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Genus-zero r-spin theory

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Abstract

We provide an explicit formula for all primary genus-zero r-spin invariants. Our formula is piecewise polynomial in the monodromies at each marked point and in r. To deduce the structure of these invariants, we use a tropical realisation of the corresponding cohomological field theories. We observe that the collection of all Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) relations is equivalent to the relations deduced from the fact that genus-zero tropical CohFT cycles are balanced.

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1. Introduction

This article achieves two distinct goals: The first is to establish a connection between the combinatorial properties of the genus-zero part of any cohomological field theory (CohFT) and its tropicalisation. The second is to apply this analysis to the CohFT of r-spin Witten classes (see Section 2 for a discussion) and obtain a detailed understanding of the corresponding numerical invariants. We first discuss this second goal, which allows for simple and concrete statements, and then we enlarge the scope to discuss the more general combinatorial journey that led to these results.

A *genus-zero r-spin invariant* $w_r(\vec{m})$, often denoted $(\tau^{m_1}\cdots\tau^{m_n})$ in the literature, is an intersection number on a moduli stack of n-pointed, genus-zero r-spin curves obtained by integrating the Euler class of the Witten bundle over a connected component of the moduli stack. Here, r is a positive integer, and $\vec{m}=(m_i)\in(1,2,\dots,r)^n$, with $\sum_{i=1}^n m_i=(n-2)(r+1)$, which is called a *numerical monodromy vector*. These intersection numbers have a rich tradition: originally proposed by Witten [Wit93], they were developed for many reasons, including their CohFT structure [JKV01], their setting in mirror symmetry as an enumerative theory for the Landau-Ginzburg model $(\mathbb{C},\mu_r,x^r)$ [FJR11], and their applications towards tautological relations [PPZ19]. Our first main result gives a closed formula for (primary) genus-zero r-spin invariants.

Theorem 1 (See Theorem 5.1). *The genus-zero r-spin invariant of a numerical monodromy vector* $\vec{m}=(m_1,\dots,m_n)$ *is*

$$w_r(\vec{m})=\frac{1}{2\cdot r^{n-3}}\sum_{\substack{S\subseteq[n]\\ \sum_{i\in S}m_i\geqslant(|S|-1)r+n-2}}(-1)^{1+|S|}\prod_{k=1}^{n-3}\left(\left(\sum_{i\in S}m_i\right)-(|S|-1)r-k\right). \tag{1}$$

Viewing $m_1,\dots,m_n,r$ as being a list of variables, one may think of $w_r(\vec{m})$ as a function defined on the integral lattice points of an unbounded polyhedron $\mathbf{M}_n$ in $\mathbb{R}^n\times\mathbb{R}$. Formula (1) implies that $w_r(\vec{m})$ is a piecewise-polynomial function of degree $n-3$. The chambers of polynomiality are polyhedra that overlap in affine linear strips of width $n-4$. The precise statements and the equations for the walls are given in Corollary 5.2. Previously, it was known that $w_r(\vec{m})$ is a piecewise polynomial when $n=3,4$ [JKV01, Prop. 6.1]. It was also known that $w_r(\vec{m})$ is equal to the dimension of $\mathfrak{sl}_2(\mathbb{C})$-invariant subspace of a certain tensor product of symmetric powers of the standard representation of $\mathfrak{sl}_2(\mathbb{C})$ [PPZ19, Thm. 2], and it is the genus-zero part of the r-KdV hierarchy [JKV01].

We also unveil properties of genus-zero r-spin invariants that are not apparent from their expression in (1). For a fixed value of r, the *dominance order* is a partial ordering on the monodromy vectors $\vec{m}$ that, roughly speaking, says they become smaller as they get closer to the small diagonal of $\mathbb{R}^n$, equivalently, as the entries of $\vec{m}$ get more equidistributed (see Section 6 for a precise definition). We show that genus-zero r-spin invariants are monotonic with respect to this partial ordering.

Theorem 2 (See Theorem 6.2(1)). *For any fixed* r,n, *the function* $w_r(\vec{m})$ *is weakly order-reversing with respect to the dominance ordering. That is,*

$$\vec{m}\leqslant\vec{m}' \implies w_r(\vec{m})\geqslant w_r(\vec{m}'). \tag{2}$$

In [PPZ19, Prop. 1.4] the authors use character theory of $\mathfrak{sl}_2(\mathbb{C})$ to show that for a monodromy vector of length n, $w_r(\vec{m})=0$ if $m_i\leqslant n-3$ for any of the entries of $\vec{m}$; we give a direct

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combinatorial proof of this fact. This vanishing, together with Theorem 2, implies the following positivity statement:

Theorem 3 (See Theorem 6.2(2)). *For any r and $\vec{m}$, we have $w_r(\vec{m}) \geqslant 0$, with $w_r(\vec{m}) \neq 0$ if and only if $n-2 \leqslant m_i \leqslant r-1$ for all i.*

Notably, the formula for genus-zero r-spin invariants is more complicated than the closed formula for *open* r-spin invariants found in [BCT18, Theorem 1.2].

All these structure results about genus-zero r-spin invariants follow from a collection of linear recursive relations, proved in Theorem 4.1. Such relations may be derived in two equivalent ways, which leads us to discussing the first stated goal of this article.

Genus-zero r-spin invariants are degrees of the zero-dimensional cycles for the CohFT of Witten classes and comprise the *numerical part* of the CohFT. Very roughly speaking, a CohFT (see [Pan18] for precise definitions) is an infinite collection of Chow classes on moduli spaces of curves that are self-referential with respect to restriction to boundary strata; this means that the intersection of a CohFT class with a stratum

$$\Delta = \mathrm{gl}_* \left(\prod_i \overline{M}_{g_i,n_i} \right)$$

is equal to the pushforward via the corresponding gluing morphism of (a linear combination of products of pullbacks via the coordinate projections of) CohFT classes from the factors. The basic linear equivalence among the three boundary points of $\overline{M}_{0,4} \cong \mathbb{P}^1$ may be pulled-back via forgetful morphisms and/or pushed-forward via gluing morphisms and then intersected with a CohFT class and integrated to obtain a collection of relations among the numerical CohFT classes. Such relations are called *WDVV relations*, and they are familiar tools in and around Gromov-Witten theory, where they are, for example, responsible for the associativity of the quantum product [FP97]. One way to obtain the recursions in Theorem 4.1 is via WDVV. Another way is to study the *tropicalisation* of the CohFT. We summarize here the discussion in Section 3:

Tropical geometry [BIMS15, Mik06] provides a combinatorialisation of standard algebraic geometry concepts, and in the last few decades, it has found interesting applications to enumerative geometry (e.g. [Mik05, GM07, GKM09]) and tautological intersection theory of moduli spaces of curves (e.g. [Mik07, Rau16, KM08, Gro18, CGM22]). The moduli space of tropical curves $M^{\mathrm{trop}}_{g,n}$ [ACP15, CCUW20] may be identified with the boundary complex of $\overline{M}_{g,n}$. For any Chow class $\alpha \in A^k(\overline{M}_{g,n})$, its tropicalization α^{trop} is a codimension-k weighted cone subcomplex of $M^{\mathrm{trop}}_{g,n}$, where each codimension-k cone of $M^{\mathrm{trop}}_{g,n}$ is weighted by the intersection number of α with the corresponding k-dimensional stratum of $\overline{M}_{g,n}$. If Ω is a CohFT class, then it follows from the strata restriction properties that the coefficients of Ω^{trop} are functions of numerical CohFT invariants. In other words, the tropicalization of a CohFT is completely controlled by its numerical part (Proposition 3.10).

In genus zero, the moduli space of curves is a *tropical compactification* [Tev07, GM10], meaning that $\overline{M}_{0,n}$ admits an embedding into a (non-proper) toric variety X_Σ such that the boundary stratification of $\overline{M}_{0,n}$ coincides with the restriction of the toric boundary of X_Σ. It follows that $M^{\mathrm{trop}}_{0,n}$ may be identified with Σ and, therefore, viewed as a *balanced fan* inside the vector space $N_{\mathbb{R}}$, spanned by the cocharacter lattice of the torus of X_Σ. There is a natural isomorphism $\varphi : A^*(\overline{M}_{0,n}) \to A^*(X_\Sigma)$, and the tropicalisation α^{trop} of a cycle $\alpha \in A^*(\overline{M}_{0,n})$ coincides with the Minkowski weight presentation of $\varphi(\alpha)$ [FS97, Kat09].

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This in turn implies that α^{trop} satisfies the *balancing condition* (12), which translates into a collection of linear equations on the weights of the cones of α^{trop}. When α is a CohFT class, one then obtains equations among the numerical CohFT invariants. Our next result relates these equations to the relations obtained from WDVV.

Theorem 4 (See Theorem 3.12). *The collection of balancing equations for all tropical cycles* Ω^{trop} *imposes the same constraints as all WDVV relations on the numerical invariants of* Ω.

The statement of this theorem can be made precise (but probably unnecessarily confusing) by saying that the two sets of equations cut down the same subvariety in some countable dimensional affine space coordinatised by the discrete invariants indexing the classes of *arbitrary* CohFTs. It could also be explained in working terms: If one is trying to reconstruct numerical CohFT invariants recursively given some initial conditions, doing so is equivalent to using the set of relations coming from WDVV or the balancing equations. The proof of Theorem 3.12 shows just how tight the connection is: Balancing at the cone point for 1-dimensional tropical cycles corresponds to WDVV relations obtained by pull-back via forgetful morphisms, and balancing along faces of higher dimensional cycles correspond to WDVV relations pushed-forward via appropriate gluing morphisms.

The tropicalisation of the moduli space of r-spin curves has been studied in [CMP20, APS22], where the authors identify a skeleton of the Berkovich analytification of algebraic r-spin curves with a cone complex parameterising tropical r-spin curves. Our perspective is to directly tropicalise the cycles of the r-spin CohFT, bypassing the need for a tropical version of the theory of Witten classes on moduli spaces of tropical r-spin curves. Once a theory of tropical vector bundles and Chern classes is established, it would be natural to try to exhibit the tropicalisation of a genus-zero r-spin Witten class as the Euler class of a tropical version of the Witten bundle.

This article is written with the intention of its being accessible to readers from different mathematical communities, with the goal of stimulating hopefully productive interactions. Section 2 provides background and intuition on the CohFT of r-spin Witten classes, which is aimed at readers from tropical enumerative geometry. Section 3 gives background on tropical intersection theory, which is aimed at algebraic geometers interested in intersection theory on moduli spaces. Section 4 derives recursions among genus-zero r-spin numerical invariants, and these recursions are used in Section 5 to prove Theorem 1. In Section 6, we study the monotonicity of genus-zero r-spin invariants with respect to the dominance ordering.

2. The r-spin cohomological field theory

We are interested in computing r-spin invariants, which are defined as intersection numbers over moduli spaces of roots of line bundles on curves. The literature on r-spin curves is both technical and full of conflicting conventions; hence we provide in this section a self-contained overview of this story. Since our focus is combinatorial, we sweep technicalities under the rug when possible.

2.1 Moduli of r-spin curves

Fix an integer $r \geqslant 2$. Let $(C, p_1, \ldots, p_n) \in M_{g,n}$ be a smooth n-marked curve of genus g, and fix $m_1, \ldots, m_n \in \{1, \ldots, r\}$ such that

$$2g - 2 + n - \sum_{i=1}^{n} m_i \in r\mathbb{Z}. \tag{3}$$

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An *r-spin structure of type* $\vec{m} := (m_1, \dots, m_n)$ on C is a line bundle L on C together with an isomorphism

$$\varphi : L^{\otimes r} \xrightarrow{\sim} \omega_{C,\log}\left(-\sum_{i=1}^{n} m_i p_i\right),$$

where $\omega_{C,\log} = \omega_C(\sum_{i=1}^n p_i)$ is the log canonical bundle of C. We refer to m_i as the *monodromy* of L at p_i; the terminology arises from an equivalent formulation of r-spin curves in which $p_1, \dots, p_n$ are orbifold points of C and m_i is the monodromy of L at p_i. We call $\vec{m}$ (subject to the condition (3)) a *monodromy vector* and call the data (C, p_i, L, φ) a (smooth) *r-spin curve*.

Remark 2.1. The integers m_i are off by one from some standard references on r-spin curves (e.g. [JKV01, BCT19, PPZ19]) but match others (e.g. [Chi08b, FJR11, FJR13]).

For an r-spin curve of type $\vec{m}$, by the existence of φ, we immediately see

$$\deg L = \frac{2g - 2 + n - \sum_{i=1}^n m_i}{r} \in \mathbb{Z}. \tag{4}$$

We denote by $M^r_{g,n}(\vec{m})$ the moduli stack of smooth r-spin curves of type $\vec{m}$, and let $M^r_{g,n} = \bigsqcup_{\vec{m}} M^r_{g,n}(\vec{m})$. There is a natural compactification $\overline{M}^r_{g,n} = \bigsqcup_{\vec{m}} \overline{M}^r_{g,n}(\vec{m})$ due to Abramovich-Jarvis and Chiodo [AJ03, Chi08a]—though other compactifications exist [Jar98, Jar00, CCC07]. In the language of [Chi08a], the stack $\overline{M}^r_{g,n}(\vec{m})$ is the moduli stack of *stable r-spin curves*, i.e. r-th roots L of $\omega_{C,\log}$ on nodal orbicurves C with $\mathbb{Z}/r\mathbb{Z}$-orbifold structure at marked points and nodes (and nowhere else), where m_i is the monodromy of L at p_i. This space admits a (finite flat surjective) forgetful map $\rho : \overline{M}^r_{g,n}(\vec{m}) \to \overline{M}_{g,n}$; see [Chi08a, Thm. 4.2.3].

Remark 2.2. When $g = 0$ (and necessarily $n \geqslant 3$), the forgetful map ρ is a bijection. It is not an isomorphism for stack-theoretic reasons—due to the fact that r-spin curves admit extra automorphisms—but we can largely ignore this subtlety for our purposes.

An important aspect of $\overline{M}^r_{g,n}(\vec{m})$—similarly to $\overline{M}_{g,n}$—is its recursive structure, which we briefly describe here. On a stable r-spin curve (C, p_i, L, φ), each node η of C locally looks like the quotient of an ordinary node $V(xy)$ by an action of $\mathbb{Z}/r\mathbb{Z}$ via

$$(x, y) \mapsto (\zeta x, \zeta^{-1} y),$$

where ζ is an r-th root of unity. Under this identification, L is locally the quotient of $V(xy) \times \mathbb{C}$ by $\mathbb{Z}/r\mathbb{Z}$, acting by

$$(x, y, t) \mapsto (\zeta x, \zeta^{-1} y, \zeta^a t)$$

for some $a \in \{1, \dots, r\}$. Exchanging the roles of x and y would instead give an action $t \mapsto \zeta^b t$, where $a + b = 0 \pmod r$. We thus have a well-defined notion of the monodromy at η, *after picking a branch of the node*.

Example 2.3. Consider a genus-g stable r-spin curve $(C, p_1, \dots, p_n, L, \varphi)$ of type $(m_1, \dots, m_n)$, with exactly two irreducible components C_1 and C_2 of genera g_1 and g_2, joined at a single node η. Denote by $J \subseteq [n]$ the set of marked points on C_1 (so J^c is the set of marked points on C_2). We additionally mark the two preimages $p_J \in C_1$ and $p_{J^c} \in C_2$ of η under the normalization map and pull back L to each component. This yields the following two smooth r-spin curves

$$(C_1, \{p_i\}_{i \in J} \cup \{p_J\}, L, \varphi) \qquad \text{and} \qquad (C_2, \{p_i\}_{i \in J^c} \cup \{P_{J^c}\}, L, \varphi)$$

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with monodromies $m_J, m_{J^c} \in \{1, \dots, r\}$ at p_J and p_{J^c}, respectively, where[1]

$$m_J \equiv 2g - 2 + (|J| + 1) - \sum_{i \in J} m_i \pmod r \quad \text{and} \quad m_{J^c} \equiv 2g - 2 + (|J^c| + 1) - \sum_{i \in J^c} m_i \pmod r. \tag{5}$$

Example 2.3 illustrates the *recursive* boundary stratification of $\overline{M}^r_{g,n}(\vec{m})$, which compatibly matches the well-known recursive boundary stratification of $\overline{M}_{g,n}$. If $D^r_{g_1,g_2,J} \subseteq \overline{M}^r_{g,n}(\vec{m})$ denotes the divisor of stable r-spin curves generically of the type in the example, we have a bijection

$$D^r_{g_1,g_2,J} \to \overline{M}^r_{g_1,|J|+1}(\{m_i\}_{i\in J} \cup \{m_J\}) \times \overline{M}^r_{g_2,|J^c|+1}(\{m_i\}_{i\notin J} \cup \{m_{J^c}\}).$$

This bijection is compatible with the decomposition of the corresponding boundary divisor $D_J \subseteq \overline{M}_{g,n}$ as a product[2] $D_J \cong \overline{M}_{g_1,|J|+1} \times \overline{M}_{g_2,|J^c|+1}$. (The bijection is not an isomorphism because the stack structure on the two sides is different.)

Example 2.4. We may similarly consider the locus in $\overline{M}^r_{g,n}(\vec{m})$ associated to any fixed combinatorial type of curve (i.e. fixed *dual graph* Γ). We may carry out a similar analysis to that in Example 2.3, again via the normalisation map, with one caveat, illustrated in an example as follows: If $D^{\text{loop}} \subseteq \overline{M}^r_{g,n}(\vec{m})$ denotes the closure of the locus of stable r-spin curves C with one irreducible component with a self-node η (so that Γ has a single vertex with a self-loop), then the monodromies at the branches of η are not uniquely determined; we instead get a bijection:

$$D^{\text{loop}} \to \bigsqcup_{k=1}^{r} \overline{M}^r_{g-1,n+2}(\vec{m} \cup \{k, r-k\}).$$

Adapting the ideas from Examples 2.3 and 2.4, one can characterise any stratum in $\overline{M}^r_{g,n}(\vec{m})$ recursively.

2.2 Genus-zero Witten classes and r-spin invariants

In this section we describe the natural class of intersection numbers on $\overline{M}^r_{g,n}(\vec{m})$ that we are interested in, restricting to the case $g = 0$. The *Witten bundle* is the derived pushforward

$$\mathcal{W}^r_{0,n} = (\mathbf{R}^1\pi_*\mathcal{L}^r_{0,n})^\vee,$$

where $\mathcal{L}^r_{0,n}$ is the universal line bundle on the universal curve $\mathcal{C}^r_{0,n} \xrightarrow{\pi} \overline{M}^r_{0,n}$. We denote by $\mathcal{W}^r_{0,n}(\vec{m})$ the restriction of the Witten bundle to $\overline{M}^r_{0,n}(\vec{m}) \subseteq \overline{M}^r_{0,n}$.

A standard argument (see [JKV01, Prop. 4.4]) shows that $\mathbf{R}^0\pi_*\mathcal{L}^r_{0,n} = 0$. This fact depends crucially on having $m_i \geqslant 1$ for all i. Applying Riemann-Roch and (4) yields

$$\operatorname{rank}\mathcal{W}^r_{0,n}(\vec{m}) = -1 + \frac{2 - n + \sum_i m_i}{r}. \tag{6}$$

Definition 2.5. The (genus-zero) r-spin Witten class with monodromy vector $\vec{m}$ is

$$W_r(\vec{m}) = r \cdot \rho_* \left(e(\mathcal{W}^r_{0,n}(\vec{m}))\right) \in A^*(\overline{M}_{0,n}),$$

[1] There is a small subtlety here involving the dimension of the Witten class when $m_J = r$, which we may ignore by Proposition 2.10 below. See [BCT19, Sec. 3.2] for details.

[2] If $g_1 = g_2$ and $n = 0$ one must take a quotient of the product, and the same is true for r-spin curves above.

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where e denotes the Euler class, and $\rho\colon \overline{M}^r_{0,n}(\vec{m}) \to \overline{M}_{0,n}$ is the (bijective, by Remark 2.11) forgetful map. The (primary, genus-zero) r-spin invariant associated to the monodromy vector $\vec{m} = (m_1, \dots, m_n)$ is

$$w_r(\vec{m}) := \int_{[\overline{M}_{0,n}]} W_r(\vec{m}) = r \cdot \int_{[\overline{M}^r_{0,n}(\vec{m})]} e(\mathcal{W}^r_{0,n}(\vec{m})).$$

In the literature, $w_r(\vec{m})$ is often denoted $\langle \tau^{m_1} \cdots \tau^{m_n} \rangle$. Note that $w_r(\vec{m}) \neq 0$ only if $\operatorname{rank} \mathcal{W}^r_{0,n}(\vec{m}) = \dim \overline{M}^r_{0,n}(\vec{m}) = n-3$, or equivalently, only if

$$\sum_i m_i = (n-2)(r+1). \tag{7}$$

In this case, we say $\vec{m}$ is *numerical*. This is true when $W_r(\vec{m})$ is a zero-dimensional cycle on $\overline{M}^r_{0,n}(\vec{w})$; in general, $W_r(\vec{m})$ is a cycle of dimension $\frac{1}{r}((n-2)(r+1) - \sum_i m_i)$.

Remark 2.6 The factor of r in Definition 2.5 is essentially a matter of convention—see [JKV01, Cor. 3.9] and the succeeding discussion.

Remark 2.7. The Witten bundle, Witten class and r-spin invariants may also be defined in higher genus; we briefly mention some of the history. Jarvis, Kimura and Vaintrob first gave a collection of axioms for—but not a construction of—a virtual r-spin class, as a collection of cohomology classes on each boundary stratum of each moduli space $\overline{M}^r_{g,n}(\vec{m})$ of r-spin curves [JKV01, Def. 4.1]. They proved that any virtual r-spin class defines a cohomological field theory (CohFT) (see Section 2.3) by pushing forward to $\overline{M}_{g,n}$ and also that any virtual r-spin class agrees with the Witten class defined above in genus zero [JKV01, Rem. 4.2.4].

In [Wit93, Sec. 1.3], Witten had earlier outlined an analytic construction of r-spin invariants. Fan, Jarvis, and Ruan [FJR11, FJR13] carried out and generalised this construction and proved that it defines a virtual r-spin class in the sense of [JKV01]. As desired in [JKV01, Rem. 4.2.5], an algebraic construction soon followed, constructed by Polishchuk and Vaintrob [PV16] using matrix factorizations, which has been generalised to enumerative theories associated to general gauged linear sigma models [CFFG+18, FK20].

2.3 Properties of r-spin classes and numerical invariants

We now list the properties of the classes $W_r(\vec{m})$ that we will need. Propositions 2.8-2.13 come directly from [JKV01].

Fix a monodromy vector $\vec{m}$. Recalling Example 2.3, let $D_J \subseteq \overline{M}_{0,n}$ be the closure of the locus of curves with two components, C_1 and C_2, connected at a node η, with C_1 containing the marks in J. As before, let $p_J \in C_1$ and $p_{J^c} \in C_2$ be the preimages of η under normalisation, and define $m_J, m_{J^c} \in \{1, \dots, r\}$ by

$$m_J \equiv (|J|-1)(r+1) - \sum_{i \in J} m_i \pmod r \qquad m_{J^c} \equiv (|J^c|-1)(r+1) - \sum_{i \in J^c} m_i \pmod r. \tag{8}$$

Note that (8) is equivalent to (5) when $g = 0$. The particular form of (8) will be convenient; see Proposition 2.14.

Proposition 2.8. [JKV01, Axiom C2, Cor. 3.9]. *We have*

$$W_r(\vec{m})|_{D_J} = W_r((m_i)_{i \in J} \cup \{m_J\}) \boxtimes W_r((m_i)_{i \in J^c} \cup \{m_{J^c}\}),$$

where $\boxtimes$ *denotes the product of pullbacks from the factors of* $D_J \cong \overline{M}_{0,|J|+1} \times \overline{M}_{0,|J^c|+1}$.

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Remark 2.9. In genus zero, Proposition 2.8 is essentially the defining property of a cohomological field theory, so we'll refer to it as the CohFT property. See Section 3 for further discussion.

Proposition 2.10. [JKV01, Axiom 4]. *If $\vec{m}$ is a monodromy vector and $m_i = r$ for some i, then $W_r(\vec{m}) = 0$.*

Remark 2.11. In the physics literature, a marked point p_i is referred to as Ramond if $m_i = r$ (and as Neveu-Schwarz otherwise); hence Proposition 2.10 is often referred to as Ramond vanishing.

The following follows from [JKV01, Axiom 5].

Proposition 2.12. *Let $\vec{m}$ be a numerical monodromy vector of length $n \geqslant 4$ with $m_i = 1$ for some i. Then $w_r(\vec{m}) = 0$.*

Proposition 2.13. [JKV01, Prop. 6.1]. *The 3-point and 4-point r-spin invariants are as follows:*

1. *For any numerical r-spin monodromy vector $\vec{m} = (m_1, m_2, m_3)$, we have $w_r(\vec{m}) = 1$.*
2. *For any numerical r-spin monodromy vector $\vec{m} = (m_1, m_2, m_3, m_4)$, we have*

$$w_r(m_1, m_2, m_3, m_4) = \frac{1}{r} \min(m_1 - 1, m_2 - 1, m_3 - 1, m_4 - 1, r - m_1, r - m_2, r - m_3, r - m_4).$$

Proposition 2.14. *Let $\vec{m} = (m_1, \ldots, m_n)$ be an r-spin monodromy vector with $n > 4$ such that $W_r(\vec{m})$ is a 1-dimensional cycle, i.e. $\sum_i m_i = (n-2)(r+1) - r$. Then for $J \subseteq (1, \ldots, n)$ with $2 \leqslant |J| \leqslant n-2$, the restriction $W_r(\vec{m})|_{D_J}$ is nonzero only if the following two inequalities hold:*

$$\sum_{i \in J} m_i < (|J| - 1)(r+1) \qquad \text{and} \qquad \sum_{i \in J^c} m_i < (|J^c| - 1)(r+1).$$

Proof. Let m_J, m_{J^c} be as in (8). Without loss of generality, suppose that $\sum_{i \in J} m_i \geqslant (|J| - 1)(r+1)$, so, in particular,

$$m_J + \sum_{i \in J} m_i > (|J| - 1)(r+1).$$

Then, by (6), we have

$$\operatorname{rank} \mathcal{W}_r((m_i)_{i \in J} \cup \{m_J\}) > |J| - 2 = \dim \overline{M}_{0,|J|+1},$$

and so $W_r((m_i)_{i \in J} \cup \{m_J\}) = 0$ for dimension reasons. By Proposition 2.8, $W_r(\vec{m})|_{D_J} = 0$. □

3. Tropical realizations of classes from cohomological field theories.

In this section we recall some notions about the tropical intersection theory of $M_{0,n}^{\mathrm{trop}}$ and its relation to the intersection theory of $\overline{M}_{0,n}$. We define the tropicalisation of a CohFT, and then we restrict our attention to genus zero and describe the relationship between the balancing condition for tropicalised CohFT cycles and the WDVV relations. Basics of tropical moduli spaces that are assumed here may be found in [SS04, Mik07, GKM09, ACP15].

3.1 Tropicalization of cycle classes in $\overline{M}_{0,n}$

The moduli space $\overline{M}_{0,n}$ of rational, stable, n-pointed curves is a tropical compactification of $M_{0,n}$ [Tev07]: The noncompact space $M_{0,n}$ may be realised as a closed subvariety of a torus T. Denote the cocharacter lattice of T by N_T. The tropicalisation of $M_{0,n}$ lives inside the vector

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space $Q_{[n]} := N_T \otimes_{\mathbb{Z}} \mathbb{R}$ as a balanced fan, which is naturally identified with $M_{0,n}^{\mathrm{trop}}$, the moduli space of rational, stable, tropical n-pointed curves (Fact 3.2). Given a Chow class in $\overline{M}_{0,n}$, one may define (Definition 3.4) its *tropicalisation*, a weighted subfan of $M_{0,n}^{\mathrm{trop}}$ that satisfies the balancing condition. We review the aspects of this story that we will be using and refer the reader to [KM08, Kat09] for proofs. Some extended examples are included for the benefit of the unfamiliar readers.

The ambient space. The vector space $Q_{[n]}$ has dimension $\binom{n-1}{2} - 1$; we describe it by giving it a (non-canonical) set of generators and relations. Fix $k \in [n]$. Then the set of pairs $\{i, j\}$ with $i, j \neq k$ gives a system of $\binom{n-1}{2}$ vectors $\mathbf{r}^k_{\{i,j\}}$ generating $Q_{[n]}$. Their $\mathbb{Z}$-span gives the integral lattice in $Q_{[n]}$, and they are subject to the unique relation (up to scaling):

$$\sum_{\{i,j\} \in [n] \setminus \{k\}} \mathbf{r}^k_{\{i,j\}} = 0 \in Q_{[n]}. \tag{9}$$

In other words, the vectors $\mathbf{r}^k_{\{i,j\}}$ may be thought of as the primitive vectors generating the rays for the fan of projective space $\mathbb{P}^{\binom{n-1}{2}-1}$.

The embedding of $M_{0,n}^{\mathrm{trop}}$ in $Q_{[n]}$. The space $M_{0,n}^{\mathrm{trop}}$ is a cone complex parameterising stable metric trees with "legs" labeled by $[n]$. (A *leg* is a labeled half-edge incident to a vertex, usually taken to have infinite length.) For each topological type of tree Γ, one has a cone isomorphic to $\mathbb{R}_{\geqslant 0}^{|E(\Gamma)|}$ corresponding to all possible edge lengths, and the cones are glued together along faces by declaring a graph with an edge of length zero equivalent to the graph obtained by contracting of that edge. The abstract cone complex $M_{0,n}^{\mathrm{trop}}$ admits a natural embedding into $Q_{[n]}$, which we now describe: We first give the image $\mathbf{v}_I \in Q_{[n]}$ of primitive vectors generating the rays of $M_{0,n}^{\mathrm{trop}}$; these correspond to trees with a single edge of length one, separating the set of legs into two parts—I, I^c—both of size at least 2. Without loss of generality, we assume that $k \notin I$. We declare

$$\mathbf{v}_I = \sum_{\{i,j\} \subseteq I} \mathbf{r}^k_{\{i,j\}}. \tag{10}$$

Given this information, the image of $M_{0,n}^{\mathrm{trop}}$ in $Q_{[n]}$ is determined by multi-linearity. A point $\mathbf{x} \in Q_{[n]}$ of (the image of) $M_{0,n}^{\mathrm{trop}}$ corresponds to an n-marked metric tree Γ. Each edge $e \in E(\Gamma)$ produces a two-part partition I_e, I_e^c of the set of indices, where we again assume $k \notin I_e$. If l_e denotes the length of the edge e, then

$$\mathbf{x} = \sum_{e \in E} l_e \mathbf{v}_{I_e}. \tag{11}$$

Remark 3.1. Both the vector space $Q_{[n]}$ and the embedding of $M_{0,n}^{\mathrm{trop}}$ may be described in a canonical way (i.e. without choosing a distinguished index k); see [KM08, Section 2]. The choice of k provides a non-canonical system of generators that makes both statements and computations more concrete.

Balancing. A pure d-dimensional weighted fan Σ in a vector space Q with an integral structure is called *balanced* if for every face τ of dimension $d-1$, one has

$$\sum_{\sigma \succ \tau} w(\sigma)\ \mathbf{u}_{\tau/\sigma} = 0 \in Q/\langle \tau \rangle_{\mathbb{R}}, \tag{12}$$

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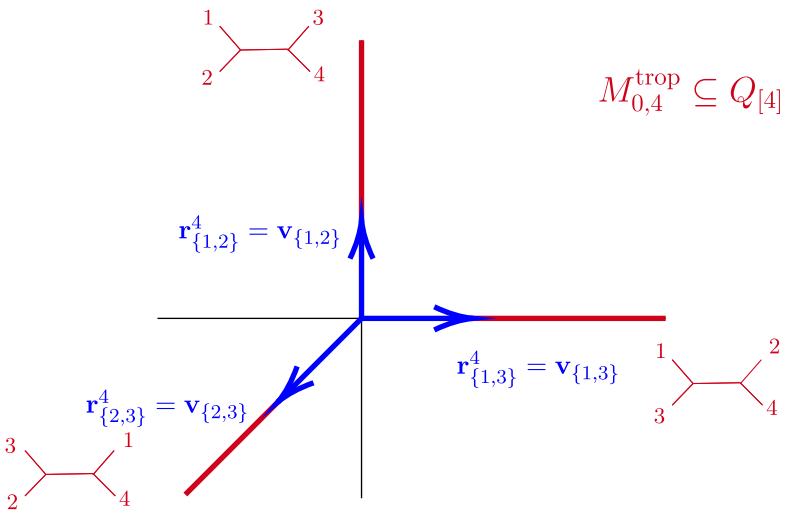


Figure 1. The space $M^{\mathrm{trop}}_{0,4}$ inside $Q_{[4]}$ as a balanced fan.

where the sum ranges over all d-dimensional cones σ of Σ that contain τ as a face; $w(\sigma)$ is the weight of the cone σ; and $\mathbf{u}_{\tau/\sigma}$ is a primitive normal vector to τ in σ, i.e. a primitive vector in the quotient space $\langle\sigma\rangle_{\mathbb{R}}/\langle\tau\rangle_{\mathbb{R}} \subseteq Q/\langle\tau\rangle_{\mathbb{R}}$.

Fact 3.2. (For a proof, see [GKM09, Thm. 3.7]). *The space $M^{\mathrm{trop}}_{0,n} \subseteq Q_{[n]}$, with the weight function $w(\sigma) = 1$ for all top dimensional cones σ, is a balanced fan of pure dimension $n-3$.*

Example 3.3. We illustrate the ideas presented in the simplest nontrivial example, $M^{\mathrm{trop}}_{0,4}$, drawn in Figure 1. The vector space $Q_{[4]}$ is 2-dimensional, presented as generated by three vectors. Choosing $k = 4$, the generators are $\mathbf{r}^4_{\{1,2\}}$, $\mathbf{r}^4_{\{1,3\}}$, and $\mathbf{r}^4_{\{2,3\}}$, and the relation is

$$\mathbf{r}^4_{\{1,2\}} + \mathbf{r}^4_{\{1,3\}} + \mathbf{r}^4_{\{2,3\}} = 0. \tag{13}$$

One may define a linear isomorphism $Q_{[4]} \xrightarrow{\cong} \mathbb{R}^2$ by identifying the first two generators with the standard basis vectors; the third generator becomes the vector $(-1,-1)$. The rays of $M^{\mathrm{trop}}_{0,4}$ are spanned by the primitive vectors $\mathbf{v}_{\{1,2\}}, \mathbf{v}_{\{1,3\}}, \mathbf{v}_{\{2,3\}}$, which in this simple case are equal to the corresponding vectors $\mathbf{r}^4_{\{1,2\}}, \mathbf{r}^4_{\{1,3\}}, \mathbf{r}^4_{\{2,3\}}$, generating $Q_{[4]}$. Since $M^{\mathrm{trop}}_{0,4}$ is a one-dimensional fan, balancing needs to be checked only at the vertex (the unique 0-dimensional face). Since all rays are given weight one, the balancing equation (12) is readily seen to reduce to (13) in this case.

Tropicalisation of Chow Classes. While tropical intersection theory is an actively developing field [Mik06, AR10, Kat09, Sha13, Gro18] we make use of a limited portion of it, which we recall in the context of $\overline{M}_{0,n}$. We refer the reader to [Kat09] for proofs and for a more complete treatment.

If $Y \subseteq \overline{M}_{0,n}$ is a pure k-dimensional cycle that meets each boundary stratum in the expected dimension, then its tropicalisation is supported on the k-dimensional cones of $M^{\mathrm{trop}}_{0,n}$, with the coefficient of a given cone $\sigma \subseteq M^{\mathrm{trop}}_{0,n}$ equal to the intersection number of Y with the corresponding boundary stratum $\Delta_\sigma \subseteq \overline{M}_{0,n}$ [Kat09, Prop. 9.4]. By the moving lemma, one may apply this construction directly to Chow classes, to obtain the following.

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DEFINITION 3.4. Let $\alpha \in A^k(\overline{M}_{0,n})$ be a codimension-k Chow class. Define the tropicalization α^{trop} of α to be

$$\alpha^{\mathrm{trop}} = \sum_{\sigma} \left(\int_{\overline{M}_{0,n}} \Delta_\sigma \cdot \alpha \right) \sigma, \tag{14}$$

where one may sum over all cones of $M_{0,n}^{\mathrm{trop}}$, but where the coefficients are nonzero only for codimension-k cones.

FACT 3.5. [Kat09, Lem. 8.13]. *The weighted fan $\alpha^{\mathrm{trop}} \subseteq Q_{[n]}$ from Definition 3.4 is balanced.*

We refer to a balanced weighted subfan of $M_{0,n}^{\mathrm{trop}}$ as a *tropical cycle*.

Example 3.6. We consider the case $n = 5$ and give an example of how to check the balancing condition along a ray. We then construct the tropicalisation of the boundary divisor class $D_{\{1,2\}} \in A^1(\overline{M}_{0,5})$ and show that it is a pure 1-dimensional weighted balanced fan.

The vector space $Q_{[5]}$ is 5-dimensional; choosing $k = 5$, we have six generators $\mathbf{r}^5_{\{i,j\}}$, with $i, j \neq 5$, which agree with six of the ten primitive vectors spanning the rays of $M_{0,5}^{\mathrm{trop}}$, i.e.

$$\mathbf{v}_{\{i,j\}} = \mathbf{r}^5_{\{i,j\}} \quad \text{for} i, j \neq 5. \tag{15}$$

By (10), the primitive vectors for the remaining four rays of $M_{0,5}^{\mathrm{trop}}$ are

$$\begin{aligned} \mathbf{v}_{\{1,2,3\}} &= \mathbf{r}^5_{\{1,2\}} + \mathbf{r}^5_{\{1,3\}} + \mathbf{r}^5_{\{2,3\}}, \\ \mathbf{v}_{\{1,2,4\}} &= \mathbf{r}^5_{\{1,2\}} + \mathbf{r}^5_{\{1,4\}} + \mathbf{r}^5_{\{2,4\}}, \\ \mathbf{v}_{\{1,3,4\}} &= \mathbf{r}^5_{\{1,3\}} + \mathbf{r}^5_{\{1,4\}} + \mathbf{r}^5_{\{3,4\}}, \\ \mathbf{v}_{\{2,3,4\}} &= \mathbf{r}^5_{\{2,3\}} + \mathbf{r}^5_{\{2,4\}} + \mathbf{r}^5_{\{3,4\}}. \end{aligned} \tag{16}$$

Each 2-dimensional cone of $M_{0,5}^{\mathrm{trop}}$ is spanned by two rays, and its image in $Q_{[5]}$ is determined by the images of the rays. We next check that given all 2-dimensional cones of weight 1, the fan $M_{0,5}^{\mathrm{trop}}$ is balanced along the ray τ spanned by $\mathbf{v}_{\{1,2,3\}}$; see Figure 2. There are three 2-dimensional cones containing τ, denoted $\sigma_{\{1,2\}\{4,5\}}, \sigma_{\{1,3\}\{4,5\}}, \sigma_{\{2,3\}\{4,5\}}$. One may check that for each $\sigma_{\{i,j\}\{4,5\}}$, a normal vector to τ is $\mathbf{u}_{\tau/\sigma_{\{i,j\}\{4,5\}}} = \mathbf{v}_{\{i,j\}}$. Then (12) becomes

$$\mathbf{v}_{\{1,2\}} + \mathbf{v}_{\{1,3\}} + \mathbf{v}_{\{2,3\}} = 0 \in Q_{[5]}/\langle \mathbf{v}_{\{1,2,3\}} \rangle_{\mathbb{R}}, \tag{17}$$

which follows from (15), (16). Thus $M_{0,5}^{\mathrm{trop}}$ is balanced along $\mathbf{v}_{\{1,2,3\}}$.

Consider the class $D_{\{1,2\}} \in A^1(\overline{M}_{0,5})$ of the divisor generically parameterizing stable curves with two components, one containing the first two marks, the other containing the other marks. We construct its tropicalisation $D^{\mathrm{trop}}_{\{1,2\}}$ and check that it is balanced.

The intersections of boundary strata in $\overline{M}_{0,n}$ are well understood (see, for example, [Cav16]). The intersection numbers of $D_{\{1,2\}}$ with all boundary divisors are as follows:

$$\begin{gathered} \deg(D^2_{\{1,2\}}) = -1, \\ \deg(D_{\{1,2\}} \cdot D_{\{3,4\}}) = \deg(D_{\{1,2\}} \cdot D_{\{3,5\}}) = \deg(D_{\{1,2\}} \cdot D_{\{4,5\}}) = 1, \end{gathered}$$

and the intersection numbers of $D_{\{1,2\}}$ with the remaining 6 boundary divisors are all zero.

By Definition 3.4, we have

$$D^{\mathrm{trop}}_{\{1,2\}} = \langle \mathbf{v}_{\{3,4\}} \rangle_{\mathbb{R}_{\geqslant 0}} + \langle \mathbf{v}_{\{1,2,3\}} \rangle_{\mathbb{R}_{\geqslant 0}} + \langle \mathbf{v}_{\{1,2,4\}} \rangle_{\mathbb{R}_{\geqslant 0}} - \langle \mathbf{v}_{\{1,2\}} \rangle_{\mathbb{R}_{\geqslant 0}}. \tag{18}$$

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TABLE 1. The computation that $D^{\mathrm{trop}}_{\{1,2\}}$ is a balanced cycle. The first four rows of the tables present the coefficients of the linear combination that expresses each of the weighted primitive vectors in terms of the generators $\mathbf{r}^5_{\{i,j\}}$. The last row, which adds up the previous ones, computes equation (12) at the origin: one obtains the relation among the $\mathbf{r}^5_{\{i,j\}}$'s.

	$\mathbf{r}^5_{\{1,2\}}$	$\mathbf{r}^5_{\{1,3\}}$	$\mathbf{r}^5_{\{1,4\}}$	$\mathbf{r}^5_{\{2,3\}}$	$\mathbf{r}^5_{\{2,4\}}$	$\mathbf{r}^5_{\{3,4\}}$
$\mathbf{v}_{\{3,4\}}$	0	0	0	0	0	1
$\mathbf{v}_{\{1,2,3\}}$	1	1	0	1	0	0
$\mathbf{v}_{\{1,2,4\}}$	1	0	1	0	1	0
$-\mathbf{v}_{\{1,2\}}$	−1	0	0	0	0	0
sum	1	1	1	1	1	1

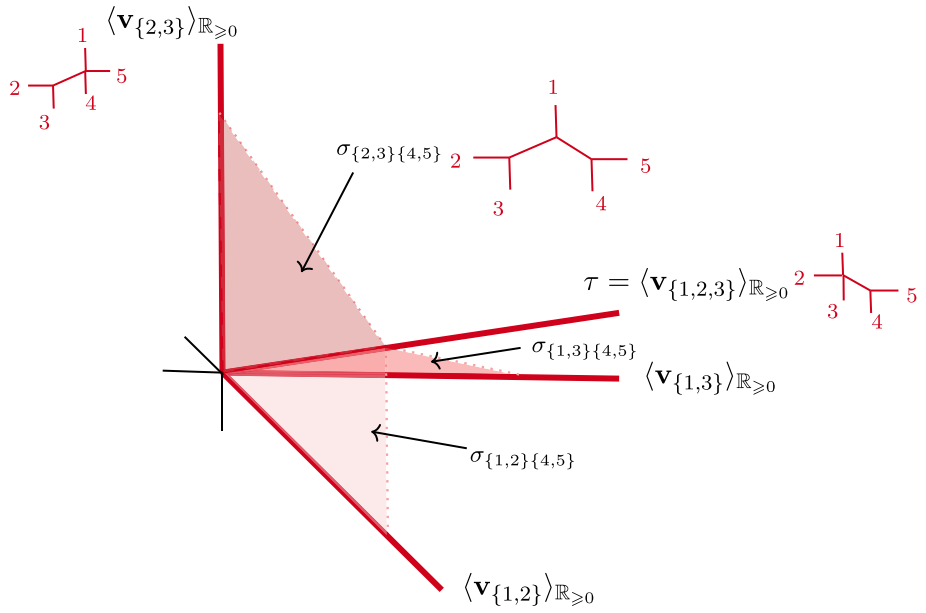


FIGURE 2. A local picture of $M^{\mathrm{trop}}_{0,5}$ around the ray $\tau = \langle \mathbf{v}_{\{1,2,3\}} \rangle_{\mathbb{R}_{\geqslant 0}}$. The red trees drawn next to the rays $\tau, \langle \mathbf{v}_{\{2,3\}} \rangle_{\mathbb{R}_{\geqslant 0}}$ and the two dimensional cone $\sigma_{\{2,3\}\{4,5\}}$ show the tropical curves parameterized by those cones.

To check that $D^{\mathrm{trop}}_{\{1,2\}}$ satisfies the balancing condition at the origin, we write each of the primitive vectors of the rays of $D^{\mathrm{trop}}_{\{1,2\}}$ as linear combinations of the generators $\mathbf{r}^5_{\{i,j\}}$ using (10). The computation is done in Table 1. Indeed, Table 1 shows that the sum of the weighted primitive normal vectors of the rays of $D^{\mathrm{trop}}_{\{1,2\}}$ can be expressed as $\sum_{\{i,j\} \in [5] \setminus \{5\}} \mathbf{r}^5_{\{i,j\}}$, which is equal to zero by (9). See also Example 3.7 below.

We now show how to use the balancing condition to obtain explicit linear relations among the coefficients of a tropical cycle on $M^{\mathrm{trop}}_{0,n}$. We describe the process for a 1-dimensional tropical cycle, which is the case that will be used the most later on.

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Consider a 1-dimensional tropical cycle $A = \sum_I a_I \langle \mathbf{v}_I \rangle_{\mathbb{R}_{\geqslant 0}}$ on $M_{0,n}^{\mathrm{trop}}$. We are using the notation introduced in this section; that is, we have fixed $k \in [n]$, and we write $\mathbf{v}_I = \sum_{\{i,j \subseteq I\}} \mathbf{r}_{\{i,j\}}^k$ for the image in $Q_{[n]}$ of the tropical curve with one edge of length one, separating the marks into sets I, I^c with $k \in I^c$. The origin is the only zero-dimensional cone of A, so checking balancing for this cycle amounts to verifying that

$$\sum_{\substack{I \subseteq [n] \smallsetminus \{k\} \\ 2 \leqslant |I| \leqslant n-2}} a_I \mathbf{v}_I = 0 \in Q_{[n]}. \tag{19}$$

We use the same mark k to pick a set of generators for $Q_{[n]}$, and we rewrite each of the $\mathbf{v}_I$ using equation (10), so the left-hand side of (19) becomes:

$$\sum_{\substack{I \subseteq [n] \setminus \{k\} \\ 2 \leqslant |I| \leqslant n-2}} a_I \left(\sum_{\{i,j\} \subseteq I} \mathbf{r}_{\{i,j\}}^k \right) = \sum_{\{i,j\} \in [n] \setminus \{k\}} \left(\sum_{\substack{\{i,j\} \subseteq I \subseteq [n] \setminus \{k\} \\ |I| \leqslant n-2}} a_I \right) \mathbf{r}_{\{i,j\}}^k = \sum_{\{i,j\} \in [n] \setminus \{k\}} B_{\{i,j\}}^k \mathbf{r}_{\{i,j\}}^k, \tag{20}$$

where the notation $B_{\{i,j\}}^k$ is defined by the second equality. Equation (20) equals the zero vector if and only if for every pair $\{i_1, j_1\}, \{i_2, j_2\}$ with $i_1, j_1, i_2, j_2 \neq k$, we have

$$B_{\{i_1,j_1\}}^k = B_{\{i_2,j_2\}}^k. \tag{21}$$

We observe that (21) is a linear homogeneous equation in the coefficients of the tropical cycle A. We call any such equation a *balancing relation*.

There is a simple description for the coefficient $B_{\{i,j\}}^k$: It is obtained by adding the coefficient $a_{\{i,j\}}$ of the ray $\langle \mathbf{v}_{\{i,j\}} \rangle_{\mathbb{R}_{\geqslant 0}}$ and the coefficients of the $2^{n-3} - 2$ rays $\langle \mathbf{v}_I \rangle_{\mathbb{R}_{\geqslant 0}}$ adjacent to it[3] such that $i, j \in I$.

Example 3.7. We illustrate how to use (21) concretely. As in Example 3.6, let $n = 5$ and choose $k = 5$. By (20) we have

$$B_{\{1,2\}}^5 = a_{\{1,2\}} + a_{\{1,2,3\}} + a_{\{1,2,4\}},$$
$$B_{\{3,4\}}^5 = a_{\{3,4\}} + a_{\{1,3,4\}} + a_{\{2,3,4\}},$$

so (21) in this case yields

$$a_{\{1,2\}} + a_{\{1,2,3\}} + a_{\{1,2,4\}} = a_{\{3,4\}} + a_{\{1,3,4\}} + a_{\{2,3,4\}}.$$

We observe that this is consistent with our calculations in Example 3.6; for $D_{\{1,2\}}^{\mathrm{trop}}$, the above equation reads:

$$-1 + 1 + 1 = 1 + 0 + 0.$$

We illustrate $B_{\{1,2\}}^5$ and $B_{\{3,4\}}^5$ in Figure 3, and note it amounts to computing the sums of the first and last columns of Table 1.

3.2 Balancing and reconstruction

In this section we apply the techniques of Section 3.1 to the genus-zero part of any cohomological field theory (CohFT). At a basic level, a CohFT should be thought of as a way to obtain a

[3] Here $\mathbf{v}_I$ and $\mathbf{v}_{\{i,j\}}$ being adjacent means that they span a two dimensional cone in $M_{0,n}^{\mathrm{trop}}$.

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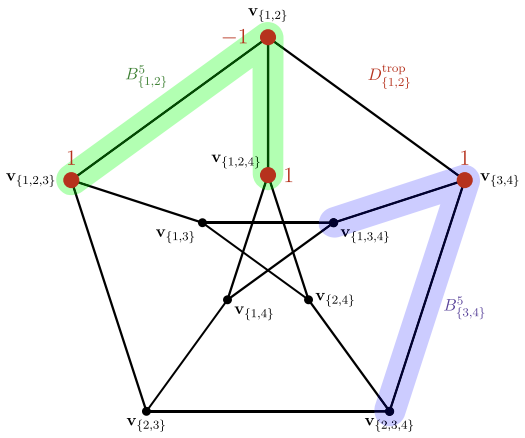


Figure 3. The Petersen graph gives a slice of the cone complex of $M^{\mathrm{trop}}_{0,5}$, parameterising tropical curves where the total length of all the edges is equal to one (unfortunately it is not possible to draw a 2-dimensional projection of $M^{\mathrm{trop}}_{0,5}$ that would allow us simultaneously to observe the combinatorial structure of the cone complex as well as the structure of a balanced fan). The vertices correspond to the primitive vectors spanning the rays of $M^{\mathrm{trop}}_{0,5}$ and are labelled by the corresponding $\mathbf{v}_I$'s. In red we have the cycle $D^{\mathrm{trop}}_{\{1,2\}}$, where the coefficients written next to the dots are the weights of the corresponding rays. Rays without a red dot have weight 0. The sum of the coefficients in the shaded areas in green (resp. in blue) gives the coefficient $B^5_{\{1,2\}}$ (resp. $B^5_{\{3,4\}}$). One can observe the balancing relation $B^5_{\{1,2\}} = B^5_{\{3,4\}} = 1$ in this case.

collection of Chow classes $\Omega_{g,n}(v_1,\dots,v_n) \in A^*(\overline{M}_{g,n})$, indexed by elements v_i of some vector space, that behave recursively when restricted to any boundary stratum. We refer the reader to the introductory paper [Pan18] for the precise definitions. Let Ω be an arbitrary CohFT over a vector space V. In order to simplify the exposition and notation, we impose the following assumption:

Assumption 3.8. *There exists a basis $\{e_\alpha\}_{\alpha\in A}$ of V such that any class of the form $\Omega_{g,n}(e_{\alpha_1},\dots,e_{\alpha_n})$ is of pure dimension, denoted $d_{g,n,\alpha_1,\dots,\alpha_n}$.*

Assumption 3.8 is not conceptually necessary; one may decompose any mixed-degree class $\Omega_{g,n}(v_1,\dots,v_n)$ into its homogeneous parts, apply the constructions that follow to each homogeneous part and then formally add everything up. Assumption 3.8 essentially prevents having to carry around these sums. Many CohFTs that are constructed from geometric properties of curves satisfy Assumption 3.8, including the main example in this work, the CohFT of Witten's r-spin classes (see Section 2).

Definition 3.9. The **numerical part** ω of a CohFT Ω records the degree of the zero-dimensional classes of Ω, i.e.

$$\omega_{g,n}(v_1,\dots,v_n) = \int_{\overline{M}_{g,n}} \Omega_{g,n}(v_1,\dots,v_n). \tag{22}$$

We call elements of the numerical part of a CohFT **numerical CohFT invariants**.

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Given a CohFT class $\Omega_{g,n}(v_1,\dots,v_n)$ of pure dimension $d_{g,n,v_1,\dots,v_n}$, one may construct its tropicalisation following Definition 3.4:

$$\Omega^{\mathrm{trop}}_{g,n}(v_1,\dots,v_n)=\sum_{\sigma}\left(\int_{\overline{M}_{g,n}}\Delta_\sigma\cdot\Omega_{g,n}(v_1,\dots,v_n)\right)\ \sigma, \tag{23}$$

where the sum runs over all cones σ of $M^{\mathrm{trop}}_{g,n}$.

PROPOSITION 3.10. *The class $\Omega^{\mathrm{trop}}_{g,n}(v_1,\dots,v_n)$ is supported on the $(d_{g,n,v_1,\dots,v_n})$-dimensional skeleton of $M^{\mathrm{trop}}_{g,n}$, and the nonzero coefficients are determined by the numerical part of Ω.*

Proof. The first part of the statement is simply a dimension count: $\Omega^{\mathrm{trop}}_{g,n}(v_1,\dots,v_n)$ can intersect only nontrivial strata of codimension $d_{g,n,v_1,\dots,v_n}$, which in turn correspond to cones of $M^{\mathrm{trop}}_{g,n}$ of dimension $d_{g,n,v_1,\dots,v_n}$. The second part follows from the splitting axioms for CohFTs (see [Pan18, Sec. 0.3]), which imply that the restriction of the class $\Omega^{\mathrm{trop}}_{g,n}(v_1,\dots,v_n)$ to a boundary stratum $\Delta_\sigma=(\mathrm{gl}_\sigma)_*([\prod_i\overline{M}_{g_i,n_i}])$ is equal to the pushforward of a linear combination of CohFT classes pulled-back from the factors of $\prod_i\overline{M}_{g_i,n_i}$. By Fubini's theorem, the integral over $\overline{M}_{g,n}$ of each summand splits as a product of integrals over the individual factors. Thus, for each factor, we obtain a corresponding numerical CohFT invariant. □

Moduli spaces of tropical curves of positive genus cannot be given the structure of a balanced fan in a vector space. Therefore for the remainder of the paper, we restrict to $g=0$, where we can exploit the additional structure of the balancing of tropical cycles.

Example 3.11. We illustrate a balancing relation for the r-spin CohFT with $r=10$, using the notation of Section 2 and Examples 3.6 and 3.7. Consider the Witten class $W_{10}(3,4,5,5,6)$. By (6), $W_{10}(3,4,5,5,6)$ is a cycle on $\overline{M}_{0,5}$ of codimension 1 (and dimension 1).

We use the balancing condition for $W_{10}(3,4,5,5,6)^{\mathrm{trop}}$ to deduce a relationship between 3- and 4-point 10-spin invariants. As in Example 3.6, we choose $k=5$ to obtain six generators $\mathbf{r}^5_{\{i,j\}}$ with $i,j\in[4]=[5]\setminus\{5\}$. We have by Definition 3.4

$$W_{10}(3,4,5,5,6)^{\mathrm{trop}}=\sum_{\substack{I\subseteq[4]\\2\leqslant|I|\leqslant3}}a_I\mathbf{v}_I,$$

where $a_I=\int_{\overline{M}_{0,5}}D_I\cdot W_{10}(3,4,5,5,6)$. By (20) and (21), with $(i_1,j_1)=(1,2)$ and $(i_2,j_2)=(3,4)$, we have $B^5_{\{1,2\}}=B^5_{\{3,4\}}$, where

$$B^5_{\{1,2\}}=\sum_{\substack{I\subseteq[4]\\1,2\in I,\ |I|\leqslant3}}a_I=a_{\{1,2\}}+a_{\{1,2,3\}}+a_{\{1,2,4\}}, \tag{24}$$

and

$$B^5_{\{3,4\}}=\sum_{\substack{I\subseteq[4]\\3,4\in I,\ |I|\leqslant3}}a_I=a_{\{3,4\}}+a_{\{1,3,4\}}+a_{\{2,3,4\}}. \tag{25}$$

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By the CohFT Axiom (Proposition 2.8), we have e.g.

$$\begin{aligned} a_{\{2,3,4\}} = \int_{\overline{M}_{0,5}} W_{10}(3,4,5,5,6)|_{D_{\{2,3,4\}}} &= \int_{\overline{M}_{0,5}} (W_{10}(4,5,5,8) \boxtimes W_{10}(3,6,2)) \\ &= \left(\int_{\overline{M}_{0,4}} W_{10}(4,5,5,8)\right)\left(\int_{\overline{M}_{0,3}} W_{10}(3,6,2)\right) \\ &= w_{10}(4,5,5,8)w_{10}(3,6,2). \end{aligned}$$

Here in the notation of Proposition 2.8, we have $m_J = 8$ and $m_{J^c} = 2$. Similarly we compute the other terms of (24) and (25):

$$\begin{aligned} a_{\{1,2\}} &= w_{10}(3,4,4)w_{10}(5,5,6,6) & a_{\{1,2,3\}} &= w_{10}(3,4,5,10)w_{10}(5,6,10) \\ a_{\{1,2,4\}} &= w_{10}(3,4,5,10)w_{10}(5,6,10) & a_{\{3,4\}} &= w_{10}(5,5,1)w_{10}(3,4,6,9) \\ a_{\{1,3,4\}} &= w_{10}(3,5,5,9)w_{10}(4,6,1). \end{aligned}$$

Thus the equation $B^5_{\{1,2\}} = B^5_{\{3,4\}}$ gives us a polynomial relation between genus-zero 10-spin invariants:

$$\begin{aligned} w_{10}(3,4,4)w_{10}(5,5,6,6) + w_{10}(3,4,5,10)w_{10}(5,6,10) + w_{10}(3,4,5,10)w_{10}(5,6,10) \\ = w_{10}(5,5,1)w_{10}(3,4,6,9) + w_{10}(3,5,5,9)w_{10}(4,6,1) + w_{10}(4,5,5,8)w_{10}(3,6,2). \end{aligned} \tag{26}$$

We can generate a huge number of relations of this form between genus-zero 10-spin invariants, both by varying i_1, j_1, i_2, j_2, and k and by considering the balancing of other (including higher-dimensional) 10-spin Witten classes along various cones of $M^{\mathrm{trop}}_{0,n}$. Theorem 3.12 shows that this collection of relations is equivalent to the WDVV equations, and in Section 4, we use the balancing relations to give an efficient recursive algorithm for reconstructing any genus-zero r-spin invariant. For the moment, we note that (26) is consistent with Proposition 2.13; indeed, (26) reads

$$1 \cdot 4 + 0 \cdot 1 + 0 \cdot 1 = 1 \cdot 1 + 1 \cdot 1 + 2 \cdot 1.$$

In Figure 4, we have computed a_I for all I and labeled the rays of $M^{\mathrm{trop}}_{0,5}$ accordingly. We have then drawn the quantities $B^k_{\{i,j\}}$ for all $i, j \in [5] \setminus \{k\}$, for $k = 5$ (left picture) and for $k = 1$ (right picture). Pictorially, one can immediately confirm the balancing relations for $B^5_{\{i,j\}}$ and $B^1_{\{i,j\}}$.

WDVV relations. In $\overline{M}_{0,4} \cong \mathbb{P}^1$ we have the simple fact that any boundary divisor is equivalent in the Chow ring to the class of a point. Choosing any two of the three boundary divisors and subtracting them one obtains a relation, which we call a basic relation. Any relation obtained by pulling back via forgetful morphisms or by pushing forward via gluing morphisms of a basic relation is called a WDVV relation. Given a CohFT Ω on a vector space V, restricting a class $\Omega_{0,n}(v_1, \ldots, v_n)$ to a WDVV relation and integrating produces a relation among the numerical CohFT invariants that we also call a WDVV relation. Explicitly, the relation obtained from pulling back a basic relation via some forgetful morphism has the following form: Denote by $\{e_\alpha\}$ a basis for V and by $\{e^\alpha\}$ the dual basis with respect to the CohFT metric. Choose four numbers, a, b, c, d, in the index set $[n]$ and for a set I denote by $\vec{v}_I = \{v_i\}_{i \in I}$:

$$\begin{aligned} \sum_\alpha \sum_{I \subseteq \{[n] \setminus \{a,b,c,d\}\}} \big(&\omega_{0,|I|+3}(v_a, v_b, \vec{v}_I, e_\alpha)\omega_{0,|I^c|+3}(e^\alpha, \vec{v}_{I^c}, v_c, v_d) \\ &- \omega_{0,|I|+3}(v_a, v_d, \vec{v}_I, e_\alpha)\omega_{0,|I^c|+3}(e^\alpha, \vec{v}_{I^c}, v_b, v_c)\big) = 0. \end{aligned} \tag{27}$$

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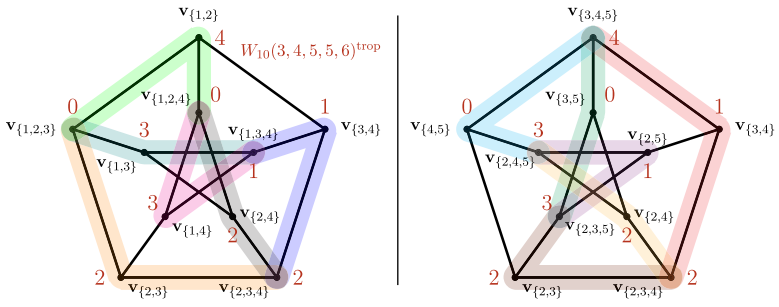


FIGURE 4. The balancing relations $B^5_{\{i,j\}} = 4$ (left picture) and $B^1_{\{i,j\}} = 7$ (right picture) for the tropicalization of $W_{10}(3,4,5,5,6)$. In each picture, the equally colored collections of vertices all have the same sum. Example 3.11 confirms that the sums of the light-green and light-purple collections of vertices in the left picture are both equal to 4. Note that we have named each ray $\mathbf{v}_I$ of $M^{\mathrm{trop}}_{0,5}$ (corresponding to a vertex of the Petersen graph) using the convention $k \notin I$; this is the only reason the vertex labels of the two pictures differ.

We now see that the numerical part of the WDVV relations has a natural analogue in the tropicalisation.

THEOREM 3.12. *For any CohFT Ω, consider the numerical CohFT invariants as unknowns to be determined; the following two collections of equations impose equivalent constraints among the invariants of ω:*

1. *the balancing equations (21) for all tropical cycles $\Omega^{\mathrm{trop}}_{0,n}(v_1,\dots,v_n)$.*
2. *the collection of all WDVV relations.*

Proof of Theorem 3.12. We prove this statement in two steps: First we show that for a CohFT class of dimension 1, the fact that its tropicalisation is balanced is equivalent to numerical WDVV relations that are pull-backs of the basic ones via forgetful morphisms (and thus take the simple form from equation (27)). Next we show that for a CohFT class of arbitrary dimension, its balancing along a cone is equivalent to WDVV relations that also involve pushforwards with respect to gluing morphisms.

STEP I. Assume $\dim \Omega_{0,n}(v_1,\dots,v_n) = 1$. Its tropicalisation can be written as a linear combination of rays:

$$\Omega^{\mathrm{trop}}_{0,n}(v_1,\dots,v_n) = \sum_{\substack{a\in I\subseteq[n]\smallsetminus k\\ 2\leqslant|I|\leqslant n-2}} a_I \langle \mathbf{v}_I\rangle_{\mathbb{R}_{\geqslant 0}} \subseteq Q_{[n]}, \tag{28}$$

with

$$a_I = \sum_{\alpha} \omega_{0,|I|+1}(\vec{v}_I, e_\alpha)\omega_{0,|I^c|+1}(e^\alpha, \vec{v}_{I^c}), \tag{29}$$

where $\omega_{0,|I|+1}(\vec{v}_I, e_\alpha)$ and $\omega_{0,|I^c|+1}(e^\alpha, \vec{v}_{I^c})$ are numerical CohFT invariants.

Assume $\Omega^{\mathrm{trop}}_{0,n}(v_1,\dots,v_n)$ is balanced, choose $a,b,c,d \in [n]$ and observe that the linear projection $Q_{[n]} \to Q_{\{a,b,c,d\}}$ restricts to the forgetful morphism $F : M^{\mathrm{trop}}_{0,n} \to M^{\mathrm{trop}}_{0,4}$, where the marks that are remembered are a,b,c,d. A useful fact used in tropical intersection theory (see e.g. [AR10, Sec. 4]) is that the pushforward of a weighted, balanced fan via a map of fans is also a

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weighted, balanced fan. We apply this fact to the map F above. Here, as $\Omega_{0,n}(v_1,\dots,v_n)$ is a sum of rays, its pushforward along F is a sum of rays in $M_{0,4}^{\mathrm{trop}}$. Since $M_{0,4}^{\mathrm{trop}}$ is an irreducible fan, the only balanced weight functions are constant on all three rays, hence $F_*(\Omega_{0,n}(v_1,\dots,v_n))$ is a constant times the sum of the three rays of $M_{0,4}^{\mathrm{trop}}$.

The coefficient of the ray $\langle \mathbf{v}_{\{a,b\}}\rangle_{\mathbb{R}_{\geqslant 0}}$ in $F_*(\Omega_{0,n}^{\mathrm{trop}}(v_1,\dots,v_n))$ equals the sum of the coefficients a_I, where $a,b\in I$ and $c,d\in I^c$, and similarly, the coefficient of the ray $\langle \mathbf{v}_{\{a,d\}}\rangle_{\mathbb{R}_{\geqslant 0}}$ in $F_*(\Omega_{0,n}^{\mathrm{trop}}(v_1,\dots,v_n))$ equals the sum of the coefficients a_I where $a,d\in I$ and $b,c\in I^c$. Imposing the equality of these coefficients and using (29), one obtains the WDVV relation (27). Thus we have proven that satisfying balancing implies satisfying WDVV.

Conversely, assume that the WDVV relations are satisfied among the numerical invariants of Ω. We want to show that $\Omega_{0,n}^{\mathrm{trop}}(v_1,\dots,v_n)$ is balanced. By construction, $\mathbf{v}_I$ is the primitive vector for the ray $\langle \mathbf{v}_I\rangle_{\mathbb{R}_{\geqslant 0}}$, and therefore this means showing that

$$\sum_{\substack{I\subseteq[n]\smallsetminus k\\ 2\leqslant|I|\leqslant n-2}} a_I\mathbf{v}_I = 0\in Q_{[n]}. \tag{30}$$

Note that we have fixed a $k\in[n]$ to not double-count each ray. We may use (10) to rewrite:

$$\sum_{\substack{I\subseteq[n]\setminus\{k\}\\ 2\leqslant|I|\leqslant n-2}} a_I\mathbf{v}_I = \sum_{\{i,j\}\in[n]\setminus\{k\}}\left(\sum_{\substack{\{i,j\}\subseteq I\subseteq[n]\smallsetminus\{k\}\\ |I|\leqslant n-2}} a_I\right)\mathbf{r}_{i,j}^k. \tag{31}$$

Subtracting the coefficient of $\mathbf{r}_{a,b}^c$ and the coefficient of $\mathbf{r}_{a,d}^c$ in (31), one obtains the WDVV relation (27), which we are assuming holds. Since this happens for all choices of $a,b,c,d\in[n]$, all the coefficients in (31) are equal so (30) holds, concluding the first part of the proof.

Step II. Let $\Omega_{0,n}(v_1,\dots,v_n)$ be a CohFT class of pure dimension d, and let τ be a $(d-1)$-dimensional cone in $M_{0,n}^{\mathrm{trop}}$ corresponding to a graph Γ. The corresponding stratum Δ_τ is isomorphic to a product $\prod_{\mathrm{v}\in V(\Gamma)}\overline{M}_{0,n_\mathrm{v}}$, where n_v denotes the valence of the vertex v of Γ. Denote by $\phi:\bigcup_{\mathrm{v}\in V(\Gamma)}\{\mathrm{v}\}\times[n_\mathrm{v}]\to\mathcal{P}([n])$ is the function that assigns to each half-edge h of Γ and is the set of marks in the connected component of Γ v that contains h.

Choose a $k\in[n]$, and for every vertex v denote by k_v the unique element of $[n_\mathrm{v}]$ such that $k\in\phi([k_\mathrm{v}])$.

Claim 3.13. For every $I_\mathrm{v}\subseteq[n_\mathrm{v}]$ not containing k_v, the assignment

$$\mathbf{v}_{I_\mathrm{v}}\mapsto\mathbf{v}_{\cup_{i\in I_\mathrm{v}}\phi(\mathrm{v},\mathrm{i})}, \tag{32}$$

gives a linear injection

$$\Phi:\bigoplus_{\mathrm{v}\in V(\Gamma)} Q_{[n_\mathrm{v}]}\to Q_{[n]}/\langle\tau\rangle_{\mathbb{R}}. \tag{33}$$

We defer the proof of the claim to not break the flow of this proof. For dimension reasons, the CohFT class of dimension d restricted to the stratum associated to the cone τ can be nonzero only if it decomposes as a sum over the vertices v of Γ, where each summand is a product of a 1-dimensional CohFT class Ω_v^1 supported on the vertex v with a multiple of the class of a point on all the other vertices. In formulas, we have:

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$$\Omega_{0,n}(v_1,\dots,v_n)|_{\Delta_\tau} = \sum_{\mathrm{v}\in V(\Gamma)} \mathrm{gl}_{\Gamma *}\left(c_{\mathrm{v}} \pi_{\mathrm{v}}^*(\Omega^1_{\mathrm{v}}) \boxtimes \prod_{\mathrm{w}\neq \mathrm{v}} \pi_{\mathrm{w}}^*(pt) \right), \tag{34}$$

where $\mathrm{gl}_\Gamma : \prod_{\mathrm{v}\in V(\Gamma)} \overline{M}_{0,n_{\mathrm{v}}} \to \overline{M}_{0,n}$ is the gluing morphism, π_{v} denotes the projection on the factor corresponding to vertex v and the coefficients c_{v} are real numbers determined by the splitting axioms. These coefficients are obtained from numerical CohFT invariants and the CohFT metric, but this information will not be explicitly used in this argument.

Any d-dimensional cone σ in $M^{\mathrm{trop}}_{0,n}$ of which τ is a face corresponds to a graph Γ_σ with an edge contraction $c_{e_\sigma} : \Gamma_\sigma \to \Gamma$. Denote by $\hat{\mathrm{v}} : \{\sigma \succ \tau\} \to V(\Gamma)$ the function that assigns to each d-dimensional cone σ the vertex $c_{e_\sigma}(e_\sigma)$ of Γ that is the image of the contracted edge. The edge e_σ also determines a two-part partition of the set of indices in $[n_{\hat{\mathrm{v}}(\sigma)}]$, i.e. a ray ρ_σ with primitive vector $\mathbf{v}_{I_\sigma}$ in $M^{\mathrm{trop}}_{0,n_{\hat{\mathrm{v}}}(\sigma)} \subseteq Q_{[n_{\hat{\mathrm{v}}(\sigma)}]}$. By the CohFT splitting axioms, the restriction of $\Omega_{0,n}(v_1,\dots,v_n)$ to Δ_σ satisfies

$$\int_{\Delta_\sigma} \Omega_{0,n}(v_1,\dots,v_n) = c_{\mathrm{v}} \int_{\overline{M}_{0,n_{\hat{\mathrm{v}}(\sigma)}}} \Omega^1_{\mathrm{v}} \cdot \Delta_{\rho_\sigma}. \tag{35}$$

We can now analyse the balancing condition along the face τ:

$$\sum_{\sigma\succ\tau} \Omega_{0,n}(v_1,\dots,v_n)|_{\Delta_\sigma} \mathbf{u}_{\tau/\sigma} = \sum_{\mathrm{v}\in V(\Gamma)} c_{\mathrm{v}} \Phi \left(\sum_{\sigma\in \hat{\mathrm{v}}^{-1}(\mathrm{v})} \left(\int_{\overline{M}_{0,n_{\hat{\mathrm{v}}}(\sigma)}} \Omega^1_{\mathrm{v}} \cdot \Delta_{\rho_\sigma} \right) \mathbf{v}_{I_\sigma} \right). \tag{36}$$

The vanishing of the left-hand side of (36) gives the balancing of $\Omega_{0,n}(v_1,\dots,v_n)$ along the higher-dimensional face τ; on the right-hand side, this quantity is expressed as a sum over the vertices v of Γ; inside the large parentheses, we recognise the weighted sum of normal vectors, whose vanishing gives balancing for the 1-dimensional CohFT class Ω^1_{v} at the cone point of $\overline{M}_{0,n_{\hat{\mathrm{v}}}(\sigma)}$. Then we take the image via Φ, which is a linear injection by Claim 3.13. Thus (36) shows that the balancing equation for a CohFT class along a higher-dimensional face τ reduces to a collection of balancing equations for 1-dimensional CohFT classes. The theorem is proved by invoking Step I.

Proof of Claim 3.13. This statement is well-known to the experts, but we include a proof here since we are not aware of a reference to it in the literature. We begin by proving the codimension-1 case. Let τ be a ray in $M^{\mathrm{trop}}_{0,n}$ corresponding to a graph Γ with only one edge separating the legs into two subsets, I and I^c. Without loss of generality, we assume that $k \notin I$ and that the marks in I are adjacent to v_1: those in I^c to v_2. In this case, $n_{\mathrm{v}_1} = |I| + 1$, and we denote the additional mark (corresponding to the germ of the edge) by k_{v_1}, following the notation in the proof of Theorem 3.12. The 1-dimensional linear space of $Q_{[n]}$ containing the ray τ is spanned by the vector $\mathbf{v}_\tau = \sum_{i,j\in I} \mathbf{r}^k_{i,j}$. The map $\Phi|_{Q_{[n_{\mathrm{v}_1}]}}$ is a linear isomorphism onto its image in $Q_{[n]}/\langle\tau\rangle_{\mathbb{R}}$: It is defined by $\mathbf{r}^{k_{\mathrm{v}_1}}_{i,j} \mapsto \mathbf{r}^k_{i,j}$, and the image of the relation among the generators of $Q_{[n_{\mathrm{v}_1}]}$ is precisely the vector $\mathbf{v}_\tau$. Denoting by x the mark corresponding to the germ of the edge attaching to v_2, the map $\Phi|_{Q_{[n_{\mathrm{v}_2}]}}$ is defined as follows:

$$\begin{aligned} \mathbf{r}^k_{i,j} &\mapsto \mathbf{r}^k_{i,j} \qquad \text{if } i,j \text{ in } I^c \\ \mathbf{r}^k_{j,x} &\mapsto \sum_{i\in I} \mathbf{r}^k_{i,j} \quad \text{if } j \text{ in } I^c, \end{aligned}$$

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where we have already set $\mathbf{v}_\tau = 0$. It is immediately necessary to check that the restriction $\Phi|_{Q_{[n_{v_2}]}}$ is injective and that $\Phi(Q_{[n_{v_1}]} \oplus 0) \cap \Phi(0 \oplus Q_{[n_{v_1}]}) = 0$, proving the claim in the case $\dim \tau = 1$. To prove the claim for a general cone τ of dimension d (corresponding to a graph Γ), one can choose arbitrarily an ordering of the edges of Γ and can iterate this construction d times, adding one edge at a time until Γ is obtained, thus composing all the resulting linear injections. □

4. Recursive structure of r-spin invariants

In this section we introduce a recursion for r-spin invariants, deducible from either the WDVV equations or (equivalently by Proposition 3.10) the balancing relations of tropical cycles. We will use the recursion extensively in proving Theorems 5.1 and 6.2.

THEOREM 4.1. *Let $\vec{m} = (m_1, \dots, m_n)$ be a numerical r-spin monodromy vector. Fix distinct elements $i, j, k \in [n]$ with $m_i > 1$ and $m_j < r$. Then*

$$w_r(\vec{m}) - w_r(m_1, \dots, m_i - 1, \dots, m_j + 1, \dots, m_n) = (n-3) \cdot T_r^{i,j,k}(m_1, \dots, m_n), \tag{37}$$

where, for distinct elements $i, j, k \in [n]$,

$$\begin{aligned} T_r^{i,j,k}(m_1, \dots, m_n) := \frac{1}{r}\Big(&\delta_{m_j+m_k \geqslant r+1} w_r(m_i - 1, m_1, \dots, \hat{m}_i, \hat{m}_j, \hat{m}_k, \dots, m_n, m_j + m_k - r) \\ &- \delta_{m_i+m_k \geqslant r+2} w_r(m_j, m_1, \dots, \hat{m}_i, \hat{m}_j, \hat{m}_k, \dots, m_n, m_i + m_k - r - 1)\Big), \end{aligned} \tag{38}$$

the symbol $\delta_{a \geqslant b}$ equals one when the inequality is satisfied and zero otherwise.

Remark 4.2. The left hand side of (37) is independent of k, which implies that $T_r^{i,j,k}(m_1, \dots, m_n)$ is independent of k.

Proof. Since $w_r(\vec{m})$ is invariant under permuting the entries of $\vec{m}$, we may assume that $i = 1$ and $j = 2$. By Theorem 3.12, we may equivalently use either the WDVV equations or the balancing condition for tropical cycles to carry out the proof. Let

$$\vec{m}^* = (m_1^*, \dots, m_{n+1}^*) = (m_1 - 1, m_2, m_3, \dots, m_n, 2),$$

and consider the 1-dimensional tropical cycle $W_r(\vec{m}^*)$. The balancing condition implies

$$B^{n+1}_{\{1,3\}}(W_r(\vec{m}^*)) = \sum_{\substack{J \subseteq \{1,\dots,n+1\} \\ 2 \leqslant |J| \leqslant n-1 \\ 1,3 \in J,\ n+1 \notin J}} \int_{\overline{M}_{0,n}} W_r(\vec{m}^*)|_{D_J} = \sum_{\substack{J \subseteq \{1,\dots,n+1\} \\ 2 \leqslant |J| \leqslant n-1 \\ 2,3 \in J,\ n+1 \notin J}} \int_{\overline{M}_{0,n}} W_r(\vec{m}^*)|_{D_J} = B^{n+1}_{\{2,3\}}(W_r(\vec{m}^*)).$$

Terms $W_r(\vec{m}^*)|_{D_J}$, with $\{1,2,3\} \subseteq J$, appear on both sides of the above equation, so we may remove them, thus obtaining a WDVV equation:

$$\sum_{\substack{J \subseteq \{1,\dots,n+1\} \\ 1,3 \in J,\ 2,n+1 \notin J}} \int_{\overline{M}_{0,n}} W_r(\vec{m}^*)|_{D_J} = \sum_{\substack{J \subseteq \{1,\dots,n+1\} \\ 2,3 \in J,\ 1,n+1 \notin J}} \int_{\overline{M}_{0,n}} W_r(\vec{m}^*)|_{D_J}. \tag{39}$$

By the CohFT property (Proposition 2.8), we have:

$$\int_{\overline{M}_{0,n}} W_r(\vec{m}^*)|_{D_J} = w_r((m_i^*)_{i \in J}, m_J^*) w_r((m_i^*)_{i \notin J}, m_{J^c}^*), \tag{40}$$

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where m_J, m_{J^c} are as in (8). Note that by Proposition 2.14, (40) vanishes unless

$$(|J|-1)(r+1)-r<\sum_{i\in J} m_i^*<(|J|-1)(r+1).$$

We now apply the following additional vanishing property, which we prove after finishing this proof.

Claim 4.3. Let $n\geqslant 5$, and suppose that $\vec{m}=(m_1,\dots,m_n)$ is a numerical r-spin monodromy vector with $m_i=2$ for some i. Then $w_r(\vec{m})=0$.

Since $m_{n+1}=2$, Claim 4.3 and (40) imply that any term on either side of (39) with $|J^c|>4$, or equivalently $|J|<n-3$, vanishes. The remaining terms on the left-hand side are:

- The term with $J=[n+1]\setminus\{2,n+1\}$, i.e.

  $$w_r(m_1-1,m_3,\dots,m_n,m_2+1)w_r(m_2,2,r-1-m_2)=w_r(m_1-1,m_2+1,m_3,\dots,m_n),$$

  and
- The terms with $J=[n+1]\setminus\{2,k,n+1\}$ for $k\in\{4,\dots,n\}$, namely

  $$\sum_{k=4}^{n} w_r(m_1-1,m_3,m_4,\dots,\hat{m}_k,\dots,m_n,m_2+m_k-r)w_r(m_2,m_k,2,2r-m_2-m_k)$$
  $$=\frac{1}{r}\sum_{k=4}^{n}\delta_{m_2+m_k\geqslant r+1}\cdot w_r(m_1-1,m_3,\dots,\hat{m}_k,\dots,m_n,m_2+m_k-r).$$

  Identical calculations show that the right-hand side of (39) is

$$w_r(\vec{m})+\frac{1}{r}\sum_{k=4}^{n}\delta_{m_1+m_k\geqslant r+2}\cdot w_r(m_2,m_3,\dots,\hat{m}_k,\dots,m_n,m_1+m_k-r-1).$$

Combining all terms above yields

$$w_r(\vec{m})-w_r(m_1-1,m_2+1,m_3,\dots,m_n)=\sum_{k=4}^{n}T^{1,2,k}(\vec{m}). \tag{41}$$

Finally, we observe that the left-hand side of (41) is invariant under permuting $m_3,\dots,m_n$, hence the right-hand side is also. It follows that the $n-3$ summands on the right-hand side of (41) are all equal, yielding (37). □

Proof of Claim 4.3. If $m_i=r$ for some $i\in[n]$, then $w_r(\vec{m})=0$, so assume $m_i\leqslant r-1$ for all $i\in[n]$. Suppose $m_1=2$. Define the monodromy vector

$$\vec{m}^{**}=(m_1^{**},\dots,m_{n+1}^{**})=(2,m_2-1,m_3,\dots,m_n,2).$$

Analogously to (39), the balancing condition $B^3_{\{1,2\}}(W_r(\vec{m}^{**}))=B^3_{\{1,4\}}(W_r(\vec{m}^{**}))$ for the 1-dimensional tropical cycle $W_r(\vec{m}^{**})$ implies the WDVV equation

$$\sum_{\substack{J\subseteq\{1,\dots,n+1\}\\ 1,2\in J,\ 3,4\notin J}}\int_{\overline{M}_{0,n}} W_r(\vec{m}^{**})|_{D_J}=\sum_{\substack{J\subseteq\{1,\dots,n+1\}\\ 1,4\in J,\ 2,3\notin J}}\int_{\overline{M}_{0,n}} W_r(\vec{m}^{**})|_{D_J}. \tag{42}$$

First note that using the constraints $m_i\leqslant r-1$, $m_1=2$, $n\geqslant 5$, and $\sum_{i=1}^n m_i=(n-2)(r+1)$, one can deduce that

$$m_i+m_j\geqslant r+3 \text{ for all distinct } i,j\in\{2,\dots,n\}. \tag{A}$$

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Second, using $m_1^{**} = m_{n+1}^{**} = 2$, $m_i^{**} \leqslant r-1$ and $\sum_{i=1}^{n+1} m_i = (n-1)(r+1) - r$, we have that if $1, n+1 \in J$, then

$$\sum_{i \in J^c} m_i \geqslant (n-1)(r+1) - r - 4 - (|J|-2)(r-1) = (|J^c|-1)(r+1) + (2|J|-7).$$

Proposition 2.14 implies $W_r(\vec{m}^{**})|_{D_J} = 0$ if $|J| \geqslant 4$. If $|J| = 3$, then $W_r(\vec{m}^{**})|_{D_J}$ could be nonzero only if $m_{J^c} = 1$. But in this case, $n \geqslant 5$ implies $|J^c| \geqslant 3$, so Propositions 2.8 and 2.12 imply $W_r(\vec{m}^{**})|_{D_J} = 0$. To summarize:

$$\text{If } |J| \geqslant 3 \text{ and } \{1, n+1\} \subseteq J, \text{ then } \int_{\overline{M}_{0,n}} W_r(\vec{m}^{**})|_{D_J} = 0. \tag{B}$$

We now show that $w_r(\vec{m}) = 0$ by induction on n, with base case $n = 5$. If $n = 5$, then a straightforward analysis using Proposition 2.8, Proposition 2.13, and Observations (A) and (B) imply the following about the terms in the left-hand side of (42):

- When $J = \{1, 2\}$, we have $\int_{\overline{M}_{0,n}} W_r(\vec{m}^{**})|_{D_J} = w_r(2, m_2 - 1, r - m_2) \cdot w_r(\vec{m}) = w_r(\vec{m})$.
- When $J = \{1, 2, 5\}$, we use (A) to see that $\int_{\overline{M}_{0,n}} W_r(\vec{m}^{**})|_{D_J} = (1/r^2) \cdot \delta_{m_2+m_5 \geqslant r+3} = 1/r^2$.
- When $J = \{1, 2, 6\}$, we use (B) to see that $\int_{\overline{M}_{0,n}} W_r(\vec{m}^{**})|_{D_J} = 0$.
- When $J = \{1, 2, 5, 6\}$, we use (B) to see that $\int_{\overline{M}_{0,n}} W_r(\vec{m}^{**})|_{D_J} = 0$.

By the same arguments, the right side of (42) has the following terms:

- When $J = \{1, 4\}$, we have $\int_{\overline{M}_{0,n}} W_r(\vec{m}^{**})|_{D_J} = w_r(2, m_2 - 1, m_3, m_4 + 1, m_5)$.
- When $J = \{1, 4, 5\}$, we use (A) to see that $\int_{\overline{M}_{0,n}} W_r(\vec{m}^{**})|_{D_J} = (1/r^2) \cdot \delta_{m_2+m_3 \geqslant r+2} = 1/r^2$.
- When $J = \{1, 4, 6\}$, we use (B) to see that $\int_{\overline{M}_{0,n}} W_r(\vec{m}^{**})|_{D_J} = 0$.
- When $J = \{1, 4, 5, 6\}$, we use (B) to see that $\int_{\overline{M}_{0,n}} W_r(\vec{m}^{**})|_{D_J} = 0$.

Thus (42) reduces to

$$w_r(\vec{m}) = w_r(2, m_2 - 1, m_3, m_4 + 1, m_5).$$

Iterating, we have

$$w_r(\vec{m}) = w_r(m_1, m_2 - j, m_3, m_4 + j, m_5),$$

for all $0 \leqslant j \leqslant \min(m_2 - 1, r - m_4)$. In fact (1) implies $\min(m_2 - 1, r - m_4) = r - m_4$. Taking $j = r - m_4$, we conclude $w_r(\vec{m}) = 0$ from Proposition 2.10. This completes the base case.

Now assume $n > 5$ and that the claim holds for all $5 \leqslant n' < n$. The inductive hypothesis implies that every term on both sides of (42) with $4 \leqslant |J| \leqslant n - 2$ vanishes. We now account for the remaining terms on the left side of (42):

- When $J = \{1, 2\}$, we have $\int_{\overline{M}_{0,n}} W_r(\vec{m}^{**})|_{D_J} = w_r(\vec{m})$.
- When $J = \{1, 2, i\}$ for $i \in \{5, \ldots, n\}$, the inductive hypothesis shows $\int_{\overline{M}_{0,n}} W_r(\vec{m}^{**})|_{D_J} = 0$.
- When $J = \{1, 2, n+1\}$, we use (B) to see that $\int_{\overline{M}_{0,n}} W_r(\vec{m}^{**})|_{D_J} = 0$.
- When $J = [n] \setminus \{3, 4\}$, we use (B) to see that $\int_{\overline{M}_{0,n}} W_r(\vec{m}^{**})|_{D_J} = 0$.

We then account for the right side:

- When $J = \{1, 4\}$, we have that $\int_{\overline{M}_{0,n}} W_r(\vec{m}^{**})|_{D_J} = w_r(2, m_2 - 1, m_3, m_4 + 1, m_5, \ldots, m_n)$.

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- When $J = \{1, 4, i\}$ for $i \in \{5, \dots, n\}$, the inductive hypothesis shows $\int_{\overline{M}_{0,n}} W_r(\vec{m}^{**})|_{D_J} = 0$.
- When $J = \{1, 4, n+1\}$, we use (B) to see that $\int_{\overline{M}_{0,n}} W_r(\vec{m}^{**})|_{D_J} = 0$.
- When $J = [n+1] \setminus \{2, 3\}$, we use (B) to see that $\int_{\overline{M}_{0,n}} W_r(\vec{m}^{**})|_{D_J} = 0$.

Thus (42) reads

$$w_r(\vec{m}) = w_r(2, m_2 - 1, m_3, m_4 + 1, m_5, \dots, m_n).$$

As in the base case, we iterate to see $w_r(\vec{m}) = w_r(2, m_2 - j, m_3, m_4 + j, m_5, \dots, m_n)$ for $0 \leqslant j \leqslant \min(m_2 - 1, r - m_4)$. Again, $\min(m_2 - 1, r - m_4) = r - m_4$ by (A). Taking $j = r - m_4$, we obtain $w_r(\vec{m}) = 0$. □

We next use Theorem 4.1 to give direct proofs of two statements proved by Pandharipande-Pixton-Zvonkine [PPZ19] using the representation theory of $\mathfrak{sl}_2(\mathbb{C})$. The first, which follows from [PPZ19, Thm. 2], is the following divisibility statement.

COROLLARY 4.4. *For any numerical monodromy vector $\vec{m}$ of length n, $w_r(\vec{m})$ is an integer multiple of $\frac{(n-3)!}{r^{n-3}}$.*

Proof. We induct on n. The base case $n = 3$ follows from 2.13. If $n > 3$, (37) expresses $w_r(\vec{m})$ as $\frac{n-3}{r} \cdot F$, where F is a $\mathbb{Z}$-linear combination in invariants $w_r(\vec{m}')$ with $|m'| = n - 1$. By the inductive hypothesis, F is an integer multiple of $\frac{(n-4)!}{r^{n-4}}$, and the result follows. □

In Theorem 6.2, we will see that in fact $w_r(\vec{m})$ is a *nonnegative* integer multiple of $\frac{(n-3)!}{r^{n-3}}$. The second statement is the following generalization of Claim 4.3, which is [PPZ19, Prop. 1.4].

COROLLARY 4.5. *Let $n \geqslant 4$, and suppose $\vec{m} = (m_1, \dots, m_n)$ is an r-spin monodromy vector with $\sum_{i=1}^n m_i = (n-2)(r+1)$. If $m_i \leqslant n - 3$ for some i, then $w_r(\vec{m}) = 0$.*

Proof. We may assume $m_1 \leqslant n - 3$. We proceed by induction on n and m_1, with base cases $m_1 = 1$ (for all n) following from Proposition 2.13.

If $n = 4$ the base case is the only case, so let $n > 4$. Assume that the Corollary holds for $n' < n$ and for $n' = n$, $m_1' < m_1$. Since $\sum_i m_i = (n-2)(r+1)$, we may also assume without loss of generality that $m_2 < r$. By Theorem 4.1,

$$w_r(\vec{m}) - w_r(m_1 - 1, m_2 + 1, m_3, \dots, m_n) = (n-3)T_r^{1,2,3}(m_1, \dots, m_n).$$

Both terms of $T_r^{1,2,3}(m_1, \dots, m_n)$ contain an $(n-1)$-pointed r-spin invariant. By induction on n, both of these invariants are zero: the first term has an insertion $m_1 - 1 \leqslant (n-1) - 3$, and the second term has an insertion $m_1 + m_3 - r - 1 \leqslant m_1 - 1 \leqslant (n-1) - 3$. Thus

$$w_r(\vec{m}) = w_r(m_1 - 1, m_2 + 1, m_3, \dots, m_n).$$

By induction on m_1, $w_r(\vec{m}) = 0$. □

Our next goal is to prove Lemma 4.9, a reconstruction result for $w_r(\vec{m})$, which we will use in Section 5 to give a recursive proof of Theorem 5.1.

DEFINITION 4.6. *Let $\vec{m} = (m_i)$ and $\vec{m}' = (m_i')$ be two monodromy vectors of length n. We say that $\vec{m}$ and $\vec{m}'$ are neighbors if there are $i, j \in \{1, \dots, n\}$ so that*

$$\vec{m} - \vec{m}' = (d_1, \dots, d_n), \text{ where } d_k = \delta_{ik} - \delta_{jk},$$

or, equivalently, $\vec{m} - \vec{m}'$ is some permutation of the vector $(-1, 1, 0, \dots, 0)$.

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Example 4.7. The two r-spin monodromy vectors $(2,2,r-1,r-1)$ and $(2,3,r-2,r-1)$ are neighbors.

We need the following fact.

LEMMA 4.8. *Given two r-spin monodromy vectors $\vec{m}$ and $\vec{m'}$ of length n, there exists a sequence of monodromy vectors $\vec{m}_0=\vec{m},\vec{m}_1,\dots,\vec{m}_{n-1},\vec{m}_n=\vec{m'}$ such that $\vec{m}_i$ and $\vec{m}_{i+1}$ are neighbors for all i.*

Proof. This follows from a standard inductive argument. □

LEMMA 4.9. *Let D be the set of numerical monodromy vectors for a given r, and let $\widetilde{w}_r : D \to \mathbb{C}$ be a function such that:*

(i) $\widetilde{w}_r(\vec{m})=1$ *when* $|\vec{m}|=3$,

(ii) $\widetilde{w}_r(\vec{m})=0$ *if* $|\vec{m}|\geqslant 4$ *and* $m_i=r$ *for some* $i\in\{1,\dots,|\vec{m}|\}$,

(iii) $\widetilde{w}_r$ *satisfies the recursion (37).*

Then $\widetilde{w}_r = w_r$.

Remark 4.10. Lemma 4.9 is a slight variation of previous reconstruction lemmas in the literature. In [JKV01, Prop. 6.2], property (iii) is replaced by the WDVV relations and a nonvanishing 4-point invariant. In [PPZ19, Lem. 1.3], property (ii) is replaced with vanishing of $\widetilde{w}_r(\vec{m})$ if $m_i=1$ for some i. Lemma 4.9 is in a form particularly suited to proving Theorem 5.1.

Proof of Lemma 4.9. We prove that $\widetilde{w}_r = w_r$ by induction on the length n of the monodromy vector. The base case $n=3$ holds by Proposition 2.13. Let $k>3$ and assume $w_r=\widetilde{w}_r$ holds for all monodromy vectors of length less than k. By (ii) and Proposition 2.10, $\widetilde{w}_r(\vec{m})=w_r(\vec{m})$ if $\vec{m}$ is a length-k monodromy vector with $m_i=r$ for some i. (It is easy to see that there must exist such a monodromy vector.)

By Theorem 4.1, if $\vec{m}$ and $\vec{m}'$ are neighbors such that $w_r(\vec{m})=\widetilde{w}_r(\vec{m})$, and such that $w_r(\vec{a})=\widetilde{w}_r(\vec{a})$ for all monodromy vectors $\vec{a}$ of length less than $|\vec{m}|$, then $w_r(\vec{m}')=\widetilde{w}_r(\vec{m}')$. Combining this with Lemma 4.8 implies that $w_r(\vec{m})=\widetilde{w}_r(\vec{m})$ for all monodromy vectors $\vec{m}$ of length k, proving the claim. □

5. Explicit piecewise-polynomial formula

In this section, we prove a closed formula for genus-zero primary r-spin invariants.

THEOREM 5.1. *For $\vec{m}=(m_1,\dots,m_n)$ a numerical genus-zero r-spin monodromy vector, we have the formula*

$$w_r(\vec{m})=\frac{1}{2r^{n-3}}\sum_{\substack{S\subseteq[n]\\ \sum_{i\in S}m_i\geqslant(|S|-1)r+n-2}}(-1)^{1+|S|}\prod_{k=1}^{n-3}\Big(\Big(\sum_{i\in S}m_i\Big)-(|S|-1)r-k\Big). \tag{43}$$

This immediately implies:

COROLLARY 5.2. *$w_r(m_1,\dots,m_n)$ is a piecewise-polynomial function of the inputs $m_1,\dots,m_n,r$. Precisely, the set*

$$\mathbf{K}_n := \{(m_1,\dots,m_n,r)\mid 1\leqslant m_i\leqslant r-1, \textstyle\sum_i m_i=(n-2)(r+1)\}\subseteq\mathbb{Z}_{>0}^n\times\mathbb{Z}_{>0},$$

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of numerical n-pointed monodromy vectors is the set of lattice points of an unbounded polyhedron $\mathbf{M}_n$, and there is a finite wall-chamber decomposition of $\mathbf{M}_n$ such that in each chamber, $w_r(m_1,\dots,m_n)$ is a polynomial.

Proof. This is immediate from the form of (43); the walls are the affine-linear spaces

$$L_S := \{(m_1,\dots,m_n,r) \mid \textstyle\sum_{i\in S} m_i = (|S|-1)r+n-3\},$$

for subsets $S\subseteq[n]$. □

Remark 5.3. In fact, a stronger version of piecewise-polynomiality holds. The walls of the above wall-chamber decomposition have "width $n-3$" in the sense that if E and E' are adjacent chambers separated by a single wall L_S and if $P, P'\in\mathbb{Q}[m_1,\dots,m_n,r]$ are such that $w_r(\vec{m})=P$ after restriction to E and $w_r(\vec{m})=P'$ after restriction to E', then P and P' agree not only on the wall L_S but also on the set

$$\{(m_1,\dots,m_n,r)\in\mathbf{K}_n \mid (|S|-1)r+1\leqslant\textstyle\sum_{i\in S} m_i\leqslant(|S|-1)r+n-3\}.$$

For the proof of Theorem 5.1, we introduce two pieces of notation:

Notation 5.4. For $\vec{m}=(m_1,\dots,m_n)$, a numerical r-spin monodromy vector, let $\widetilde{w}_r(\vec{m})$ denote the right-hand side of (43):

$$\widetilde{w}_r(\vec{m}) := \frac{1}{2r^{n-3}} \sum_{\substack{S\subseteq[n]\\ \sum_{i\in S} m_i\geqslant(|S|-1)r+n-2}} (-1)^{1+|S|}\prod_{k=1}^{n-3}\Big(\Big(\sum_{i\in S} m_i\Big)-(|S|-1)r-k\Big). \tag{44}$$

Therefore, our goal for the remainder of this section is to prove $\widetilde{w}_r(\vec{m})=w_r(\vec{m})$ for all r and $\vec{m}$.

Notation 5.5. Let $\vec{m}$ be a fixed n-pointed genus-zero r-spin monodromy vector. For a subset $S\subseteq[n]$, we say that S satisfies (∗1) (resp. (∗2)) if

$$\sum_{i\in S} m_i \geqslant (|S|-1)r+n-2, \tag{∗1}$$

$$\sum_{i\in S} m_i \geqslant (|S|-1)r+n-3. \tag{∗2}$$

We will say that, e.g. S, satisfies (∗1) with respect to $\vec{m}$ if $\vec{m}$ is unclear from context. From (44), it is clear why Condition (∗1) is useful: the usefulness of (∗2) will become clear in the proof of Theorem 5.1.

Observe that if $|S|\leqslant 1$, then S necessarily satisfies (∗1), while if $|S|\geqslant n-1$, then S cannot satisfy (∗1).

We now state two lemmas that we will use in the proof, starting with the following well-known combinatorial fact:

LEMMA 5.6. *Let $T\subseteq A$ be finite sets, and let $F(x)\in\mathbb{C}[x]$ with $\deg(F(x))<|A|-|T|$. Then*

$$\sum_{T\subseteq S\subseteq A} (-1)^{|S|}F(|S|)=0. \tag{45}$$

Proof. We reduce to the case $T=\emptyset$ by summing over $S\subseteq A\setminus T$ and performing the substitution $|S|\mapsto|S|+|T|$ (which does not affect the degree of F). We then reduce to the case $F(x)=x^j$

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since the left-hand side of (45) is linear in F. Finally, the left-hand side of (45) is equal to $\sum_{k=0}^{|A|}(-1)^k\binom{|A|}{k}k^j$, which (up to sign) counts surjections $[j]\to A$ by a famous inclusion-exclusion argument. There are clearly no such surjections. □

In proving Theorem 5.1, we will apply Lemma 5.6 in the following form:

Lemma 5.7. *Let n be a positive integer, let $m_1,\dots,m_{n-2}\in\mathbb{C}$ and let $\ell_k(x)\in\mathbb{C}[x]$ be linear functions for $k=1,\dots,n-4$. Then*

$$\sum_{S\subseteq\{1,2,\dots,n-2\}}(-1)^{1+|S|}\prod_{k=1}^{n-4}\Big(\Big(\sum_{i\in S}m_i\Big)-\ell_k(|S|)\Big)=0. \tag{46}$$

Proof. Consider the left-hand side of (46) as a degree-$(n-4)$ polynomial in *formal variables* $m_1,m_2,\dots,m_{n-2}$, and denote this polynomial by $G(m_1,m_2,\dots,m_{n-2})$. Fix integers $a_1,\dots,a_{n-2}\geqslant 0$ with $\sum_{i=1}^{n-2}a_i\leqslant n-4$, and let $M=\prod_{i=1}^{n-2}(m_i)^{a_i}$. The coefficient of the monomial M in $G(m_1,\dots,m_{n-2})$ is equal to

$$\sum_{\substack{S\subseteq\{1,\dots,n-2\}\\ S\supseteq\{i:a_i>0\}}}(-1)^{|S|}F_M(|S|),$$

where

$$F_M(x)=\sum_{\substack{B\subseteq\{1,\dots,n-4\}\\ |B|=\sum_i a_i}}\binom{|B|}{a_1,\dots,a_{n-2}}\prod_{k\in\{1,\dots,n-4\}\setminus B}\ell_k(x).$$

Then $F_M(x)$ has degree

$$\deg(F_M(x))=n-4-\sum_{i=1}^{n}a_i<n-2-|\{i:a_i>0\}|,$$

By Lemma 5.6, the coefficient of M in $G(m_1,\dots,m_{n-2})$ is zero. Since M is arbitrary, G vanishes identically as desired. □

Proof of Theorem 5.1 It suffices to show that $\widetilde{w}_r(\vec{m})$ satisfies conditions (i), (ii) and (iii) of Lemma 4.9.

If $n=3$, then for any $\vec{m}$, the only subsets S satisfying (∗1) are those with $|S|\leqslant 1$. Then $\widetilde{w}_r(\vec{m})=\frac{3}{2}-\frac{1}{2}=1$ as desired, proving condition (i).

Next, suppose $\vec{m}$ is a numerical r-spin monodromy vector of length $n\geqslant 4$, with $m_i=r$ for some $i\in[n]$. Suppose without loss of generality that $i=n$. Let $S\subseteq[n-1]$. Observe that S satisfies (∗1) if and only if $S\cup\{n\}$ satisfies (∗1) and the summands corresponding to S and $S\cup\{n\}$ of $\widetilde{w}_r(\vec{m})$ are identical with opposite signs since $|S\cup\{n\}|=|S|+1$. This shows that $\widetilde{w}_r(\vec{m})=0$, proving (ii).

Next, suppose $\vec{m}$ is a numerical r-spin monodromy vector of length $n\geqslant 4$ with $m_i=r$ for some $i\in[n]$. Suppose without loss of generality that $i=n$. Let $S\subseteq[n-1]$. Observe that S satisfies (∗1) if and only if $S\cup\{n\}$ satisfies (∗1), and the summands corresponding to S and $S\cup\{n\}$ of $\widetilde{w}_r(\vec{m})$ are identical with opposite signs since $|S\cup\{n\}|=|S|+1$. This shows $\widetilde{w}_r(\vec{m})=0$, proving (ii).

Finally, we must prove that $\widetilde{w}_r$ satisfies (37). We do so by induction on n, with base case $n=3$. When $n=3$, the calculation $\widetilde{w}_r(\vec{m})=1$ above implies that both sides of (37) vanish. For $n\geqslant 4$, let

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$$\vec{m}^* = (m_1^*, \dots, m_n^*) = (m_1 - 1, m_2 + 1, m_3, \dots, m_n). \tag{47}$$

Then we compute

$$\widetilde{w}_r(\vec{m}) - \widetilde{w}_r(\vec{m}^*) = \frac{n-3}{2r^{n-3}} \left(\sum_{\substack{S \subseteq [n] \\ (*1) \\ 1 \in S,\ 2 \notin S}} (-1)^{1+|S|} \prod_{k=2}^{n-3} \left(\left(\sum_{i \in S} m_i \right) - (|S|-1)r - k \right) \right. \tag{48}$$
$$\left. - \sum_{\substack{S \subseteq [n] \\ (*2) \\ 2 \in S,\ 1 \notin S}} (-1)^{1+|S|} \prod_{k=1}^{n-4} \left(\left(\sum_{i \in S} m_i \right) - (|S|-1)r - k \right) \right).$$

The factor $n-3$ appears as the difference of a $k=1$ factor and a $k=n-3$ factor, from the definition of $\widetilde{w}_r$. By induction, we have

$$T_r^{1,2,3}(\vec{m}) = \frac{1}{r} \Big(\delta_{m_2+m_3 \geqslant r+1} \widetilde{w}_r(m_1 - 1, m_4, \dots, m_n, m_2 + m_3 - r) \tag{49}$$
$$- \delta_{m_1+m_3 \geqslant r+2} \widetilde{w}_r(m_2, m_4, \dots, m_n, m_1 + m_3 - r - 1) \Big).$$

Let $\vec{m}^{**} = (m_1 - 1, m_4, \dots, m_n, m_2 + m_3 - r)$, where we index this tuple by the set $\{1, 4, \dots, n, \dagger\}$ and let $\vec{m}^{***} = (m_2, m_4, \dots, m_n, m_1 + m_3 - r - 1)$, where we index this tuple by the set $\{2, 4, \dots, n, \bullet\}$. Plugging in (44) to (49), we have four cases, according to whether $m_2 + m_3 \geqslant r+1$ and whether $m_1 + m_3 \geqslant r+2$.

Case 1: $m_2 + m_3 \geqslant r+1$ **and** $m_1 + m_3 \geqslant r+2$**.** By (49) and the definition of $\widetilde{w}_r$, we have

$$T_r^{1,2,3}(\vec{m}) = \frac{1}{2r^{n-3}} \left(\sum_{S' \subseteq \{1,4,\dots,n,\dagger\} (*2) \text{ w.r.t. } \vec{m}^{**}} (-1)^{1+|S'|} \prod_{k=1}^{n-4} \Big(\Big(\sum_{i \in S'} m_i^{**} \Big) - (|S'|-1)r - k \Big) \right.$$
$$\left. - \sum_{S'' \subseteq \{2,4,\dots,n,\bullet\} (*2) \text{ w.r.t. } \vec{m}^{***}} (-1)^{1+|S''|} \prod_{k=1}^{n-4} \Big(\Big(\sum_{i \in S''} m_i^{***} \Big) - (|S''|-1)r - k \Big) \right). \tag{50}$$

To clarify, e.g., in the first summation, we sum over all S' that satisfy $(*2)$ with respect to $\vec{m}^{**}$, as defined in Notation 5.5.

A subset $S' \subseteq \{4, \dots, n\}$ (satisfying $(*2)$ with respect to $\vec{m}^{**}$, and equivalently with respect to $\vec{m}^{***}$) contributes to both sums in (50) with opposite signs. The contribution from a subset $S' \supseteq \{1, \dagger\}$ (satisfying (2) with respect to $\vec{m}^{**}$) cancels with an identical contribution from the corresponding subset $S'' = (S' \cup \{2, \bullet\}) \setminus \{1, \dagger\}$ (which necessarily satisfies $(*2)$ with respect to $\vec{m}^{***}$).

We now translate the sums over S' and S'' in (50) into sums over S. If a subset $S' \subseteq \{1, 4, \dots, n, \dagger\}$ contains $\dagger$, then S' satisfies $(*2)$ with respect to $\vec{m}^{**}$ if and only if $S := (S' \setminus \{\dagger\}) \cup \{2, 3\}$ satisfies $(*2)$ with respect to $\vec{m}$. If instead S' contains 1, then S' satisfies $(*2)$ with respect to $\vec{m}^{**}$ if and only if $S := S'$ satisfies $(*1)$ with respect to $\vec{m}$. Using these observations (and similar calculations for S''), we rewrite the sums in (50) as sums over subsets of $[n]$:

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$$T_r^{1,2,3}(\vec{m}) = \frac{1}{2r^{n-3}} \Bigg(\sum_{\substack{S\subseteq[n] \\ (*1) \\ 1\in S,\ 2,3\notin S}} (-1)^{1+|S|} \prod_{k=2}^{n-3} \Big(\Big(\sum_{i\in S} m_i \Big) - (|S|-1)r - k \Big)$$
$$+ \sum_{\substack{S\subseteq[n] \\ (*2) \\ 2,3\in S,\ 1\notin S}} (-1)^{|S|} \prod_{k=1}^{n-4} \Big(\Big(\sum_{i\in S} m_i \Big) - (|S|-1)r - k \Big)$$
$$- \sum_{\substack{S\subseteq[n] \\ (*2) \\ 2\in S,\ 1,3\notin S}} (-1)^{1+|S|} \prod_{k=1}^{n-4} \Big(\Big(\sum_{i\in S} m_i \Big) - (|S|-1)r - k \Big)$$
$$- \sum_{\substack{S\subseteq[n] \\ 1,3\in S,\ 2\notin S}} (-1)^{|S|} \prod_{k=2}^{n-3} \Big(\Big(\sum_{i\in S} m_i \Big) - (|S|-1)r - k \Big) \Bigg). \tag{51}$$

Combining the first and last sums in (51), as well as the second and third sums, and comparing with (48), shows that (37) is satisfied. This completes Case 1.

Case 2: $m_2 + m_3 \leqslant r$ and $m_1 + m_3 \geqslant r + 2$. In this case, if a subset $S \subseteq [n]$ satisfies $2, 3 \notin S$, then by an easy calculation, S must satisfy $(*1)$ with respect to $\vec{m}$. Similarly, if $2, 3 \in S$, then S does not satisfy $(*1)$. Thus (48) reads:

$$\widetilde{w}_r(\vec{m}) - \widetilde{w}_r(\vec{m}^*) = \frac{n-3}{2r^{n-3}} \Bigg(\sum_{\substack{S\subseteq[n] \\ (*1) \\ 1,3\in S,\ 2\notin S}} (-1)^{1+|S|} \prod_{k=2}^{n-3} \Big(\Big(\sum_{i\in S} m_i \Big) - (|S|-1)r - k \Big)$$
$$+ \sum_{\substack{S\subseteq[n] \\ 1\in S,\ 2,3\notin S}} (-1)^{1+|S|} \prod_{k=2}^{n-3} \Big(\Big(\sum_{i\in S} m_i \Big) - (|S|-1)r - k \Big)$$
$$- \sum_{\substack{S\subseteq[n] \\ (*2) \\ 2\in S,\ 1,3\notin S}} (-1)^{1+|S|} \prod_{k=1}^{n-4} \Big(\Big(\sum_{i\in S} m_i \Big) - (|S|-1)r - k \Big) \Bigg). \tag{52}$$

On the other hand, (49) reads as

$$T_r^{1,2,3}(\vec{m}) = -\frac{1}{r}\widetilde{w}_r(m_2, m_4, \ldots, m_n, m_1 + m_3 - r - 1)$$
$$= \frac{1}{2r^{n-3}} \sum_{\substack{S''\subseteq\{2,4,\ldots,n,\bullet\} \\ (*1) \text{ w.r.t. } \vec{m}^{***}}} (-1)^{1+|S''|} \prod_{k=1}^{n-4} \Big(\Big(\sum_{i\in S''} m_i^{***} \Big) - (|S''|-1)r - k \Big). \tag{53}$$

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If a subset S'' contains both 2 and $\bullet$, then by the assumption $m_2+m_3\leqslant r$, S'' does not contribute to the sum. Using this and rewriting all sums in terms of subsets of $[n]$ as above, we obtain:

$$T_r^{1,2,3}(\vec{m})=\frac{-1}{2r^{n-3}}\Bigg(\sum_{\substack{S\subseteq[n]\\(*2)\\2\in S,\ 1,3\notin S}}(-1)^{1+|S|}\prod_{k=1}^{n-4}\Big(\Big(\sum_{i\in S}m_i\Big)-(|S|-1)r-k\Big)$$
$$+\sum_{\substack{S\subseteq[n]\\(*1)\\1,3\in S,\ 2\notin S}}(-1)^{|S|}\prod_{k=2}^{n-3}\Big(\Big(\sum_{i\in S}m_i\Big)-(|S|-1)r-k\Big)$$
$$+\sum_{\substack{S\subseteq[n]\\1,2,3\notin S}}(-1)^{1+|S|}\prod_{k=1}^{n-4}\Big(\Big(\sum_{i\in S}m_i\Big)-(|S|-1)r-k\Big)\Bigg). \tag{54}$$

Note that the first two sums in (54) match the first and third sums in (52). To prove that (37) holds, it remains to prove

$$\sum_{S\subseteq\{1,4,\ldots,n\}}(-1)^{1+|S|}\prod_{k=2}^{n-3}\Big(\Big(\sum_{i\in S}m_i^*\Big)-(|S|-1)r-k\Big)=0, \tag{55}$$

where m_i^* is as in (47). Here we have written m_i^* because the second sum in (52), and the third sum in (54) can be combined cleanly only after the substitution $m_1\mapsto m_1-1$. One sees that (55) is a special case of Lemma 5.7. This completes Case 2.

Case 3: $m_2+m_3\geqslant r+1$ **and** $m_1+m_3\leqslant r+1$**.** This case is almost identical to Case 2. Similar computations to those in Case 2 give

$$\widetilde{w}_r(\vec{m})-\widetilde{w}_r(\vec{m}^*)=\frac{n-3}{2r^{n-3}}\Bigg(\sum_{\substack{S\subseteq[n]\\(*1)\\1\in S,\ 2,3\notin S}}(-1)^{1+|S|}\prod_{k=2}^{n-3}\Big(\Big(\sum_{i\in S}m_i\Big)-(|S|-1)r-k\Big)$$
$$-\sum_{\substack{S\subseteq[n]\\(*2)\\2,3\in S,\ 1\notin S}}(-1)^{1+|S|}\prod_{k=1}^{n-4}\Big(\Big(\sum_{i\in S}m_i\Big)-(|S|-1)r-k\Big)$$
$$-\sum_{\substack{S\subseteq[n]\\2\in S,\ 1,3\notin S}}(-1)^{1+|S|}\prod_{k=1}^{n-4}\Big(\Big(\sum_{i\in S}m_i\Big)-(|S|-1)r-k\Big)\Bigg). \tag{56}$$

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and

$$T_r^{1,2,3}(\vec{m}) = \frac{1}{2r^{n-3}} \Bigg(\sum_{\substack{S\subseteq[n] \\ (*1) \\ 1\in S,\ 2,3\notin S}} (-1)^{1+|S|} \prod_{k=2}^{n-3} \Big(\Big(\sum_{i\in S} m_i \Big) - (|S|-1)r - k \Big)$$
$$+ \sum_{\substack{S\subseteq[n] \\ (*2) \\ 2,3\in S,\ 1\notin S}} (-1)^{|S|} \prod_{k=1}^{n-4} \Big(\Big(\sum_{i\in S} m_i \Big) - (|S|-1)r - k \Big)$$
$$+ \sum_{\substack{S\subseteq[n] \\ 1,2,3\notin S}} (-1)^{1+|S|} \prod_{k=1}^{n-4} \Big(\Big(\sum_{i\in S} m_i \Big) - (|S|-1)r - k \Big) \Bigg). \tag{57}$$

From (56) and (57), we see that (37) is equivalent to

$$\sum_{S\subseteq\{2,4,\ldots,n\}} (-1)^{1+|S|} \prod_{k=1}^{n-4} \Big(\Big(\sum_{i\in S} m_i \Big) - (|S|-1)r - k \Big) = 0,$$

which is a special case of Lemma 5.7. This completes Case 3.

Case 4: $m_2 + m_3 \leqslant r$ **and** $m_1 + m_3 \leqslant r+1$**.** In this case, if $S \subseteq [n]$ satisfies $1, 3 \notin S$ or $2, 3 \notin S$, then S must satisfy $(*1)$. Similarly, if $2, 3 \in S$ or $1, 3 \in S$, then S does not satisfy $(*1)$. Thus (48) simplifies to:

$$\widetilde{w}_r(\vec{m}) - \widetilde{w}_r(\vec{m}^*) = \frac{n-3}{2r^{n-3}} \Bigg(\sum_{S\subseteq\{1,4,\ldots,n\}} (-1)^{1+|S|} \prod_{k=2}^{n-3} \Big(\Big(\sum_{i\in S} m_i \Big) - (|S|-1)r - k \Big)$$
$$- \sum_{S\subseteq\{2,4,\ldots,n\}} (-1)^{1+|S|} \prod_{k=1}^{n-4} \Big(\Big(\sum_{i\in S} m_i \Big) - (|S|-1)r - k \Big) \Bigg), \tag{58}$$

while (49) is equal to zero. Again, Lemma 5.7 implies that both sums in (58) are equal to zero, so (37) is satisfied, completing Case 4.

We have now checked that $\widetilde{w}_r$ satisfies all of the conditions of Lemma 4.9, so we conclude $\widetilde{w}_r = w_r$. □

6. Monotonicity of genus 0 r-spin invariants

The techniques introduced also imply that genus 0 r-spin invariants satisfy a monotonicity property, which is not obvious from the closed formula in the previous section.

Recall the *dominance* partial ordering on partitions of an integer N, where for partitions $\vec{p} = (p_1, p_2, \ldots)$ and $\vec{p}\,' = (p_1', p_2', \ldots)$ of N with $p_1 \leqslant p_2 \leqslant \cdots$ and $p_1' \leqslant p_2' \leqslant \cdots$, we say that $\vec{p} \leqslant \vec{p}\,'$ if for all $i \geqslant 0$, we have $p_1 + \cdots + p_i \leqslant p_1' + \cdots + p_i'$. That is, $\vec{p} \leqslant \vec{p}\,'$ if $\vec{p}$ is a "more balanced distribution of N' than $\vec{p}\,'$. If $\vec{p} \leqslant \vec{p}\,'$, we say $\vec{p}\,'$ *dominates* $\vec{p}$.

Remark 6.1. The dominance partial ordering is a fundamental structure of partitions (or, more generally, vectors of real numbers with a fixed sum) and thus arises in many contexts inside (and

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outside) mathematics. See [MOA79] for a survey of such appearances. In algebraic geometry, the dominance partial ordering is known for (among other things) its connections to the geometry of Hilbert schemes of points on surfaces; see [Nak16].

THEOREM 6.2. *Genus-zero r-spin invariants satisfy the following properties:*

1. *For any r and n, w_r is a weakly order-reversing function on n-part partitions of $(n-2)(r+1)$ with respect to the dominance order. That is, if $\vec{m}$ and $\vec{m}'$ are two numerical r-spin monodromy vectors and $\vec{m} \leqslant \vec{m}'$, then $w_r(\vec{m}) \geqslant w_r(\vec{m}')$.*
2. *For any numerical r-spin monodromy vector $\vec{m}$, $w_r(\vec{m}) \geqslant 0$, with $w_r(\vec{m}) \neq 0$ if and only if $n-2 \leqslant m_i \leqslant r-1$ for all i.*

We prove the theorem in several steps, beginning with the following standard fact about the dominance order (see [Bry73, Prop. 2.3]).

FACT 6.3. *A partition $\vec{p}$ is dominated by $\vec{p}'$ if and only if there is a finite sequence*

$$\vec{p}' = \vec{p}^{\,0}, \vec{p}^{\,1}, \ldots, \vec{p}^{\,k} = \vec{p},$$

such that $\vec{p}^{\,i}$ is obtained from $\vec{p}^{\,i-1}$ by replacing two parts $a, b \in \vec{p}^{\,i-1}$, where $a < b-1$, with nonnegative integers $c = a+1$ and $d = b-1$, respectively. That is, $\vec{p}^{\,i}$ and $\vec{p}^{\,i+1}$ are neighbours, and $\vec{p}^{\,i+1}$ dominates $\vec{p}^{\,i}$. Intuitively, a "richer' part (b) donates one "cookie' to a "poorer' part (a) but only if doing so would not make b poorer than a.

LEMMA 6.4. *Suppose $\vec{m}$ and $\vec{m}'$ are numerical r-spin monodromy vectors. If $\vec{m}$ and $\vec{m}'$ are neighbours and if $\vec{m} \geqslant \vec{m}'$, then $w_r(\vec{m}) \leqslant w_r(\vec{m}')$.*

Proof. We proceed by induction on n. The base case is $n = 4$, which follows from Proposition 2.36.

Fix n, and assume that both statements hold for k-pointed monodromy vectors if $k < n$. Without loss of generality, write $\vec{m} = (m_1, \ldots, m_n)$ and $\vec{m}' = (m_1 - 1, m_2 + 1, m_3, \ldots, m_n)$, with $1 < m_1 \leqslant m_2 < r$ as $\vec{m} \leqslant \vec{m}'$. For $j \in \{4, \ldots, n\}$, we have, by definition,

$$T_r^{1,2,j}(m_1, \ldots, m_n) = w_r(\vec{m}^{\,1}) w_r(\vec{m}^{\,2}) - w_r(\vec{m}^{\,3}) w_r(\vec{m}^{\,4}),$$

where:

$$\vec{m}^{\,1} = (m_1 - 1, m_3, \ldots, \hat{m_j}, \ldots, m_n, m_2 + m_j - r) \quad \vec{m}^{\,2} = (2, m_2, m_j, 2r - m_2 - m_j)$$
$$\vec{m}^{\,3} = (m_2, m_3, \ldots, \hat{m_j}, \ldots, m_n, m_1 + m_j - r - 1) \quad \vec{m}^{\,4} = (2, m_1 - 1, m_j, 2r + 1 - m_1 - m_j).$$

Note first that

$$(m_1 - 1) + (m_2 + m_j - r) = (m_1 + m_j - r - 1) + (m_2),$$

and that

$$m_2 > \max\{m_1 - 1, m_2 + m_j - r\} \geqslant \min\{m_1 - 1, m_2 + m_j - r\} > m_1 + m_j - r - 1.$$

Thus, there is a finite sequence such as that in Fact 6.3 between $\vec{m}^{\,3}$ and $\vec{m}^{\,1}$. This proves $\vec{m}^{\,1} \leqslant \vec{m}^{\,3}$, so by the inductive hypothesis, we have

$$w_r(\vec{m}^{\,1}) \geqslant w_r(\vec{m}^{\,3}). \tag{59}$$

Next, we prove $w_r(\vec{m}^{\,2}) \geqslant w_r(\vec{m}^{\,4})$. As both are 4-point invariants with an insertion of 2, both are either 0 or $\frac{1}{r}$. We need only to show that if $w_r(\vec{m}^{\,2}) = 0$, then $w_r(\vec{m}^{\,4}) = 0$. Indeed, using Proposition 2.13 and the fact that $m_2 < r$, $w_r(\vec{m}^{\,2}) = 0$ implies that either $2r - m_2 - m_j \geqslant r$ or

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$m_j = 1$, which in turn implies that either $2r + 1 - m_1 - m_j \geqslant r$ or $m_j = 1$. Thus, $w_r(\vec{m}^4) = 0$. In conclusion,

$$w_r(\vec{m}^2) \geqslant w_r(\vec{m}^4). \tag{60}$$

By the inductive hypothesis, the expressions $w_r(\vec{m}^1)$, $w_r(\vec{m}^2)$, $w_r(\vec{m}^3)$ and $w_r(\vec{m}^4)$ are nonnegative, so (59) and (60) imply $T_r^{1,2,j} \geqslant 0$. Thus,

$$w_r(m_1, \ldots, m_n) \geqslant w_r(m_1 - 1, m_2 + 1, m_3, \ldots, m_n), \tag{61}$$

completing the proof. □

Lemma 6.5. *Let $\vec{m} = (m_1, \ldots, m_n)$ be a monodromy vector with $m_1 = n - 2$ and let $n - 2 \leqslant m_i \leqslant r - 1$ for all i. Then $w_r(\vec{m}) = \frac{(n-3)!}{r^{n-3}}$.*

Proof. We proceed by induction on n, with base case $n = 4$ following from Proposition 2.13.

For $n > 4$, we apply Theorem 4.1 to $\vec{m}$, with $(i, j, k) = (1, 2, 3)$. Since $m_1 = n - 2$, by Corollary 4.5, $w_r(m_1 - 1, m_2 + 1, m_3, \ldots, m_n) = 0$, so Theorem 4.1 reads

$$w_r(\vec{m}) = (n - 3)T_r^{1,2,3}(\vec{m}).$$

We thus have

$$\begin{aligned} m_2 + m_3 - r = (n-2)(r+1) - (n-2) - r - \sum_{i=4}^{n} m_i \geqslant (n-2)(r+1) \\ - (n-2) - r - (n-3)(r-1) = n - 3, \end{aligned}$$

and

$$m_1 + m_3 - r - 1 \leqslant (n-2) + (r-1) - r - 1 = n - 4,$$

which imply

$$T_r^{1,2,3}(\vec{m}) = \frac{1}{r}(1 \cdot w_r(m_1 - 1, m_4, \ldots, m_n, m_2 + m_3 - r) - 0) = \frac{1}{r} \cdot \frac{(n-4)!}{r^{n-4}}$$

by induction. Thus $w_r(\vec{m}) = (n - 3)T_r^{1,2,3}(\vec{m}) = \frac{(n-3)!}{r^{n-3}}$. □

Proof of Theorem 6.2 Fix an n-pointed r-spin monodromy vector $\vec{m}'$ with $\vec{m}' \geqslant \vec{m}$. According to Fact 6.3, there is a finite sequence $\vec{m}' = \vec{m}^{\,0} > \vec{m}^{\,1} > \cdots > \vec{m}^{\,k} = \vec{m}$ of neighbors, where $\vec{m}^{\,i}$ is dominated by $\vec{m}^{\,i+1}$. By Lemma 6.4, $w_r(\vec{m}^{\,i}) \leqslant w_r(\vec{m}^{\,i+1})$ for all i, so we conclude $W_r(\vec{m}) \geqslant W_r(\vec{m}')$. This proves statement (i).

Applying Lemma 6.4 repeatedly, we have

$$w_r(\vec{m}) - w_r(m_1 - k, m_2 + k, m_3, \ldots, m_n) \geqslant 0,$$

for any $k \leqslant \min\{m_1 - 1, r - m_2\}$. Taking $k = \min\{m_1 - 1, r - m_2\}$ gives $w_r(\vec{m}) \geqslant 0$. This proves the first half of statement (ii).

For the second half of statement (ii), the "if' direction follows from Proposition 2.10 and Corollary 4.5. For the "only if' direction, suppose $\vec{m} = (m_1, \ldots, m_n)$ is a monodromy vector such that $n - 2 \leqslant m_i \leqslant r - 1$ for all i, with $m_1 \leqslant \cdots \leqslant m_n$. It is easy to check, using Fact 6.3, that $\vec{m}$ is dominated by a monodromy vector $\vec{m}'$ with $m_1 = n - 2$ and $n - 2 \leqslant m_i \leqslant r - 1$ for $i \in \{2, \ldots, n\}$. (Roughly, m_1 repeatedly "donates a cookie' to a part $m_i < r - 1$; such a part is guaranteed to exist by $\sum_i m_i = (n-2)(r+1)$, and $\vec{m}' \geqslant \vec{m}$ follows from the assumption that $m_1 \leqslant m_2 \leqslant \cdots \leqslant m_n$). Thus, by statement (i) proved just above, it is sufficient to prove $w_r(\vec{m}) > 0$

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in the case where $m_1 = n-2$ and $n-2 \leqslant m_i \leqslant r-1$ for all i. This is the immediate conclution from Lemma 6.5. □

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Conflicts of Interest

None.

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