrak{p}}^\beta$.

For the rest of this paper, we assume that $G$ is connected, and we fix a $G$-invariant locally closed submanifold $X$ of $Z$. From now on, we denote the restriction of $\mu_{\mathfrak{p}}$ to $X$ by $\mu_{\mathfrak{p}}$. Then,

$$\mathrm{grad}\mu_{\mathfrak{p}}^\beta = \beta_X,$$

where grad is computed with respect to the induced Riemannian metric on $X$.

Let $\beta \in \mathfrak{p}$. It is well known that $G^\beta$ is compatible and

$$G^\beta = K^\beta \exp(\mathfrak{p}^\beta),$$

where $K^\beta = K \cap G^\beta = \{g \in K : \mathrm{Ad}(g)(\beta) = \beta\}$ and $\mathfrak{p}^\beta = \{v \in \mathfrak{p} : [v, \beta] = 0\}$ (see [13]).

COROLLARY 2.1.1. *If $x \in X$ and $\mu_{\mathfrak{p}}(x) = \beta$, there are a $G^\beta$-invariant decomposition $T_x X = \mathfrak{g}^\beta \cdot x \oplus W$, open $G^\beta$-invariant subsets $S \subset W$, $\Omega \subset X$, and a $G^\beta$-equivariant diffeomorphism $\Psi : G^\beta \times^{G_x} S \to \Omega$, such that $0 \in S, x \in \Omega$, and $\Psi([e,0]) = x$.*

*Proof.* See [9, p. 169]. □

$\beta_X$ is a vector field on $X$, that is, a section of $TX$. For $x \in X$, the differential is a map $T_x X \to T_{\beta_X(x)}(TX)$. If $\beta_X(x) = 0$, there is a canonical splitting $T_{\beta_X(x)}(TX) = T_x X \oplus T_x X$. Accordingly, $d\beta_X(x)$ splits into a horizontal part and a vertical part. The horizontal part is the identity map. We denote the vertical part by $d\beta_X(x)$. It belongs to $\mathrm{End}(T_x X)$. Let $\{\varphi_t = \exp(t\beta)\}$ be the flow of $\beta_X$. There is a corresponding flow on $TX$. Since $\varphi_t(x) = x$, the flow on $TX$ preserves $T_x X$, and there it is given by $d\varphi_t(x) \in \mathrm{Gl}(T_x X)$. Thus, we get a linear $\mathbb{R}$-action on $T_x X$ with infinitesimal generator $d\beta_X(x)$.

https://doi.org/10.1017/nmj.2021.14 Published online by Cambridge University Press

COROLLARY 2.1.2. *If $\beta \in \mathfrak{p}$ and $x \in X$ is a critical point of $\mu_{\mathfrak{p}}^{\beta}$, then there are open invariant neighbourhoods $S \subset T_xX$ and $\Omega \subset X$ and an $\mathbb{R}$-equivariant diffeomorphism $\Psi : S \to \Omega$, such that $0 \in S, x \in \Omega$, and $\Psi(0) = x$. (Here, $t \in \mathbb{R}$ acts as $d\varphi_t(x)$ on $S$ and as $\varphi_t$ on $\Omega$.)*

*Proof.* See [9]. □

Let $x \in \mathrm{Crit}(\mu_{\mathfrak{p}}^{\beta})$. Let $D^2\mu_{\mathfrak{p}}^{\beta}(x)$ denote the Hessian, which is a symmetric operator on $T_xX$ such that

$$(D^2\mu_{\mathfrak{p}}^{\beta}(x)v, v) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mu_{\mathfrak{p}}^{\beta} \circ \gamma)(0),$$

where $\gamma$ is a smooth curve, $\gamma(0) = x$, and $\dot{\gamma}(0) = v$. Denote by $V_-$ (resp. $V_+$) the sum of the eigenspaces of the Hessian of $\mu_{\mathfrak{p}}^{\beta}$ corresponding to negative (resp. positive) eigenvalues. Denote by $V_0$ the kernel. Since the Hessian is symmetric, we get an orthogonal decomposition

$$T_xX = V_- \oplus V_0 \oplus V_+. \tag{3}$$

Let $\alpha : G \to X$ be the orbit map: $\alpha(g) := gx$. The differential $d\alpha_e$ is the map $\xi \mapsto \xi_X(x)$.

PROPOSITION 2.2. *If $\beta \in \mathfrak{p}$ and $x \in \mathrm{Crit}(\mu_{\mathfrak{p}}^{\beta})$, then*

$$D^2\mu_{\mathfrak{p}}^{\beta}(x) = d\beta_X(x).$$

*Moreover, $d\alpha_e(\mathfrak{r}^{\beta\pm}) \subset V_{\pm}$ and $d\alpha_e(\mathfrak{g}^{\beta}) \subset V_0$. If $X$ is $G$-homogeneous, these are equalities.*

*Proof.* See [5], [9, Prop. 2.5]. □

COROLLARY 2.2.1. *For every $\beta \in \mathfrak{p}$, $\mu_{\mathfrak{p}}^{\beta}$ is a Morse–Bott function.*

Using an $\mathrm{Ad}(K)$-invariant inner product of $\mathfrak{p}$, we define $\nu_{\mathfrak{p}}(z) := \frac{1}{2} \parallel \mu_{\mathfrak{p}}(z) \parallel^2$. The function $\nu_{\mathfrak{p}}$ is $K$-invariant, and it is called the *norm square function*. The following result is proved in [9, Corollaries 6.11 and 6.12, p. 21].

PROPOSITION 2.3. *Let $x \in M$. Then:*

- *If $\nu_{\mathfrak{p}}$ restricted to $G \cdot x$ has a local maximum at $x$, then $G \cdot x = K \cdot x$.*
- *If $G \cdot x$ is compact, then $G \cdot x = K \cdot x$.*

A strategy to analyzing the $G$ action on $M$ is to view $\nu_{\mathfrak{p}}$ as a generalized Morse function. In [9], the authors proved the existence of a smooth $G$-invariant stratification of $M$, and they studied its properties.

## §3. Closed orbit of parabolic subgroups

Let $(Z, \omega)$ be a Kähler manifold, and $U^{\mathbb{C}}$ acts holomorphically on $Z$ with a momentum map $\mu : Z \to \mathfrak{u}$. Let $G \subset U^{\mathbb{C}}$ be a closed and compatible subgroup. Hence, $G = K\exp(\mathfrak{p})$, where $K := G \cap U$ is a maximal compact subgroup of $G$ and $\mathfrak{p} := \mathfrak{g} \cap \mathrm{i}\mathfrak{u}; \mathfrak{g}$ is the Lie algebra of $G$. Let $X$ be a connected $G$-stable submanifold of $Z$, and let $\mu_{\mathfrak{p}} : X \longrightarrow \mathfrak{p}$ the corresponding $G$-gradient map.

https://doi.org/10.1017/nmj.2021.14 Published online by Cambridge University Press

REMARKS. Let $Q \subset G$ be a parabolic subgroup. The following facts are easy to check:

a) If $Q \cdot p$ is compact, then $G \cdot p$ is closed since $G = KQ$.
b) Let $\mathcal{O}$ be a compact $G$-orbit. By Proposition 2.3, it follows that $\mathcal{O} = G \cdot p = K \cdot p$. Since $\mu_{\mathfrak{p}}$ is $K$-equivariant, the restricted gradient map $\mu_{\mathfrak{p}} : K \cdot p \to K \cdot \mu_{\mathfrak{p}}(x)$ is a smooth $K$-equivariant submersion.

Let $\beta \in \mathfrak{p}$, and let

$$Y = \{z \in X : \max_{x \in X} \mu_{\mathfrak{p}}^{\beta}(x) = \mu_{\mathfrak{p}}^{\beta}(z)\}.$$

Assume that $Y$ is not empty. By Corollary 2.1.2, $Y$ is a smooth, possibly disconnected, submanifold of $X$.

LEMMA 3.1. *$Y$ is $G^{\beta+}$ invariant.*

*Proof.* Let $g \in G$, and let $\xi \in \mathfrak{p}$. It is easy to check that

$$(\mathrm{d}g)_p(\xi_X) = (\mathrm{Ad}(g)(\xi))_X(gp),$$

and so $G^{\beta}$ preserves $X^{\beta}$. We claim that $Y$ is $G^{\beta}$-stable. In fact, $G^{\beta} = K^{\beta}\exp(\mathfrak{p}^{\beta})$, and $Y$ is $K^{\beta}$-invariant by $K$-invariant property of the gradient map. For each $y \in Y$, let $\xi \in \mathfrak{p}^{\beta}$, and let $\gamma(t) = \exp(t\xi) \cdot y$. Since $\beta_X(\gamma(t)) = 0$, it follows that $\mu_{\mathfrak{p}}^{\beta}(\gamma(t))$ is constant, and so $\exp(t\xi) \cdot y \in Y$. Now, $G^{\beta+} = G^{\beta}R^{\beta+}$ where $R^{\beta+}$ is connected and the unipotent radical of $G^{\beta+}$. By Proposition 2.2, $\mathfrak{r}^{\beta+} \subset V_+$, and so $\mathfrak{r}^{\beta+} \cdot z \subset G_z$, for all $z \in Y$. Since $R^{\beta+}$ is connected, this implies $R^{\beta+}$ does not act on $Y$, and the result follows. ∎

LEMMA 3.2. *Let $\mathcal{O}$ be a compact $G$-orbit. Let $\beta \in \mathfrak{p}$. If $x \in \mathcal{O}$ is a local maximum of $\mu_{\mathfrak{p}}^{\beta} : \mathcal{O} \to \mathbb{R}$, then $x$ is a global maximum of $\mu_{\mathfrak{p}}^{\beta} : \mathcal{O} \to \mathbb{R}$.*

*Proof.* If $x \in \mathcal{O}$ is a local maximum of $\mu_{\mathfrak{p}}^{\beta}$, then $\mu_{\mathfrak{p}}(x)$ is a local maximum of the height function

$$K \cdot \mu_{\mathfrak{p}}(x) \to \mathbb{R}, \quad z \mapsto \langle z, \beta \rangle.$$

But it was noted in the proof of Proposition 3.9 in [5] that $\langle \cdot, \beta \rangle$ has only global maximum when restricted to $K \cdot \mu_{\mathfrak{p}}(x)$. Then, a local maximum is a global maximum, and this implies that $\mu_{\mathfrak{p}}(x)$ is a global maximum of the height function $\langle \cdot, \beta \rangle$. Since

$$\max_{\mathcal{O}}(\beta) = \max\{\langle z, \beta \rangle, z \in K \cdot \mu_{\mathfrak{p}}(x)\},$$

$x$ is a global maximum of $\mu_{\mathfrak{p}}^{\beta}$. ∎

PROPOSITION 3.3. *Let $p \in \mathcal{O}$ be such that $G^{\beta+} \cdot p$ is closed. Then, $G^{\beta+} \cdot p$ is a finite union of connected components of* $\max_{\mathcal{O}}(\beta)$.

*Proof.* Since $G^{\beta+} \cdot p$ is compact, $\mu_{\mathfrak{p}}^{\beta}|_{G^{\beta+} \cdot p}$ has a maximum. Let $q \in G^{\beta+} \cdot p$ denote a maximum of $\mu_{\mathfrak{p}}^{\beta}|_{G^{\beta+} \cdot p}$. By Proposition 2.2, $q$ is a $R^{\beta+}$ fixed point. Applying again, Proposition 2.2, keeping in mind that $\mathcal{O}$ is $G$ homogeneous, $q$ is a local maximum of $\mu_{\mathfrak{p}}^{\beta} : \mathcal{O} \to \mathbb{R}$. By Lemma 3.2, $q$ is a global maximum of $\mu_{\mathfrak{p}}^{\beta}$. By Lemma 3.1, the unipotent group $R^{\beta+}$ acts trivially on $\max_{\mathcal{O}}(\beta)$, and $G^{\beta+} \cdot p \subset \max_{\mathcal{O}}(\beta)$.

Let $x \in \max_{\mathcal{O}}(\beta)$. By Proposition 2.2 and Corollary 2.2.1, keeping in mind that $\mathcal{O}$ is $G$ homogeneous and $R^{\beta+}$ acts trivially on $\max_{\mathcal{O}}(\beta)$, it follows that $T_x\max_{\mathcal{O}}(\beta) = T_xG^{\beta} \cdot x$. By Lemma 3.1, $(G^{\beta})^o$ preserves any connected component of $\max_{\mathcal{O}}(\beta)$. Moreover, the

https://doi.org/10.1017/nmj.2021.14 Published online by Cambridge University Press

restriction of $\mu_{\mathfrak{p}}$ to any connected component of $\max_{\mathcal{O}}(\beta)$ defines the gradient map of $(G^{\beta})^o$ (see [9]). By Proposition 2.3, it follows that $(G^{\beta})^o$ has a closed orbit on any connected component of $\max_{\mathcal{O}}(\beta)$. Since any $(G^{\beta})^o$ orbit is open in $\max_{\mathcal{O}}(\beta)$, it follows that the connected component of $\max_{\mathcal{O}}(\beta)$ containing $x$ is $(G^{\beta})^o$ homogeneous. The connected components of $G^{\beta}$ are finite and intersect the connected components of $K^{\beta}$. Therefore, keeping in mind that $R^{\beta+}$ acts trivially on $\max_{\mathcal{O}}(\beta)$, $G^{\beta+}\cdot x$ is a finite union of connected components of $\max_{\mathcal{O}}(\beta)$. The same result holds for $G^{\beta+}\cdot p$, concluding the proof. □

COROLLARY 3.3.1. *Let $x \in \max_{\mathcal{O}}(\beta)$. Then, $G^{\beta+}\cdot x$ is closed and is a finite union of connected components of $\max_{\mathcal{O}}(\beta)$.*

Summing up, we have proved our first main result.

THEOREM 3.4. *Let $\beta \in \mathfrak{p}$. Then:*

- *If $G^{\beta+}\cdot x$ is compact, then $\mathcal{O} = G\cdot x$ is compact, and $G^{\beta+}\cdot x$ is a finite union of connected components of $\max_{\mathcal{O}}(\beta)$.*
- *If $\mathcal{O}$ is a compact $G$-orbit, then $\max_{\mathcal{O}}(\beta)$ is a finite union of compact $G^{\beta+}$-orbits.*

*In particular, the number of compact $G^{\beta+}$-orbits is equal or bigger than the number of compact $G$-orbits.*

Let $Q \subset U^{\mathbb{C}}$ be a parabolic subgroup. There exists $\beta \in i\mathfrak{u}$ such that $Q = (U^{\mathbb{C}})^{\beta+}$. If $\tilde{\mathcal{O}}$ is a compact $U^{\mathbb{C}}$-orbit, then it is a complex $U$-orbit, and so it is a flag manifold (see [8]). By definition of the gradient map,

$$\max_{\tilde{\mathcal{O}}}(\beta) = \max\{p \in \tilde{\mathcal{O}} : \langle \mu(p), -i\beta\rangle = \max_{p\in\tilde{\mathcal{O}}}\mu^{-i\beta}\},$$

and so it is connected (see [2], [7]). This means that $(U^{\mathbb{C}})^{\beta+}$ has a unique closed orbit in $\tilde{\mathcal{O}}$.

COROLLARY 3.4.1. *The number of compact $(U^{\mathbb{C}})^{-i\beta+}$-orbits is equal to the number of compact $U^{\mathbb{C}}$-orbits. Any closed $(U^{\mathbb{C}})^{-i\beta+}$-orbit arises as $\max_{\mathcal{O}}(\beta)$, where $\tilde{\mathcal{O}}$ is a compact $U^{\mathbb{C}}$-orbit.*

Assume that $G$ is a real form of $U^{\mathbb{C}}$. Then, $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$, $\mathfrak{u} = \mathfrak{k}\oplus i\mathfrak{p}$, and so $\mathfrak{g}^{\mathbb{C}} = \mathfrak{u}^{\mathbb{C}}$ (see [10], [13]).

Assume that there exists $x \in Z$ such that $U^{\mathbb{C}}\cdot x$ is compact. Then, $U^{\mathbb{C}}\cdot x = U\cdot x$ and so a flag manifold. The following result is essentially an old theorem of Wolf [16] (see also [9]).

THEOREM 3.5 (Wolf). *There exists a unique closed $G$-orbit in $U^{\mathbb{C}}\cdot x$.*

*Proof.* Let $G_{ss}$ denote the connected subgroup of $G$ with Lie algebra $[\mathfrak{g},\mathfrak{g}]$. Then, $G_{ss}$ is closed and compatible, and $G = Z(G)^o\cdot G_{ss}$, where $Z(G)^o$ is the connected component of the center (see [13, p. 442]). By Proposition 2.3, $G$ has a closed orbit in $U^{\mathbb{C}}\cdot x$. The center of $U$ does not act on $U\cdot x$, and $G_{ss}$ is a real form of $(U^{\mathbb{C}})_{ss} = (U_{ss})^{\mathbb{C}}$. By a theorem of Wolf [16], $G_{ss}$ has a unique closed orbit in $U^{\mathbb{C}}\cdot x$. On the other hand, by Proposition 2.3, it follows that $G_{ss}$ has a closed orbit on any closed orbit of $G$. Therefore, $G$ has a unique closed orbit in $U^{\mathbb{C}}\cdot x$. □

Let $\mathcal{O}$ denote the unique compact $G$-orbit in $U^{\mathbb{C}}\cdot x$. Let $\beta \in \mathfrak{p}$. We denote by

$$\max_{U^{\mathbb{C}}\cdot x}(\beta) = \left\{p \in U^{\mathbb{C}}\cdot x : \mu_{\mathfrak{p}}^{\beta}(p) = \max_{p\in U^{\mathbb{C}}\cdot x}\mu_{\mathfrak{p}}^{\beta}\right\}.$$

https://doi.org/10.1017/nmj.2021.14 Published online by Cambridge University Press

LEMMA 3.6. *For any* $\beta \in \mathfrak{p}, \max_{U^{\mathbb{C}}.x}(\beta) \cap \mathcal{O} \neq \emptyset$. *Hence,*

$$\max_{U^{\mathbb{C}}.x}(\beta) \cap \mathcal{O} = \max_{\mathcal{O}}(\beta).$$

*Proof.* Since

$$\max_{U^{\mathbb{C}}.x}(\beta) = \max\left\{p \in U^{\mathbb{C}} \cdot x : \langle \mu(p), -i\beta\rangle = \max_{p \in U^{\mathbb{C}}.x} \mu^{-i\beta}\right\},$$

by Corollary 3.4.1, it follows that $\max_{U^{\mathbb{C}}.x}(\beta)$ is the unique closed orbit of $(U^{\mathbb{C}})^{\beta+}$. Since

$$G^{\beta+} = G \cap (U^{\mathbb{C}})^{\beta+},$$

we get $R^{\beta+} \subset R((U^{\mathbb{C}})^{-i\beta+})$, and so $R^{\beta+}$ acts trivially on $\max_{U^{\mathbb{C}}.x} \mu^{\beta}_{\mathfrak{p}}$. Applying the same arguments of Proposition 3.3, it follows that $G^{\beta+}$ has a closed orbit in $\max_{U^{\mathbb{C}}.x} \mu^{\beta}_{\mathfrak{p}}$. By Theorem 3.5, we get $\max_{U^{\mathbb{C}}.x}(\beta) \cap \mathcal{O} \neq \emptyset$, and the result follows. □

As a consequence, we obtain the following result.

PROPOSITION 3.7. *Let* $\mathfrak{a} \subset \mathfrak{p}$ *be an abelian subalgebra. Then,*

$$\mu_{\mathfrak{a}}(U^{\mathbb{C}} \cdot x) = \mu_{\mathfrak{a}}(\mathcal{O}).$$

*Proof.* It is well known that both $\mu_{\mathfrak{a}}(U^{\mathbb{C}} \cdot x)$ and $\mu_{\mathfrak{a}}(\mathcal{O})$ are polytopes (see [9], [14]). Applying the above lemma and Proposition 2.1, we get $\mu_{\mathfrak{a}}(U^{\mathbb{C}} \cdot x) = \mu_{\mathfrak{a}}(\mathcal{O})$. □

Now, we are ready to prove our second main result.

THEOREM 3.8. *The set* $\max_{\mathcal{O}}(\beta)$ *is the unique closed orbit of* $G^{\beta+}$ *in* $\mathcal{O}$. *Moreover, it is connected and is a* $(K^{\beta})^o$*-orbit.*

*Proof.* Let $(G^{\beta})^o_{ss}$ denote the connected subgroup whose Lie algebra is $[\mathfrak{g}^{\beta}, \mathfrak{g}^{\beta}]$. It is closed, semisimple, and compatible (see [13]). By Lemma 3.1, it preserves any connected components of $\max_{\mathcal{O}}(\beta)$. By Proposition 2.3, $(G^{\beta})^o_{ss}$ has a closed orbit on any connected component $\max_{\mathcal{O}}(\beta)$. On the other hand, $\max_{U^{\mathbb{C}}.x}(\beta)$ is connected and, by Proposition 2.2, is a closed orbit of $(U^{\mathbb{C}})^{\beta}$. Note that $(G^{\beta})^o_{ss}$ is a real form of $(U^{\beta})_{ss}$, and $\max_{U^{\mathbb{C}}.x}(\beta)$, keeping in mind that it is a flag manifold and the center of $(U^{\mathbb{C}})^{\beta}$ does not act on it, is a compact $(U^{\beta})_{ss}$ orbit. Applying a theorem of Wolf [16], it follows that $(G^{\beta})^o_{ss}$ has a unique closed orbit in $\max_{U^{\mathbb{C}}.x}(\beta)$. Since both $(G^{\beta})^o$ act trivially on $\max_{U^{\mathbb{C}}.x}(\beta)$, the unique closed orbit of $(G^{\beta})^o_{ss}$ is contained in a closed $(G^{\beta})^o_{ss}$ is contained in a closed orbit of $G^{\beta+}$, and so it is contained in $\mathcal{O}$. By Theorem 3.4, this orbit is contained in $\max_{\mathcal{O}}(\beta)$. Since $(G^{\beta})^o_{ss}$ preserves $\max_{\mathcal{O}}(\beta)$ and it has a closed orbit on any connected component of $\max_{\mathcal{O}}(\beta)$, it follows that $\max_{\mathcal{O}}(\beta)$ is connected. This means $\max_{\mathcal{O}}(\beta)$ is the unique closed orbit of $G^{\beta+}$. In particular, keeping in mind $\max_{\mathcal{O}}(\beta)$ is $(G^{\beta})^o$ homogeneous, applying Proposition 2.3, we get $\max_{\mathcal{O}}(\beta)$ is a $(K^{\beta})^o$-orbit, concluding the proof. □

## REFERENCES


[1] D. N. Akhiezer, *Lie Group Actions in Complex Analysis*, Asp. Math. **E27**, Friedr. Vieweg, Braunschweig, 1995.

[2] M. F. Atiyah, *Convexity and commuting Hamiltonians*, Bull. Lond. Math. Soc. **14** (1982), 1–15.

[3] L. Biliotti, *Satake–Furstenberg compactifications and gradient map*, preprint, arXiv:2012.14858

[4] L. Biliotti, A. Ghigi, and P. Heinzner, *Coadjoint orbitopes*, Osaka J. Math. **51** (2014), 935–968.

[5] L. Biliotti, A. Ghigi, and P. Heinzner, *Polar orbitopes*, Comm. Ann. Geom. **21** (2013), 1–28.

[6] A. Borel and L. Ji, *Compactifications of Symmetric and Locally Symmetric Spaces*, Math. Theory Appl., Birkhäuser Boston, Boston, MA, 2006.


https://doi.org/10.1017/nmj.2021.14 Published online by Cambridge University Press

[7] V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), 491–513.
[8] V. Guillemim and S. Sternberg, *Symplectic Techniques in Physics*, 2nd ed., Cambridge Univ. Press, Cambridge, 1990.
[9] P. Heinzner, G. W. Schwarz, and H. Stötzel, *Stratifications with respect to actions of real reductive groups*, Compos. Math. **144** (2008), 163–185.
[10] P. Heinzner and H. Stötzel, *Semistable points with respect to real forms*, Math. Ann. **338** (2007), 1–9.
[11] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Pure Appl. Math. **80**, Academic Press, New York, 1978.
[12] F. Kirwan, *Cohomology of Quotients in Symplectic and Algebraic Geometry*, Math. Notes **31**, Princeton Univ. Press, Princeton, NJ, 1984.
[13] A. W. Knapp, *Lie Groups Beyond an Introduction*, 2nd ed., Progr. Math. **140**, Birkhäuser Boston, Boston, MA, 2002.
[14] B. Kostant, *On convexity, the Weyl group and the Iwasawa decomposition*, Ann. Sci. Éc. Norm. Supér. (4) **6** (1973), 413–455.
[15] R. Schneider, *Convex Bodies: The Brunn–Minkowski Theory*, Encyclopedia Math. Appl. **44**, Cambridge Univ. Press, Cambridge, 1993.
[16] J. Wolf, *The action of real of a real semisimple group on a complex flag manifold. I: Orbit structure and holomorphic arc components*. Bull. Amer. Math. Soc. **75** (1969), 1121–1237.

Leonardo Biliotti
*Dipartimento di Scienze Matematiche, Fisiche e Informatiche*
*Università di Parma,*
*Parma, Italy*
leonardo.biliotti@unipr.it

Oluwagbenga Joshua Windare
*Dipartimento di Scienze Matematiche, Fisiche e Informatiche*
*Università di Parma,*
*Parma, Italy*
oluwagbengajoshua.windare@unipr.it

https://doi.org/10.1017/nmj.2021.14 Published online by Cambridge University Press