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Analysis of $2+1$ diffusive–dispersive PDE arising in river braiding

SALEH TANVEER[1] and CHARIS TSIKKOU[2]

[1]*Mathematics Department, The Ohio State University*
Columbus, OH 43210, USA
email: tanveer@math.ohio-state.edu
[2]*Mathematics Department, West Virginia University*
Morgantown, WV 26505, USA
email: tsikkou@math.wvu.edu

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We present local existence and uniqueness results for the following $2+1$ diffusive–dispersive equation due to P. Hall arising in modelling of river braiding:

$$u_{yyt} - \gamma u_{xxx} - \alpha u_{yyyy} - \beta u_{yy} + \left(u^2\right)_{xyy} = 0$$

for $(x, y) \in [0, 2\pi] \times [0, \pi]$, $t > 0$, with boundary condition $u_y = 0 = u_{yyy}$ at $y = 0$ and $y = \pi$ and 2π periodicity in x, using a contraction mapping argument in a Bourgain-type space $T_{s,b}$. We also show that the energy $\|u(\cdot,\cdot,t)\|_{L^2}^2$ and cumulative dissipation $\int_0^t \|u_y(\cdot,\cdot,s)\|_{L^2}^2 dt$ are globally controlled in time t.

Key words: Initial/boundary value problem, dispersive–diffusive PDE, river braiding, Bourgain-type space

1 Introduction and the main result

In a relatively recent study [14], a non-linear evolution model was developed to describe the braiding process in rivers with gravel beds and weak transport of sediments. The calculations based on this model show the natural tendency of rivers with width larger than some critical value to form braiding patterns. Despite limiting assumptions, a remarkable feature is that this model predicts the ratio of lengths and widths of islands caused by braiding process to be about three, independent of all the complex details, and this agrees qualitatively with data from many rivers throughout the world. Weakly non-linear simplifications were also derived valid in a neighbourhood of linear instability onset. The weakly non-linear model is believed to be relevant to rivers experiencing flood conditions when flood onset initiates sediment transport. If we denote the sediment depth measured from some reference point to be $u(x, y, t)$ at time t, with downstream and orthogonal directions to be x and y, then the weakly non-linear Hall equations are

$$u_{yyt} - \gamma u_{xxx} - \alpha u_{yyyy} - \beta u_{yy} + [u^2]_{xyy} = 0 \tag{1}$$

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with parameters γ, α, β, where $\alpha > 0$ and $\gamma \neq 0$. In the river context, an appropriate domain would be

$$\mathscr{D} : \{(x, y, t) : y \in (0, \pi), x \in \mathbb{R},\ t > 0\}.$$

The physically appropriate boundary condition on the river walls correspond to

$$u_y = 0 = u_{yyy} \text{ on } y = 0, \pi. \tag{2}$$

With initial conditions

$$u(x, y, 0) = u_0(x, y) \tag{3}$$

(1)–(3) constitutes the initial/boundary value problem of interest. We are not aware of any analysis of the Hall equations (1)–(3). The mathematical well-posedness of models derived under simplifying assumptions is crucial for the validity of the models, since predictions from ill-posed models cannot be physically relevant. This paper proves local well-posedness of the model (1)–(3) and determines a condition for global regularity.

In a mathematical context, this is an example of a non-linear $2+1$ evolution equation that is dispersive in one spatial direction (x), like the KdV or non-linear Schrödinger equation, while being diffusive in the other direction (y). There has been some work done on equations of mixed type in $2+1$ dimensions [5, 11, 17], but they have quite different properties than the one studied here. For instance, in the study [5] of KP-II equations, one encounters dispersion in one direction and diffusion in another as in the present study requiring a partition of Fourier modes. However, in the KP-II problem, one does not encounter the problem of controlling non-linearity with non-uniform dispersion as in the present study. This feature necessitates more delicate estimates than otherwise.

If one were to express (1) in an infinite system of $1+1$ equations using a Fourier cosine series involving $\cos(ny)$ in y, as appropriate for (2), the dispersive regularization in x is not uniform. Therefore, the analysis of this initial value problem requires adaptation of known methods (see for instance [3,4,15,21]) for dispersive PDEs.) Also, it is to be noted that a priori higher order energy bounds are not easily available in this $2+1$ system. There is a fairly large literature in the study of dispersive systems using a Bourgain type of spaces (see for instance [1–21]). Here, we present results for the periodic case of the Hall equation:

$$u(x + 2\pi, y, t) = u(x, y, t), \tag{4}$$

though it will be clear that the analysis can be adapted with minor changes (for instance, sum over m replaced by integration in the ensuing) for the non-periodic case $x \in \mathbb{R}$ as well. However, finiteness in the y-direction is essential for the method presented here to work. In particular, Lemma 24 cannot be extended when integer $n \neq 0$ is replaced by a continuous variable which can be arbitrarily small in absolute value. We also note that the analysis for the more general periodic case in x and y (different from 2π) can be brought to the case studied here merely by rescaling variables and parameters. We note at once from y integration of (1) that for any regular solution

$$\frac{\partial^3}{\partial x^3} \int_0^\pi u(x, y, t) dy = 0, \text{implying from above } \frac{1}{\pi} \int_0^\pi u(x, y, t) dy = C_0(t). \tag{5}$$

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It is to be noted that $C_0(t)$ can be specified and is not determined by the equation itself; however, the analysis only needs to be done for $C_0(t) = 0$ since if $C_0(t) \neq 0$, the change of variables $(x, y, t, u) \to \left(x - 2\int_0^t C_0(t')dt', y, t, u + C_0(t)\right)$ transforms (1) back to itself with the new u satisfying $\int_0^\pi u(x, y, t)dy = 0$. Therefore, we seek a solution with representation[1]

$$u(x, y, t) = e^{\beta t} \sum_{(m,n)\in\mathbb{Z}_0^2} u_{m,n}(t) \exp\left[imx + iny\right], \tag{6}$$

where

$$\mathbb{Z}_0^2 = \left\{(m, n) \in \mathbb{Z}^2 : n \neq 0\right\}, \tag{7}$$

with the restriction

$$u_{m,-n} = u_{m,n}, u_{-m,n} = u_{m,n}^* \tag{8}$$

that ensures that u is real and contains only $\{\cos(ny)\}_{n=1}^{\infty}$ terms that automatically satisfy boundary conditions (2) at $y = 0$ and $y = \pi$. Applying Duhamel's principle, the initial boundary value problem (1)–(3) can be formally reduced to the following integral equation for $\mathbf{U}(t) = \{u_{m,n}(t)\}_{m,n\in\mathbb{Z}_0^2}$ with $t > 0$:

$$u_{m,n}(t) = e^{-\left(\alpha n^2 - il_{m,n}\right)t} u_{m,n}(0) + \int_0^t e^{-\left(\alpha n^2 - il_{m,n}\right)(t-s)} e^{\beta s} A_{m,n}(s)ds, \tag{9}$$

where

$$l_{m,n} = \frac{\gamma m^3}{n^2} \tag{10}$$

and

$$A_{m,n} = -im \sum_{(m',n')\in\mathbb{Z}_{0,n}^2} u_{m',n'} u_{m-m',n-n'}, \tag{11}$$

where for any $n \in \mathbb{Z} \setminus \{0\}$,

$$\mathbb{Z}_{0,n}^2 = \left\{(m', n') \in \mathbb{Z}^2 : n' \neq 0, n' \neq n\right\}. \tag{12}$$

We also require that $u_{m,n}(0)$ satisfies the symmetry condition (8).

Remark 1 *With the symmetry (8) imposed initially, with appropriate transformation of summation variables (m', n'), it is clear that if $\mathbf{U}(t) = \{u_{m,n}(t)\}_{(m,n)\in\mathbb{Z}_0^2}$ is one solution, so will be $\{u_{m,-n}(t)\}_{(m,n)\in\mathbb{Z}_0^2}$, or $\left\{u_{-m,n}^*(t)\right\}_{(m,n)\in\mathbb{Z}_0^2}$. Therefore, if the solution is unique, as will be shown to be the case, the conditions (8), once satisfied initially, remain time-invariant and therefore (8) is satisfied automatically. It is of course possible to generalize representation (9) for complex periodic initial data by relaxing (8), though it is not of physical interest in the river context.*

[1] The insertion of factor $e^{\beta t}$ in (6) makes the analysis of the ensuing integral equation simpler.

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It is useful to introduce abstract operator $e^{t\mathscr{L}}$ so that integral equation (9) through basis representation (6) can be interpreted as

$$u(\cdot,\cdot,t) = e^{t\mathscr{L}}u_0 - \int_0^t e^{(t-\tau)\mathscr{L}} \quad \left[u^2\right]_x(\cdot,\cdot,\tau)d\tau \tag{13}$$

where in the basis representation (6)

$$\left[e^{t\mathscr{L}}v\right]_{m,n} = e^{\beta t}e^{(-\alpha n^2 + il_{m,n})t}v_{m,n}. \tag{14}$$

Remark 2 *Equation (13) is an integral reformulation of the original PDE initial value problem (1), (3); it does not require functions to be differentiable in any sense, except for once in x. Therefore, it is appropriate to look for solutions to (13) for which $u_{m,n}(t)$ is integrable in time and $\{(1+|m|+|n|)^s u_{m,n}\} \in l^2(\mathbb{Z}_0^2)$, for $s > 2$. Clearly, if s is large enough, it generates a classical solution of (1), (3).*

As usual for treatment of dispersive PDEs through Fourier transform in time, we introduce a smooth cut-off function ϕ with support in $(-2\delta, 2\delta)$ and with $\phi = 1$ in $[-\delta, \delta]$. We replace the original system of equations (9) by the following system that extends the solution to $t \in \mathbb{R}$.

$$\begin{aligned}\theta_{m,n}(t) = {} & \phi(t)e^{-\alpha|t|n^2 + il_{m,n}t}u_{m,n}(0) \\ & + \phi(t)\int_{\mathbb{R}} H(t-s)e^{-(\alpha n^2 - il_{m,n})(t-s)}e^{\beta s}\phi^2(s)\Lambda_{m,n}(s)ds \\ & - \phi(t)e^{(il_{m,n}t - \alpha n^2|t|)}\int_{\mathbb{R}} H(-s)e^{(\alpha n^2 - il_{m,n})s}e^{\beta s}\phi^2(s)\Lambda_{m,n}(s)ds,\end{aligned} \tag{15}$$

where H is the Heaviside function ($H := \chi_{[0,\infty)}$) and

$$\Lambda_{m,n} = -im \sum_{(m',n')\in\mathbb{Z}_{0,n}^2} \theta_{m',n'}\theta_{m-m',n-n'}. \tag{16}$$

It is clear that any solution $\boldsymbol{\theta}(t) := \{\theta_{m,n}(t)\}_{(m,n)\in\mathbb{Z}_0^2}$ satisfying (15) has compact support in $(-2\delta, 2\delta)$ and, since $\phi = 1$ for $t \in [0,\delta]$, satisfies exactly the same equation (9) in that interval as does $\mathbf{U}(t) = \{u_{m,n}(t)\}_{(m,n)\in\mathbb{Z}_0^2}$. Further, for any solution $\mathbf{U}(t)$ to (9), $\phi(t)\mathbf{U}(t)$ will be compactly supported in $(-2\delta, 2\delta)$ and for $t \in [0,\delta]$, where $\phi = 1$, will satisfy (15), though this is not true outside this time interval. It is convenient to write the system in (15) symbolically as

$$\boldsymbol{\theta} = \boldsymbol{\theta}^{(0)} + \mathscr{M}[\boldsymbol{\theta}] =: \mathscr{N}[\boldsymbol{\theta}], \tag{17}$$

where the (m,n)-th component of (17) at time $t \in \mathbb{R}$, for $(m,n) \in \mathbb{Z}_0^2$ is given by (15), with $\Lambda_{m,n}$ determined in terms of $\theta_{m,n}$ through (16).

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Definition 1 *Similar to [18], we define a Bourgain-type [3,4] norm* $\|.\|_{s,b}$ *of vector function* $\boldsymbol{\theta}(t) = \{\theta_{m,n}\}_{(m,n)\in\mathbb{Z}_0^2}$ *such that*

$$\|\boldsymbol{\theta}\|_{s,b}^2 = \sum_{(m,n)\in\mathbb{Z}_0^2} \int_{\mathbb{R}} W_{m,n}(\tau;b)\left|\mathscr{F}\left[\theta_{m,n}\right](\tau)\right|^2 d\tau, \tag{18}$$

where $\mathscr{F}[\theta_{m,n}](\tau) = \int_{\mathbb{R}} e^{-it\tau}\theta_{m,n}(t)dt$ *is the Fourier transform in time and weight*

$$W_{m,n}(\tau;b) = \rho_{m,n}\omega_{m,n}(\tau;b),\ \rho_{m,n} = \left(|n|+|m|\right)^{2s},\ \ \omega_{m,n}(\tau;b) = \left(n^2+|\tau-l_{m,n}|\right)^{2b}. \tag{19}$$

The space of vector functions $\{\boldsymbol{\theta}(t)\}$ *for which* $\|\boldsymbol{\theta}\|_{s,b} < \infty$ *is defined as* $T_{s,b}$*. We also denote by* $H_{0,b}^s$ *the space of functions* $v(x,y)$ *in* $\mathbb{T}[0,2\pi]^2$ *with* $\int_0^{2\pi} v(x,y)dy = 0$*, represented by* $v(x,y) = \sum_{(m,n)\in\mathbb{Z}_0^2} v_{m,n}e^{imx+iny}$*, equipped with the norm*

$$\|v\|_{H_{0,b}^s}^2 = \sum_{(m,n)\in\mathbb{Z}_0^2} |n|^{4b-2}\rho_{m,n}|v_{m,n}|^2.$$

H_0^s *will denote the subspace of the usual Sobolev space* H^s *satisfying zero average condition* $\int_0^{2\pi} v(x,y)dy = 0$.

Remark 3 *With the representation (6), the* $\|\cdot\|_{s,b}$ *norm could also be thought of as a norm in the space of functions* $\{v(x,y,t)\}$ *rather than the space of corresponding sequences, as defined here.*

Theorem 1 *Assume* $s > \frac{5}{2}$ *and* $u_0 \in H_{0,b}^s(\mathbb{T}^2)$*. Then, for* $b \in \left(\frac{1}{2},\frac{2}{3}\right)$*, there exists* δ *sufficiently small, that depends on* $\|u_0\|_{H_{0,b}^s}$*, so that there exists unique solution* $\boldsymbol{\theta}$ *to the integral equation* (17) *in some small ball in* $T_{s,b}$*. If initial data satisfies (8), then this corresponds to the unique continuous solution* $u(\cdot,\cdot,t) \in H_0^s(\mathbb{T}^2[0,2\pi])$ *satisfying (1)–(3) in the time interval of existence. Further, the energy* $\|u(\cdot,\cdot,t)\|_{L^2}^2$ *as well as cumulative dissipation* $\int_0^t \|u_y(\cdot,\cdot,s)\|_{L^2}^2 ds$ *are globally controlled in time.*

The proof will rely on a contraction argument in the space $T_{s,b}$ for small δ and is similar in spirit to the ones presented earlier in [16–21] for $1+1$ dispersive system like the KdV. What is new here is the introduction of the space $T_{s,b}$ and a suitable integral reformulation (15) for which dispersive regularization estimates through Fourier transform in time is good enough despite lack of uniform dispersion, and also overcomes ill-posedness in backwards time in the original formulation (9). Also, crucial to the proof are the bounds established in Section 3, which rely partially on previous estimates [16, 21] or simple adaptation of them, which are presented in the Appendix for completeness. The energy control and uniqueness rely on a more traditional energy argument in classical Sobolev space.

Remark 4 *Clearly, since* $b > 1/2$*,* $H_0^s \subset H_{0,b}^s \subset H_0^{s+(4b-2)}$*, where* H_0^s *is the subspace of the traditional Sobolev space* H^s *with zero average in* y*. Indeed, since the condition for existence*

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and uniqueness locally in time is $s > \frac{5}{2}$, with no restriction on how small $b - 1/2 > 0$ is, any initial data in $H^{s'}_0$ for $s' > \frac{5}{2}$ suffices.

Remark 5 *Global control on $\|u\|_{L^2}$ is however not enough to control $\|u(\cdot,\cdot,t)\|_{H^s_0}$ for $s > \frac{5}{2}$.*

2 Key Lemmas and Proof of Theorem 1

Notation: The symbol $\lesssim$ is used in the following sense: $|f| \lesssim |g|$ is equivalent to existence of some constant C independent of (m, n, m', n', τ), but possibly dependent on $\alpha, \beta, \gamma, s,$ and b such that $|f| \leqslant C|g|$. The value of C is not important.

Lemma 2 *For $\frac{1}{2} < b$,*

$$\|\boldsymbol{\theta}^{(0)}\|_{s,b} \lesssim \|u_0\|_{H^s_{0,b}}. \tag{20}$$

Proof First, we note from computation that

$$\mathscr{F}\left[\exp\left(-\alpha n^2|t| + il_{m,n}t\right)\right](\tau) = \frac{2\alpha n^2}{\alpha^2 n^4 + \left(\tau - l_{m,n}\right)^2}.$$

Therefore,

$$\int_{\mathbb{R}} \left(n^2 + \left|\tau - l_{m,n}\right|\right)^{2b} \left|\mathscr{F}\left[e^{-\alpha n^2|t| + il_{m,n}t}\right](\tau)\right|^2 \lesssim \int_{\mathbb{R}} \frac{n^4 d\tau}{\left(n^2 + \left|\tau - l_{m,n}\right|\right)^{4-2b}} \lesssim |n|^{4b-2}.$$

Using the above inequality and applying Lemma 25 to $\boldsymbol{\theta}^{(0)}$ (see (15) and (17)), it follows that

$$\|\boldsymbol{\theta}^{(0)}\|^2_{s,b} \lesssim \sum_{(m,n)\in\mathbb{Z}^2_0} |n|^{4b-2}\left(|m| + |n|\right)^{2s} |u_{m,n}(0)|^2 \lesssim \|u_0\|^2_{H^s_{0,b}}.$$

□

Proposition 3 *For $b \in \left(\frac{1}{2}, \frac{2}{3}\right)$, $b \leqslant b' \leqslant \min\{2b - \frac{1}{2}, \frac{2}{3}\}$, and $s > \frac{5}{2}$,*

$$\sup_{(m,n)\in\mathbb{Z}^2_0, \tau\in\mathbb{R}} \left[\sum_{(m',n')\in\mathbb{Z}^2_{0,n}} \int_{\mathbb{R}} \frac{m^2 W_{m,n}(\tau; b')d\tau_1}{|n^2 + i(\tau - l_{m,n})|^2 W_{m',n'}(\tau_1; b) W_{m-m',n-n'}(\tau - \tau_1; b)}\right] \lesssim 1. \tag{21}$$

The proof of Proposition 3 is given at the end of Section 3 after some bounds are established. It relies crucially on estimates of Kenig *et al.* [16]. However, since we make use of slight variants of them, we state with proof the required estimate in Lemma 27 in the appendix. Lemma 29 as it appears, is exactly the same as it appears in Kenig *et al* [16] and is reproduced here for completeness.

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Lemma 4 *Define*

$$b_{m,n}(t) = \sum_{(m',n')\in\mathbb{Z}^2_{0,n}} \int_{\mathbb{R}} H(t-s) e^{(-\alpha n^2 + i l_{m,n})(t-s)} \hat{\theta}_{m',n'}(s) \hat{\beta}_{m-m',n-n'}(s) ds, \tag{22}$$

where $\hat{\theta}_{m,n}(t) = \psi(t)\theta_{m,n}(t)$, $\hat{\beta}_{m,n}(t) = \psi(t)\beta_{m,n}(t)$, $\psi(t) = e^{\beta t/2}\phi(t)$. *With the restriction on b, b′, s in Proposition 3,*

$$\sum_{(m,n)\in\mathbb{Z}^2_0} \int_{\mathbb{R}} W_{m,n}(\tau; b) m^2 \left| \mathscr{F}[\phi b_{m,n}](\tau) \right|^2 d\tau \lesssim \delta^{2b'-2b} \|\boldsymbol{\theta}\|^2_{s,b} \|\boldsymbol{\beta}\|^2_{s,b}. \tag{23}$$

Proof First, note that if we apply Lemma 25 with $q = b_{m,n}$, it follows that the left-hand side of (23)

$$\lesssim \delta^{2b'-2b} \sum_{(m,n)\in\mathbb{Z}^2_0} \int_{\mathbb{R}} W_{m,n}(\tau; b') m^2 \left| \mathscr{F}[b_{m,n}](\tau) \right|^2 d\tau. \tag{24}$$

We note from convolution form of $b_{m,n}$ that

$$\mathscr{F}[b_{m,n}](\tau) = \sum_{(m',n')\in\mathbb{Z}^2_0} \frac{1}{\alpha n^2 + i(\tau - l_{m,n})} \int_{\mathbb{R}} \mathscr{F}[\hat{\theta}_{m',n'}](\tau_1) \mathscr{F}[\hat{\beta}_{m-m,n-n'}](\tau - \tau_1) d\tau_1. \tag{25}$$

So, applying Cauchy–Schwartz inequality it follows that the expression in (24)

$$\begin{aligned} &\lesssim \delta^{2b'-2b} \sup_{(m,n)\in\mathbb{Z}^2_0, \tau\in\mathbb{R}} \left\{ \sum_{(m',n')} \int_{\mathbb{R}} \frac{m^2 W_{m,n}(\tau; b') d\tau_1}{|n^2 + i(\tau - l_{m,n})|^2 W_{m',n'}(\tau_1; b) W_{m-m',n-n'}(\tau - \tau_1; b)} \right\} \\ &\quad \times \left\{ \sum_{(m,n)} \sum_{(m',n')} \int_{\mathbb{R}} \int_{\mathbb{R}} W_{m',n'}(\tau_1; b) \left| \mathscr{F}[\hat{\theta}_{m',n'}](\tau_1) \right|^2 \right. \\ &\quad \left. \times W_{m-m',n-n'}(\tau - \tau_1; b) \left| \mathscr{F}[\hat{\beta}_{m-m',n-n'}](\tau - \tau_1) \right|^2 d\tau_1 d\tau \right\}, \end{aligned} \tag{26}$$

from which the Lemma follows if we apply Proposition 3 and Lemma 25, $q = \theta_{m',n'}$ or $q = \beta_{m-m',n-n'}$ and ϕ replaced by ψ, which is also compactly supported smooth function in $(-2\delta, 2\delta)$. □

Lemma 5 *Under the conditions of Proposition 3, for* $B_{m,n}$ *with support in* $(-2\delta, 2\delta)$ *if the integral on the right side in* (27) *exists, then*

$$\begin{aligned} &\int_{\mathbb{R}} \omega_{m,n}(\tau; b) \left| \mathscr{F}\left[\phi(t) \int_{-2\delta}^{0} e^{i l_{m,n}(t-s) - \alpha n^2 (|t|-s)} B_{m,n}(s) ds \right](\tau) \right|^2 d\tau \\ &\quad \lesssim \int_{\mathbb{R}} \omega_{m,n}(\tau; b) \left| \mathscr{F}\left[\phi(t) \int_{\mathbb{R}} H(t-s) e^{[i l_{m,n} - \alpha n^2](t-s)} B_{m,n}(s) ds \right] \right|^2 (\tau) d\tau. \end{aligned} \tag{27}$$

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Proof We note from Lemma 25 and 28, that

$$\begin{aligned}
&\int_{\mathbb{R}} \omega_{m,n}(\tau;b)\left|\mathscr{F}\left[\phi(t)\int_{-2\delta}^{0} e^{il_{m,n}(t-s)-\alpha n^2(|t|-s)}B_{m,n}(s)ds\right](\tau)\right|^2 d\tau \\
&\quad\lesssim \left[\int_{\mathbb{R}} \omega_{m,n}(\tau;b)\left|\mathscr{F}\left[e^{il_{m,n}t-\alpha n^2|t|}\right](\tau)\right|^2 d\tau\right]\left|\int_{-2\delta}^{0} e^{-il_{m,n}s+\alpha n^2 s}B_{m,n}(s)ds\right|^2 \\
&\quad\lesssim \left(\int_{\mathbb{R}} \frac{\omega_{m,n}(\tau;b)n^4}{\left(n^2+|\tau-l_{m,n}|\right)^4}d\tau\right)\left|\int_{\mathbb{R}} H(-s)e^{-il_{m,n}s+\alpha n^2 s}B_{m,n}(s)ds\right|^2 \\
&\quad\lesssim \int_{\mathbb{R}} \frac{n^4 d\tau}{\left(n^2+|\tau-l_{m,n}|\right)^{4-2b}}\sup_t\left|\phi(t)\int_{\mathbb{R}} H(t-s)e^{il_{m,n}(t-s)-\alpha n^2(t-s)}B_{m,n}(s)ds\right|^2 \\
&\quad\lesssim \left(\int_0^{\infty}\frac{|n|^{6-4b}d\tau}{(n^2+\tau)^{4-2b}}\right)\int_{\mathbb{R}}\left(n^2+|\tau-l_{m,n}|\right)^{2b} \\
&\qquad\times\left|\mathscr{F}\left[\phi(t)\int_{\mathbb{R}} H(t-s)e^{il_{m,n}(t-s)-\alpha n^2(t-s)}B_{m,n}(s)ds\right](\tau)\right|^2 d\tau
\end{aligned} \tag{28}$$

from which Lemma follows readily. □

Lemma 6 *Assume that conditions of Proposition 3 hold and that* $\boldsymbol{\theta},\boldsymbol{\beta}\in T_{s,b}$. *Define*

$$d_{m,n}=e^{il_{m,n}t-\alpha n^2|t|}\int_{\mathbb{R}} H(-s)e^{(\alpha n^2-il_{m,n})s}e^{\beta s}\phi^2(s)\Lambda_{m,n}(s)ds, \tag{29}$$

where

$$\Lambda_{m,n}(t)=-im\sum_{(m',n')\in\mathbb{Z}^2_{0,n}}\theta_{m',n'}(t)\beta_{m-m',n-n'}(t). \tag{30}$$

Then,

$$\sum_{(m,n)\in\mathbb{Z}^2_0}\int_{\mathbb{R}} W_{m,n}(\tau;b)\left|\mathscr{F}[\phi d_{m,n}](\tau)\right|^2 d\tau\lesssim\delta^{2b'-2b}\|\boldsymbol{\theta}\|^2_{s,b}\|\boldsymbol{\beta}\|^2_{s,b}. \tag{31}$$

Proof We apply Lemma 5

$$B_{m,n}(t)=\phi^2(t)e^{\beta t}\Lambda_{m,n}(t)=-im\sum_{(m'n')\in\mathbb{Z}^2_{0,n}}\hat{\theta}_{m',n'}(t)\hat{\beta}_{m-m',n-n'}(t),$$

where $\hat{\theta}_{m',n'}(t)=e^{\beta t/2}\phi(t)\theta_{m',n'}(t)$ and $\hat{\beta}_{m-m',n-n'}(t)=e^{\beta t/2}\phi(t)\beta_{m-m',n-n'}(t)$, and then use Lemma 4. □

Proposition 7 *Under the conditions of Proposition 3, except we require* $b'>b$, *for* $\mathscr{M}$ *defined in* (15)–(17), *there exists constant* c_1 *independent of* δ *such that for* $\boldsymbol{\theta}^{(1)},\boldsymbol{\theta}^{(2)}\in T_{s,b}$,

$$\|\mathscr{M}[\boldsymbol{\theta}^{(1)}]-\mathscr{M}[\boldsymbol{\theta}^{(2)}]\|_{s,b}\leqslant c_1\delta^{b'-b}\|\boldsymbol{\theta}^{(1)}+\boldsymbol{\theta}^{(2)}\|_{s,b}\|\boldsymbol{\theta}^{(1)}-\boldsymbol{\theta}^{(2)}\|_{s,b}. \tag{32}$$

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Proof We note from definition in (15)–(17)

$$\left[\mathscr{M}[\boldsymbol{\theta}^{(1)}]-\mathscr{M}[\boldsymbol{\theta}^{(2)}]\right]_{m,n}(t)=\phi(t)\int_{\mathbb{R}}H(t-s)e^{-(\alpha n^2-il_{m,n})(t-s)}e^{\beta s}\phi^2(s)\Lambda_{m,n}(s)ds$$
$$-\phi(t)e^{il_{m,n}t-\alpha n^2|t|}\int_{\mathbb{R}}H(-s)e^{(\alpha n^2-il_{m,n})s}e^{\beta s}\phi^2(s)\Lambda_{m,n}(s)ds, \tag{33}$$

where

$$\Lambda_{m,n}(t)=-im\sum_{(m',n')\in\mathbb{Z}^2_{0,n}}\left(\theta^{(1)}_{m',n'}(t)+\theta^{(2)}_{m',n'}(t)\right)\left(\theta^{(1)}_{m-m',n-n'}(t)-\theta^{(2)}_{m-m',n-n'}(t)\right) \tag{34}$$

the result follows from applying Lemmas 4 and 6, with $\boldsymbol{\theta}=\boldsymbol{\theta}^{(1)}+\boldsymbol{\theta}^{(2)}$, $\boldsymbol{\beta}=\boldsymbol{\theta}^{(1)}-\boldsymbol{\theta}^{(2)}$. □

Proof of Theorem 1 The existence and uniqueness of solution in some ball containing the initial condition by applying Proposition 7 with $b'>b$ (with all other given restriction) and Lemma 2, which for $\boldsymbol{\theta}^{(1)}=\boldsymbol{\theta}$ and $\boldsymbol{\theta}^{(2)}=\mathbf{0}$ (note $\mathscr{M}[\mathbf{0}]=0$) implies $\|\mathscr{M}[\boldsymbol{\theta}]\|_{s,b}\leqslant c_1\delta^{b'-b}\|\boldsymbol{\theta}\|^2_{s,b}$. Thus, in a ball of size $2c_0\|u_0\|_{H^s_{0,b}}$ in $T_{s,b}$ for sufficiently small δ results in

$$\|\mathscr{N}[\boldsymbol{\theta}]\|_{s,b}\leqslant c_0\|u_0\|_{H^s_{0,b}}+4c_0^2c_1\delta^{b'-b}\|u_0\|^2_{H^s_{0,b}}\leqslant 2c_0\|u_0\|_{H^s_{0,b}}$$
$$\|\mathscr{N}[\boldsymbol{\theta}^{(1)}]-\mathscr{N}[\boldsymbol{\theta}^{(2)}]\|_{s,b}\leqslant 4c_0c_1\delta^{b'-b}\|u_0\|_{H^s_{0,b}}\|\boldsymbol{\theta}^{(1)}-\boldsymbol{\theta}^{(2)}\|_{s,b}\leqslant\epsilon_1\|\boldsymbol{\theta}^{(1)}-\boldsymbol{\theta}^{(2)}\|_{s,b},$$

where $\epsilon_1<1$, and therefore $\mathscr{N}$ is contractive in the ball. Further, for any time $t\in[0,\delta]$, applying Lemma 28 it follows that for $s>\frac{5}{2}$,

$$\|u(\cdot,\cdot,t)\|^2_{H^s_{0,b}}\lesssim\sum_{(m,n)\in\mathbb{Z}^2_0}W_{m,n}(\tau)\left|\mathscr{F}[\theta_{m,n}](\tau)\right|^2d\tau=\|\boldsymbol{\theta}\|^2_{s,b}.$$

Furthermore, the estimate on $\|\mathscr{M}[\boldsymbol{\theta}]\|_{s,b}$ implies from the integral equation itself that

$$\|u(\cdot,\cdot,t)-u^{(0)}(\cdot,\cdot,t)\|_{H^s_{0,b}}\lesssim\|\boldsymbol{\theta}-\boldsymbol{\theta}^{(0)}\|_{s,b}=\|\mathscr{M}[\boldsymbol{\theta}]\|_{s,b}\lesssim\delta^{b'-b}\|\boldsymbol{\theta}\|^2_{s,b}$$

which by shrinking δ implies time continuity of solution at $t=0$ in $H^s_{0,b}$ since $u^{(0)}(\cdot,\cdot,t)$ is obviously continuous. To prove continuity at a point $t_0\in[0,\delta]$, we rewrite (9) in the form

$$u_{m,n}(t)=e^{-(\alpha n^2-il_{m,n})(t-t_0)}u_{m,n}(t_0)+\int_{t_0}^{t}e^{-(\alpha n^2-il_{m,n})(t-s)}e^{\beta s}A_{m,n}(s)ds,$$

which is clearly possible, and find an equivalent integral equation similar to (15), except centring it at t_0. It is clear that the same argument shows continuity at t_0. What is left now is the global energy control and uniqueness argument of the solution to (9). This is accomplished through a more traditional energy type argument in the following subsection. □

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2.1 Energy control, uniqueness and end of Theorem 1 proof

While the argument in the ensuing can be carried out in the space of sequences $\{u_{m,n}\}$ by working directly with (9), and doing inner product through in $(m,n) \in \mathbb{Z}_0^2$, it is easier to see that this argument directly follows[2] from (1)–(3). It is convenient to introduce

$$v(x,y,t) = \int_0^y u(x,y',t)dy'. \tag{35}$$

It is to be noted that $v \in \mathbb{T}[0,2\pi]^2$ since $\int_0^{2\pi} u(x,y,t)dy = 0$. Then, on integrating (1) from 0 to y, noting $u_y = 0$ and $u_{yyy} = 0$ at $y = 0$, we obtain

$$v_{yyt} - \gamma v_{xxx} - \alpha v_{yyyy} - \beta v_{yy} + \left(v_y^2\right)_{xy} = 0. \tag{36}$$

Using inner product of (36) with v and appropriate integration by parts, we obtain the energy bound

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \alpha\|u_y\|_{L^2}^2 - \beta\|u\|_{L^2}^2 = 0. \tag{37}$$

It follows

$$\frac{1}{2}\|u(\cdot,\cdot,t)\|_{L^2}^2 + \alpha\int_0^t \|u_y(\cdot,\cdot,s)\|_{L^2}^2 ds = \frac{1}{2}\|u_0\|_{L^2}^2 + \beta\int_0^t \|u(\cdot,\cdot,s)\|_{L^2}^2 ds. \tag{38}$$

Using Poincare inequality and Gronwall's Lemma, it follows that

$$\frac{1}{2}\|u(\cdot,\cdot,t)\|_{L^2}^2 \leqslant \frac{1}{2}\|u_0\|_{L^2}^2 e^{2(\beta-\alpha)t} \tag{39}$$

and we have global exponential control of L^2 norm. If $\tilde{u}$ is another solution, and we define corresponding $\tilde{v} = \int_0^y \tilde{u}dy$, Then, $\tilde{v}$ satisfies (36) as well. Subtracting, we obtain the following equation for $w = v - \tilde{v}$:

$$w_{yyt} - \gamma w_{xxx} - \alpha w_{yyyy} - \beta w_{yy} + \left[\left(v_y + \tilde{v}_y\right) w_y\right]_{xy} = 0. \tag{40}$$

Inner product with w in (40), integration by parts and then time integration leads to

$$\frac{1}{2}\|w_y(\cdot,\cdot,t)\|_{L^2}^2 + \int_0^t \alpha\|w_{yy}(\cdot,\cdot,s)\|_{L^2}^2 ds = \beta\int_0^t \|w_y(\cdot,\cdot,s)\|_{L^2}^2 ds - \frac{1}{2}\int_0^t \left(w_y^2, [v_y + \tilde{v}_y]_x\right)(s)ds. \tag{41}$$

Recalling $w_y = u - \tilde{u}$ and $v_y = u$, $\tilde{v}_y = \tilde{u}$, using Gronwall's Lemma in the integral form, it follows $u = \tilde{u}$ and the solution to (9) is unique in $L^2\left(\mathbb{T}[0,2\pi]^2\right)$ within the class of functions for which

$$\int_0^t \|u_x(\cdot,\cdot,s)\|_\infty ds < \infty. \tag{42}$$

From Sobolev embedding theorem, it is enough to require $\|u(\cdot,\cdot,t)\|_{H_0^s}$ for $s > 2$ is time integrable. This is certainly true for the u shown to exist for $t \in [0,\delta]$ because the corresponding $\boldsymbol{\theta} \in T_{s,b}$ to (15) for $s > \frac{5}{2}$. This completes the proof of Theorem 1.

[2] While the manipulation to get energy inequality is formal in the sense that the derivatives in x and y are not assured, the end product is legitimate as it involves norms that exist.

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Remark 6 *We have made no attempts to optimize in s or b; in all likelihood, the solution exists in weaker spaces.*

3 Estimates and proof of Proposition 3

Definition 8 *For given* $(m,n) \in \mathbb{Z}_0^2$, *define*

$$\mathbb{Z}^2_{0,n,>} = \left\{ (m',n') \in \mathbb{Z}^2_{0,n} : n'/n \geqslant \frac{1}{2} \right\}, \tag{43}$$

where $\mathbb{Z}^2_{0,n}$ *was given in (12). Also, define*

$$Q_{m,n,m',n'} = |n|^{-4b} \left(n^2 + \left| l_{m,n} - l_{m',n'} - l_{m-m',n-n'} \right| \right)^{2b'-2}. \tag{44}$$

Lemma 9

$$\begin{aligned} &\sum_{(m',n')\in\mathbb{Z}^2_{0,n}} \int_{\mathbb{R}} \frac{m^2 W_{m,n}(\tau;b')d\tau_1}{\left(n^2+|\tau-l_{m,n}|\right)^2 W_{m',n'}(\tau_1;b) W_{m-m',n-n'}(\tau-\tau_1;b)} \\ &\lesssim \sum_{(m',n')\in\mathbb{Z}^2_{0,n,>}} \int_{\mathbb{R}} \frac{m^2 W_{m,n}(\tau;b')d\tau_1}{\left(n^2+|\tau-l_{m,n}|\right)^2 W_{m',n'}(\tau_1;b) W_{m-m',n-n'}(\tau-\tau_1;b)}. \end{aligned} \tag{45}$$

Proof We simply note that the summand above is invariant on change of variables $(m',n',\tau_1) \to (m-m',n-n',\tau-\tau_1)$; hence,

$$\begin{aligned} &\sum_{(m',n')\in\mathbb{Z}^2_{0,n},|n'|\leqslant|n-n'|} \int_{\mathbb{R}} \frac{m^2 W_{m,n}(\tau;b')d\tau_1}{\left(n^2+|\tau-l_{m,n}|\right)^2 W_{m',n'}(\tau_1;b) W_{m-m',n-n'}(\tau-\tau_1;b)} \\ &= \sum_{(m',n')\in\mathbb{Z}^2_{0,n},|n'|\geqslant|n-n'|} \int_{\mathbb{R}} \frac{m^2 W_{m,n}(\tau;b')d\tau_1}{\left(n^2+|\tau-l_{m,n}|\right)^2 W_{m',n'}(\tau_1;b) W_{m-m',n-n'}(\tau-\tau_1;b)}. \end{aligned} \tag{46}$$

Hence, the Lemma follows since the set $|n'| \geqslant |n'-n|$ is equivalent to $n'/n \geqslant \frac{1}{2}$. □

Definition 10 *For any* $(m,n) \in \mathbb{Z}_0^2$, *define the corresponding set*

$$\mathscr{B}_0 = \left\{ (m',n') \in \mathbb{Z}^2_{0,n,>}, n' = 2n \right\}$$

Remark 7 *We first consider the summation over this special set* $\mathscr{B}_0 \subset \mathbb{Z}^2_{0,n,>}$, *since this analysis differs substantially from the rest of the summation set* $\mathbb{Z}^2_{0,n,>}$.

Lemma 11 *For any* $s \geqslant 0$,

$$\sup_{(m',n')\in\mathscr{B}_0} \frac{\rho_{m,n}}{\rho_{m',n'}\rho_{m-m',n-n'}} \lesssim 1. \tag{47}$$

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Proof We note from expression of $\rho_{m,n}$ that

$$\frac{\rho_{m,n}}{\rho_{m',2n}\rho_{m-m',n}} = \frac{(|m|+|n|)^{2s}}{\left(|m'|+4|n|\right)^{2s}\left(|m-m'|+|n|\right)^{2s}} \lesssim 1, \tag{48}$$

since we can break up the set $\mathscr{B}_0$ into $|m'| \leqslant |m|/2$ and its complement $|m'| > |m|/2$, and the expression above is bounded in either case. □

Lemma 12 *For* $|m| \geqslant 1$, *under conditions of Proposition 3,*

$$\sum_{(m',n')\in\mathscr{B}_0} \int_{\mathbb{R}} \frac{m^2 W_{m,n}(\tau;b')d\tau_1}{\left(n^2+|\tau-l_{m,n}|\right)^2 W_{m',n'}(\tau_1;b)W_{m-m',n-n'}(\tau-\tau_1;b)} \lesssim 1. \tag{49}$$

Proof Using expressions for $W_{m,n}(\tau)$, changing integration variable $\tau_1 \to \tau-\tau_1$ and using Lemma 11, it is enough to show

$$\sum_{m'\in\mathbb{Z}} \int_{\mathbb{R}} \frac{m^2\omega_{m,n}(\tau;b'-1)d\tau_1}{\omega_{m',2n}(\tau-\tau_1;b)\omega_{m-m',-n}(\tau_1;b)} \lesssim 1. \tag{50}$$

Since we are only interested in determining supremum of the left-hand side of (50), it is convenient to change variables $(\tau,\tau_1) \to (\tau n^2, \tau_1 n^2)$, in which case, we get the above to simplify to

$$\begin{aligned}
&\frac{1}{|n|^{8b-4b'+2}} \sum_{m'\in\mathbb{Z}} \int_{\mathbb{R}} \frac{m^2\left(1+|\tau-\frac{\gamma m^3}{n^4}|\right)^{2b'-2} d\tau_1}{\left(4+|\tau-\tau_1-\frac{\gamma m'^3}{n^4}|\right)^{2b}\left(1+|\tau_1-\frac{\gamma(m-m')^3}{n^4}|\right)^{2b}} \\
&\lesssim \frac{1}{|n|^{8b-4b'+2}} \sum_{m'\in\mathbb{Z}} \int_{\mathbb{R}} \frac{m^2\left(1+|\tau-\frac{\gamma m^3}{n^4}|\right)^{2b'-2} d\tau_1}{\left(1+|\tau-\tau_1-\frac{\gamma m'^3}{n^4}|\right)^{2b}\left(1+|\tau_1-\frac{\gamma(m-m')^3}{n^4}|\right)^{2b}}.
\end{aligned} \tag{51}$$

We may assume $\gamma > 0$ without loss of generality, as otherwise for $\gamma < 0$, we replace $(\gamma,\tau) \to (-\gamma,-\tau)$ in the argument given in the ensuing. Introducing $\xi = \gamma^{1/3}m/|n|^{4/3}$, $\xi_1 = |\gamma|^{1/3}m'/|n|^{4/3}$, we obtain by using Lemma 27 that the above expression

$$\begin{aligned}
&\lesssim \frac{1}{|n|^{8b-4b'-2}} \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\xi^2\left(1+|\tau-\xi^3|\right)^{2b'-2} d\tau_1 d\xi_1}{\left(1+|\tau-\tau_1-\xi_1^3|\right)^{2b}\left(1+|\tau_1-(\xi-\xi_1)^3|\right)^{2b}} \\
&\lesssim \frac{1}{|n|^{8b-4b'-2}} \int_{\mathbb{R}} \frac{\xi^2\left(1+|\tau-\xi^3|\right)^{2b'-2} d\xi_1}{\left(1+|\tau-\xi_1^3-(\xi-\xi_1)^3|\right)^{2b}}.
\end{aligned} \tag{52}$$

Using Lemma 29 in the Appendix (originally due to Kenig, Ponce & Vega [16]), the above

$$\lesssim \frac{\xi^2}{\sqrt{|\xi|}\left(1+|4\tau-\xi^3|\right)^{1/2}\left(1+|\tau-\xi^3|\right)^{2-2b'}} \lesssim 1. \tag{53}$$

□

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Remark 8 *Having got the sum over the set $\mathscr{B}_0$ out of the way, we now consider the rest. For that purpose, it is useful to reduce the integration over τ_1 into a simpler expression as in the following Lemma.*

Lemma 13 *For $(m,n) \in \mathbb{Z}_0^2$, under conditions of Proposition 3, we have*

$$\sup_{\tau\in\mathbb{R}} \sum_{(m',n')\in\mathbb{Z}_{0,n}} \int_{\mathbb{R}} \frac{m^2 W_{m,n}(\tau;b')d\tau_1}{\left(n^2+|\tau-l_{m,n}|\right)^2 W_{m',n'}(\tau_1;b)W_{m-m',n-n'}(\tau-\tau_1;b)} \lesssim \sum_{(m',n')\in\mathbb{Z}^2_{0,n,>}} \frac{m^2\rho_{m,n}Q_{m,n,m',n'}}{\rho_{m-m',n-n'}\rho_{m',n'}}. \tag{54}$$

Proof We recall $W_{m,n}(\tau;b) = \omega_{m,n}(\tau;b)\rho_{m,n}$ and from definition of $\omega_{m,n}$,

$$\frac{\omega_{m,n}(\tau;b')}{\left(n^2+|\tau-l_{m,n}|\right)^2} = \omega_{m,n}(\tau;b'-1) = \frac{1}{\omega_{m,n}(\tau;1-b')}.$$

First, we note that if $|n'| \geqslant |n-n'|$, which is equivalent to $n'/n \geqslant 1/2$, Lemma 27 with $k_1 = (n-n')^2$, $k_3 = n'^2$, $k_4 = l_{m',n'}$, $k_2 = l_{m-m',n-n'}$, implies

$$\int_{\mathbb{R}} \frac{\omega_{m,n}(\tau;b'-1)d\tau_1}{\omega_{m',n'}(\tau_1;b)\omega_{m-m',n-n'}(\tau-\tau_1;b)} \lesssim \frac{1}{|n-n'|^{4b-2}\omega_{m,n}(\tau;1-b')\omega_{m',n'}(\tau-l_{m-m',n-n'};b)}. \tag{55}$$

Therefore,

$$\sum_{(m',n')\in\mathbb{Z}^2_{0,n,>}} \int_{\mathbb{R}} \frac{m^2 W_{m,n}(\tau;b')d\tau_1}{\left(n^2+|\tau-l_{m,n}|\right)^2 W_{m',n'}(\tau_1;b)W_{m-m',n-n}(\tau-\tau_1;b)} \lesssim \sum_{(m',n')\in\mathbb{Z}^2_{0,n,>}} \frac{m^2\rho_{m,n}}{\rho_{m',n'}\rho_{m-m',n-n'}|n-n'|^{4b-2}\omega_{m,n}(\tau;1-b')\omega_{m',n'}(\tau-l_{m-m',n-n'};b)}. \tag{56}$$

Further by switching variables $(n',m',\tau_1) \to (n-n', m-m', \tau-\tau_1)$, it follows that

$$\begin{aligned}\sum_{(m',n')\in\mathbb{Z}^2_{0,n},|n'|<|n-n'|} \int_{\mathbb{R}} \frac{m^2 W_{m,n}(\tau;b')d\tau_1}{\left(n^2+|\tau-l_{m,n}|\right)^2 W_{m',n'}(\tau_1;b)W_{m-m',n-n'}(\tau-\tau_1;b)} &\lesssim \sum_{(m',n')\in\mathbb{Z}_{0,n,>}} \int_{\mathbb{R}} \frac{m^2\rho_{m,n}\omega_{m,n}(\tau;b'-1)d\tau_1}{\rho_{m',n'}\rho_{m-m',n-n'}\omega_{m',n'}(\tau_1;b)\omega_{m-m',n-n'}(\tau-\tau_1;b)} \\ &\lesssim \sum_{(m',n')\in\mathbb{Z}^2_{0,n,>}} \frac{m^2\rho_{m,n}}{\rho_{m',n'}\rho_{m-m',n-n'}|n-n'|^{4b-2}\omega_{m,n}(\tau;1-b')\omega_{m',n'}(\tau-l_{m-m',n-n'};b)}. \end{aligned} \tag{57}$$

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We also note $n'/n \geqslant \frac{1}{2}$, implies $n'^2 \geqslant \frac{n^2}{4}$ and so in that case applying Lemma 26

$$\begin{aligned}
&\frac{1}{\omega_{m,n}(\tau;1-b')\omega_{m',n'}(\tau - l_{m-m',n-n'};b)} \\
&\lesssim \frac{1}{\left(|n|^2/4 + \left|\tau - l_{m,n}\right|\right)^{2(1-b')}\left(n'^2 + \left|\tau - l_{m',n'} - l_{m-m',n-n'}\right|\right)^{2b}} \\
&\lesssim \frac{1}{\left(|n|^2/4 + \left|\tau - l_{m,n}\right|\right)^{2(1-b')}\left(\frac{n^2}{4} + \left|\tau - l_{m',n'} - l_{m-m',n-n'}\right|\right)^{2b}} \lesssim Q_{m,n,m',n'}, \qquad (58)
\end{aligned}$$

where in the last step, we used Lemma 26. Therefore, using (56)–(58), the lemma follows. □

Lemma 14 *For $s > 1$,*

$$\sum_{(m',n')\in\mathbb{Z}^2_{0,n,>}} \frac{\rho_{m,n}}{\rho_{m',n'}\rho_{m-m',n-n'}} \lesssim 1. \qquad (59)$$

Proof We define

$$\begin{aligned}
P_1 &:= \left\{(m',n') \in \mathbb{Z}^2_{0,n} : (m')^2 + (n')^2 \geqslant \frac{1}{4}(m^2+n^2)\right\}, \\
P_2 &:= \left\{(m',n') \in \mathbb{Z}^2_{0,n} : (m-m')^2 + (n-n')^2 \geqslant \frac{1}{4}(m^2+n^2)\right\}.
\end{aligned}$$

It is clear[3] from geometry that $\mathbb{Z}^2_{0,n} = P_1 \cup P_2$. By changing indices $(m',n') \to (m-m', n-n')$, it is clear that $\sum_{(m',n')\in P_1}$ and $\sum_{(m',n')\in P_2}$ contribute equally. However, because of the equivalence of norms

$$\begin{aligned}
\sum_{(m',n')\in\mathbb{Z}^2_{0,n}} \frac{\rho_{m,n}}{\rho_{m',n'}\rho_{m-m',n-n'}} &\lesssim \sum_{(m',n')\in P_1} \frac{\rho_{m,n}}{\rho_{m',n'}\rho_{m-m',n-n'}} \lesssim \left(\sum_{(m',n')\in P_1} \frac{1}{\left(|n-n'| + |m-m'|\right)^{2s}}\right) \\
&\lesssim \sum_{(m',n')\in\mathbb{Z}^2_{0,n}} \frac{1}{\left(|n'|^2 + |m'|^2\right)^s} \lesssim \int_1^\infty r^{1-2s} dr \lesssim 1. \qquad (60)
\end{aligned}$$

□

Proposition 15 *Under conditions of Proposition 3, if $|m| < |n|^{2-2(b'-b)}$,*

$$\sup_{\tau\in\mathbb{R}} \sum_{(m',n')\in\mathbb{Z}^2_{0,n}} \int_{\mathbb{R}} \frac{m^2 W_{m,n}(\tau;b')d\tau_1}{\left(n^2 + |\tau - l_{m,n}|\right)^2 W_{m',n'}(\tau_1;b)W_{m-m',n-n'}(\tau-\tau_1;b)} \lesssim \frac{m^2}{n^{4-4(b'-b)}} \leqslant 1.$$

[3] Note that $P_1 \cap P_2 \neq \{0\}$, but this has no bearing on the proof.

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Proof This follows from applying Lemmas 13 and 14 and noting that

$$Q_{m,n,m',n'} \leqslant \frac{1}{|n|^{4-4b'+4b}}.$$

□

Remark 9 *Because of Proposition 15, in the remaining, we now only need to consider* $|m| \geqslant n^{2-2(b'-b)}$ *to complete the proof of Proposition 3. For that reason, we may assume from this point onwards,* $m \neq 0$. *For each such* (m,n), *we break up summation index set* $\mathbb{Z}^2_{0,n,>}$ *into smaller sets, each of which requires a different argument. We note that the partition set for summation indices depends on* (m,n); *however, since the bound on contribution of each such set to the summation is found independent of* (m,n), *the final result does not depend on the partition. First, consider the subset in* $\mathbb{Z}^2_{0,n,>}$ *for which* $|m'| > \frac{3}{2}|m|$ *in the following Lemma.*

Definition 16 *For each* $(m,n) \in \mathbb{Z}_{0,n}$, *define corresponding set*

$$\mathscr{B}_1 := \left\{ (m',n') \in \mathbb{Z}^2_{0,n,>} : |m'| > \frac{3}{2}|m| \right\}.$$

Lemma 17 *For* $s > 2$, $m \neq 0$,

$$\sum_{(m',n')\in\mathscr{B}_1} \frac{m^2 \rho_{m,n} Q_{m,n,m',n'}}{\rho_{m',n'}\rho_{m-m',n-n'}} \lesssim 1. \tag{61}$$

Proof We note that since $Q_{m,n,m',n'} \leqslant \frac{1}{|n|^{4-4(b'-b)}}$,

$$\begin{aligned}
&\sum_{(m',n')\in\mathscr{B}_1} \frac{m^2 \rho_{m,n} Q_{m,n,m',n'}}{\rho_{m',n'}\rho_{m-m',n-n'}} \\
&\lesssim \frac{1}{|n|^{4-4b'+4b}} \sum_{|m'|>\frac{3}{2}|m|, n'/n>\frac{1}{2}} \frac{m^2(|n|+|m|)^{2s}}{(|n|/2+|m'|)^{2s}(|n-n'|+|m-m'|)^{2s}} \\
&\lesssim \frac{m^2}{|n|^{4-4b'+4b}} \sum_{|m'|>\frac{3}{2}|m|} \int_{-\infty}^{\infty} \frac{dr}{(|n-r|+|m'|/3)^{2s}} \\
&\lesssim \frac{m^2}{|n|^{4-4b'+4b}} \sum_{|m'|>\frac{3}{2}|m|} \frac{1}{(|n|+|m'|/3)^{2s-1}} \lesssim \frac{|n|^{4b'-4b}}{n^4|m|^{2s-4}} \lesssim 1.
\end{aligned} \tag{62}$$

□

Definition 18 *We define* $\eta = m'/m$, $\zeta = n'/n$ *and*

$$f(\eta,\zeta) = \frac{(\eta-\zeta)^2}{\zeta^2(1-\zeta)^2}(2\zeta-1)(\eta-g(\zeta)), g(\zeta) = \frac{\zeta(2-\zeta)}{2\zeta-1}. \tag{63}$$

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For given $(m,n) \in \mathbb{Z}_0 \setminus \{m=0\}$, *define*

$$\mathscr{T} = \left\{(m',n') \in \mathbb{Z}^2_{0,n,>} : |\eta| \leqslant \frac{3}{2},\ \left|(2\zeta-1)\eta - (2\zeta-1)g(\zeta)\right| \leqslant k(n) \text{ or, } \quad |\eta-\zeta| \leqslant k(n)\right\}, \tag{64}$$

where

$$k(n) = \min\left\{\frac{1}{10}, |n|^{-2/3+2b'-2b}\right\}. \tag{65}$$

$\mathscr{T}^c$ *will denote the complementary set in* $\mathbb{Z}^2_{0,n} \cup \left\{(m',n') : |m'| \leqslant \frac{3}{2}|m|\right\}$, *i.e.*

$$\mathscr{T}^c = \left\{(m',n') \in \mathbb{Z}^2_{0,n,>} : |\eta| \leqslant \frac{3}{2},\ \left|(2\zeta-1)\eta - (2\zeta-1)g(\zeta)\right| > k(n) \text{ and, } |\eta-\zeta| > k(n)\right\}. \tag{66}$$

Lemma 19 *For* $(m',n') \in \mathscr{T}^c$, *there exists non-zero constant* C_0 *so that*

$$\left|l_{m,n} - l_{m-m',n-n'} - l_{m,n}\right| \geqslant \frac{C_0 k^3 |m|^3}{n^2}.$$

Proof Calculation shows

$$l_{m,n} - l_{m-m',n-n'} - l_{m',n'} = \frac{\gamma m^3}{n^2} f(\eta,\zeta).$$

First, we consider $\frac{1}{2} \leqslant \zeta \leqslant 5$. In that case,

$$\frac{1}{|f(\eta,\zeta)|} = \left|\frac{\zeta^2(1-\zeta)^2}{(\eta-\zeta)^2\left[(2\zeta-1)\eta - (2\zeta-\zeta^2)\right]}\right| \leqslant \frac{1}{k^3}\left(|\zeta|^2|1-\zeta|^2\right) \lesssim \frac{1}{k^3}.$$

Now, if $\zeta > 5$, then since $|\eta| \leqslant \frac{3}{2}$, we have

$$\frac{1}{|f(\eta,\zeta)|} \leqslant \left|\frac{\zeta^2(1-\zeta)^2}{(\zeta-\frac{3}{2})^2\left[\zeta^2 - 2\zeta - \frac{3}{2}(2\zeta-1)\right]}\right| \lesssim 1$$

from which the Lemma follows. □

Lemma 20 *Under conditions given in Proposition 3, if* $|m| \geqslant |n|^{2-2b'+2b}$,

$$\sum_{(m',n') \in \mathscr{T}^c} \frac{m^2 \rho_{m,n} Q_{m,n,m',n'}}{\rho_{m',n'} \rho_{m-m',n-n'}} \lesssim 1. \tag{67}$$

Proof First, there is nothing to prove for $m = 0$, since it is obvious left side is zero. So, we will assume $m \neq 0$. We first prove

$$\sup_{(m',n') \in \mathscr{T}^c} m^2 Q_{m,n,m',n'} \lesssim 1. \tag{68}$$

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Applying Lemma 19, the left side of (67) is bounded by a constant multiple of

$$\frac{m^2}{|n|^{4b}\left(\frac{k^3|m|^3}{n^2}\right)^{2(1-b')}} \lesssim 1, \tag{69}$$

under given restriction on b, b', implying (68). Further applying Lemma 14, the result easily follows. □

Definition 21 *For given* $(m,n) \in \mathbb{Z}_0^2 \setminus \{m=0\}$, *define*

$$\mathscr{S}_1 = \left\{(m',n') \in \mathbb{Z}^2_{0,n,>} : |1-\eta| \leqslant k_1(n)\right\},$$

$$\mathscr{S}_0 = \left\{(m',n') \in \mathbb{Z}^2_{0,n,>} : |\eta| \leqslant k_1(n)\right\},$$

$$k_1(n) = \frac{1}{10|n|},$$

and $\mathscr{S} = \mathscr{S}_0 \cup \mathscr{S}_1$. *We denote* $\mathscr{S}^c$ *to be its complement in the set* $\mathbb{Z}^2_{0,n,>} \cap \{(m',n') : |m'| \leqslant \frac{3}{2}|m|\}$, *i.e.*

$$\mathscr{S}^c = \left\{(m',n') \in \mathbb{Z}^2_{0,n,>} : \frac{3}{2} \geqslant |\eta| > k_1(n), |1-\eta| > k_1(n)|\right\}.$$

Lemma 22 *Under conditions of Proposition 3, if* $|m| \geqslant |n|^{2-2b'+2b}$,

$$\sum_{(m',n')\in\mathscr{S}^c} \frac{m^2 \rho_{m,n} Q_{m,n,m',n'}}{\rho_{m',n'}\rho_{m-m',n-n'}} \lesssim 1. \tag{70}$$

Proof We note that

$$\begin{aligned}
&\sum_{(m',n')\in\mathscr{S}^c} \frac{m^2 \rho_{m,n} Q_{m,n,m',n'}}{\rho_{m',n'}\rho_{m-m',n-n'}} \\
&\lesssim \left\{\int_{-\frac{3}{2}}^{-k_1} + \int_{k_1}^{1-k_1} + \int_{1+k_1}^{\frac{3}{2}}\right\} \int_{\frac{1}{2}}^{\infty} \frac{|m|^3|n| Q_{m,n,m',n'}\rho_{m,n} d\zeta d\eta}{(1+|\eta||m|+|\zeta||n|)^{2s}(1+|1-\eta||m|+|1-\zeta||n|)^{2s}} \\
&\lesssim \frac{|m|^3}{|n|^{3-4b'+4b}} \left\{\int_{-\frac{3}{2}}^{-k_1} + \int_{k_1}^{1-k_1} + \int_{1+k_1}^{\frac{3}{2}}\right\} \int_{\frac{1}{2}}^{\infty} \frac{\rho_{m,n} d\zeta d\eta}{(|n|+|\eta||m|)^{2s}(1+|1-\eta||m|+|1-\zeta||n|)^{2s}} \\
&\lesssim \frac{\rho_{m,n}|m|^3}{|n|^{4-4b'+4b}} \left\{\int_{-\frac{3}{2}}^{-k_1} + \int_{k_1}^{\frac{1}{2}} + \int_{\frac{1}{2}}^{1-k_1} + \int_{1+k_1}^{\frac{3}{2}}\right\} \frac{d\eta}{(|n|+|\eta||m|)^{2s}(1+|1-\eta||m|)^{2s-1}} \\
&\lesssim \frac{|n|^{4b'-4b}}{|n|^4|k_1|^{2s-1}|m|^{2s-4}} \lesssim 1,
\end{aligned} \tag{71}$$

under the restriction given where in the last step we used the fact that in each of the four integration range, only one of the $\rho_{m,n}(|n|+|\eta||m|)^{-2s} \lesssim 1$ or $\rho_{m,n}(1+|1-\eta||m|)^{-2s} \lesssim 1$, while the remaining factor in the integrand is $O((|k_1||m|)^{-2s+1})$. □

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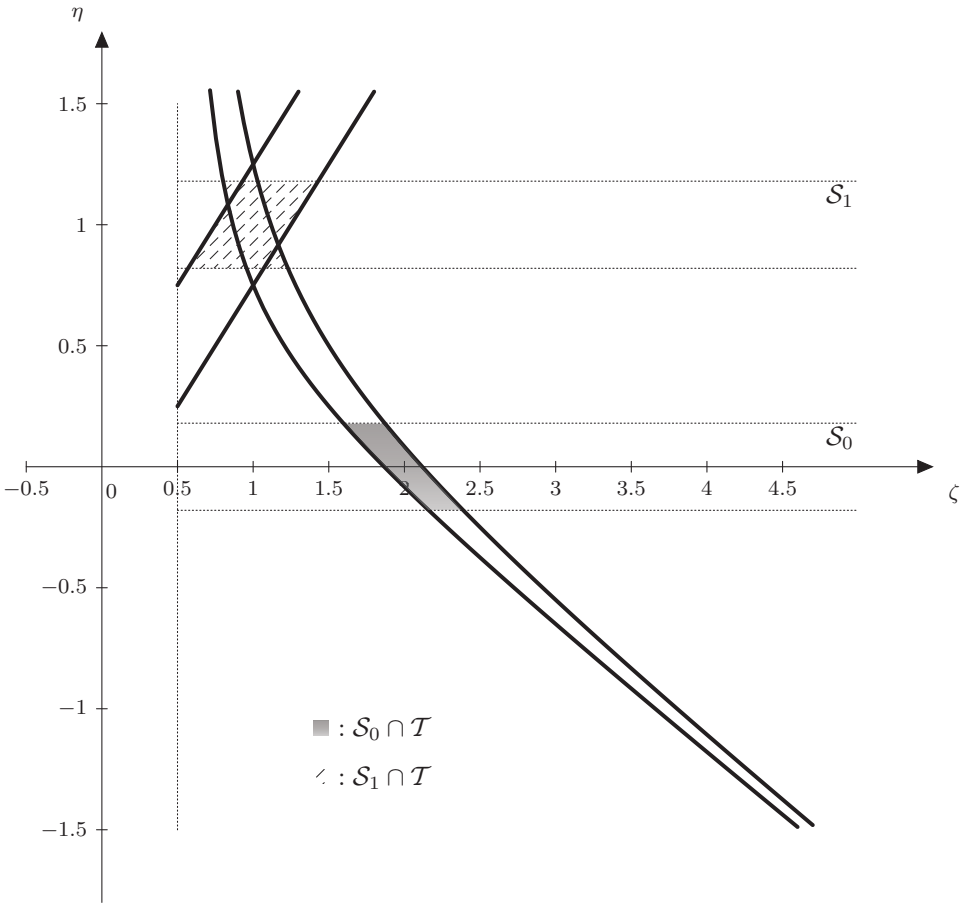


FIGURE 1. (η,ζ)-plane region corresponding to partition of $\mathbb{Z}^2_{0,n,>} \setminus \mathscr{B}_1$. $\mathscr{T}$ is the narrow region between solid lines.

Remark 10 *The previous two Lemmas give control over* $\mathscr{S}^c \cup \mathscr{T}^c$. *The complement set* $\mathscr{S} \cap \mathscr{T}$ *has two components:* $\mathscr{S}_1 \cap \mathscr{T}$, *that corresponds to a neighbourhood of* $(\eta,\zeta) = (1,1)$ *and* $\mathscr{S}_0 \cap \mathscr{T}$ *that includes a neighbourhood of* $(\eta,\zeta) = (0,2)$.

Lemma 23 *Under conditions of Proposition 3, if* $|m| \geqslant |n|^{2-2b'+2b}$,

$$\sum_{(m',n')\in\mathscr{S}_1\cap\mathscr{T}} \frac{m^2 Q_{m,n,m',n'}\rho_{m,n}}{\rho_{m',n'}\rho_{m-m',n-n'}} \lesssim 1. \tag{72}$$

Proof We note for $(m',n') \in \mathscr{S}_1 \cap \mathscr{T}$, (η,ζ) is in some small neighbourhood of $(\eta,\zeta) = (1,1)$ (see top shaded region in Figure 1). For large $|n|$, the size of this region is $O(|n|^{-2/3+2b'-2b})$ horizontally and $O(|n|^{-1})$ vertically. It is useful to write $(\eta,\zeta) = (1+\delta, 1+\Delta)$ and then it

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is clear that

$$f(\eta,\zeta)=\frac{(\Delta-\delta)^2}{\zeta^2\Delta^2}(1+2\Delta)\left(\frac{\Delta(2+\Delta)}{1+2\Delta}+\delta\right).$$

Note $|\delta|\leqslant 1/(10|n|)$ and we can assume that for (η,ζ) with corresponding $(m',n')\in\mathscr{S}_1\cap\mathscr{T}$, $|\Delta|<\frac{2}{9}$ and given the discreteness of n', $|\Delta|\geqslant\frac{1}{|n|}$; hence, $|\delta|\leqslant\frac{|\Delta|}{10}$. It is then clear from the above expression for f that $|f(\eta,\zeta)|\gtrsim|\Delta|\geqslant\frac{1}{|n|}$, implying that

$$m^2Q_{m,n,m',n'}\lesssim m^2|n|^{-4b}\left(n^2+\frac{|m|^3}{|n|^3}\right)^{2b'-2}\lesssim\frac{|n|^{6-6b'-4b}}{|m|^{4-6b'}}\lesssim 1.$$

Therefore, from applying Lemma 14, it follows that

$$\sum_{(m',n')\in\mathscr{S}_1\cap\mathscr{T}}\frac{m^2Q_{m,n,m',n'}\rho_{m,n}}{\rho_{m',n'}\rho_{m-m',n-n'}}\lesssim 1.$$

□

Lemma 24 *Under conditions of Proposition 3, if* $|m|\geqslant|n|^{2-2b'+2b}$, *then*

$$\sum_{(m',n')\in\mathscr{S}_0\cap\mathscr{T},n'\neq 2n}\frac{m^2Q_{m,n,m',n'}\rho_{m,n}}{\rho_{m',n'}\rho_{m-m',n-n'}}\lesssim 1. \tag{73}$$

Proof We note that for $(m',n')\in\mathscr{S}_0\cap\mathscr{T}$, $n'\neq 2n$, (η,ζ) is in a small neighbourhood of $(0,2)$ not including $(0,2)$ itself, and for large n, $|\eta|=O(|n|^{-1})$ and $|2-\zeta|=O(|n|^{-2/3+2b'-2b})$. It is convenient to write $(\eta,\zeta)=(\delta,2+\Delta)$, and note that from discreteness of n', $|\Delta|\geqslant\frac{1}{|n|}$. Then, we find

$$f(\eta,\zeta)=\frac{(2+\Delta-\delta)^2}{\zeta^2(\zeta-1)^2}\left(2\Delta+\delta(2\Delta+3)+\Delta^2\right). \tag{74}$$

From the choice of $k_1(n)$ and the $\frac{1}{|n|}$ lower bound of $|\Delta|$, it follows that $|\delta|\leqslant\frac{|\Delta|}{10}$, and therefore it is clear from above that $|f(\eta,\zeta)|\gtrsim|\Delta|\geqslant\frac{1}{|n|}$ and thus $Q_{m,n,m',n'}\lesssim|m|^{6b'-6}|n|^{6-6b'-4b}$. Therefore, it follows that

$$m^2Q_{m,n,m',n'}\lesssim\frac{|n|^{6-6b'-4b}}{|m|^{4-6b'}}\lesssim 1.$$

Therefore, from applying Lemma 14, it follows that

$$\sum_{(m',n')\in\mathscr{S}_0\cap\mathscr{T},n'\neq 2n}\frac{m^2Q_{m,n,m',n'}\rho_{m,n}}{\rho_{m',n'}\rho_{m-m',n-n'}}\lesssim 1.$$

□

Remark 11 *Though the set* $\{(m',n')\in\mathscr{S}_0\cap\mathscr{T}:n'=2n\}$ *is not covered by Lemma 24, we don't have to worry about this since control over the set* $\mathscr{B}_0$ *has already been shown in Lemma 12.*

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Proof of Proposition 3 Now follows by applying Lemma 9 in 12, and then using Lemma 13 in Proposition 15, Lemmas 17, 20, 22, 23 and 24. In this context, it is useful to note that

$$\mathbb{Z}^2_{0,n,>} = \mathscr{B}_0 \cup \mathscr{B}_1 \cup \mathscr{S}^c \cup \mathscr{T}^c \cup \{\mathscr{S}_1 \cap \mathscr{T}\} \cup \{\{\mathscr{S}_0 \cap \mathscr{T}\} \setminus \mathscr{B}_0\},$$

where $\mathscr{B}_0$, $\mathscr{B}_1$, $\mathscr{S}^c$, $\mathscr{T}^c$, $\mathscr{T}$, $\mathscr{S}_0$ and $\mathscr{S}_1$ are defined in Definitions 10, 16, 21 and 18. □

4 Conclusion

We study a weakly non-linear model which is relevant to rivers experiencing flood conditions when flood onset initiates sediment transport. We prove local well-posedness and determine a condition for global regularity, crucial for the validity of the model, and to its ability to provide physically relevant predictions. Dispersion in this 2+1 diffusive–dispersive equation is not uniform, and delicate estimates are needed to control non-linearity. Global control will be the subject of future work.

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Appendix A

Lemma 25 (Based on Tao [21, Lemma 2.11]) *If ϕ is any smooth function with support in $(-2\delta, 2\delta)$ and $\frac{1}{2} < b \leqslant b' < b + \frac{1}{2}$, with $b' < 1$, then*

$$\int_{\mathbb{R}} \left(n^2 + |\tau - l_{m,n}|\right)^{2b} \left|\mathscr{F}[\phi q](\tau)\right|^2 d\tau \lesssim \delta^{2b'-2b} \int_{\mathbb{R}} \left(n^2 + |\tau - l_{m,n}|\right)^{2b'} \left|\mathscr{F}[q](\tau)\right|^2 d\tau. \quad \text{(A 1)}$$

Proof We define $\Phi(\tau) = \mathscr{F}[\phi](\tau)$ and $Q(\tau) = \mathscr{F}[q](\tau)$. Then, we note that $\mathscr{F}[\phi q](\tau) = [Q * \Phi](\tau)$, where $*$ is the Fourier-Convolution. We decompose

$$Q(\tau) = U^{(1)}(\tau) + U^{(2)}(\tau), \quad \text{(A 2)}$$

where support of $U^{(2)}$ is in $n^2 + |\tau - l_{m,n}| \leqslant \frac{1}{\delta}$ while support of $U^{(1)}$ is in its complement. Then,

$$\left\{\int_{\mathbb{R}} \left(n^2 + |\tau - l_{m,n}|\right)^{2b} \left|\mathscr{F}[\phi q](\tau)\right|^2 d\tau\right\}^{1/2} \lesssim \left\{\int_{\mathbb{R}} \left(n^2 + |\tau - l_{m,n}|\right)^{2b} \left|U^{(1)} * \Phi[\tau]\right|^2 d\tau\right\}^{1/2} + \left\{\int_{\mathbb{R}} \left(n^2 + |\tau - l_{m,n}|\right)^{2b} \left|U^{(2)} * \Phi[\tau]\right|^2 d\tau\right\}^{1/2}. \quad \text{(A 3)}$$

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Now, we note that the first term on the right is given by

$$\begin{aligned}
&\left\{\int_{\mathbb{R}} \left(n^2+|\tau-l_{m,n}|\right)^{2b}\left|\int_{\mathbb{R}} U^{(1)}(\tau-\tau_1)\Phi(\tau_1)d\tau_1\right|^2 d\tau\right\}^{1/2}\\
&\quad\lesssim \int_{\mathbb{R}}\left\{\int_{\mathbb{R}} \left(n^2+|\tau-\tau_1-l_{m,n}|\right)^{2b}\left(1+|\tau_1|\right)^{2b}\left|U^{(1)}(\tau-\tau_1)\Phi(\tau_1)\right|^2 d\tau\right\}^{1/2} d\tau_1\\
&\quad\lesssim \int_{\mathbb{R}}\left\{\int_{\mathbb{R}} \frac{\left(n^2+|\tau-\tau_1-l_{m,n}|\right)^{2b'}}{\left(n^2+|\tau-\tau_1-l_{m,n}|\right)^{2(b'-b)}}\left|U^{(1)}(\tau-\tau_1)\right|^2 d\tau\right\}^{1/2} (1+|\tau_1|)^b|\Phi(\tau_1)|d\tau_1\\
&\quad\lesssim \delta^{b'-b}\left\{\int_{\mathbb{R}} \left(n^2+|\tau-\tau_1-l_{m,n}|\right)^{2b'}\left|U^{(1)}(\tau-\tau_1)\right|^2 d\tau\right\}^{1/2}. \qquad \text{(A 4)}
\end{aligned}$$

For the second part, note that

$$\begin{aligned}
&\int_{\mathbb{R}} \left(n^2+|\tau-l_{m,n}|\right)^{2b}\left|U^{(2)}*\Phi[\tau]\right|^2 d\tau\\
&\quad=\int_{\mathbb{R}} \left(n^2+|\tau-l_{m,n}|\right)^{2b}\left|\int_{\mathbb{R}} U^{(2)}(\tau_1)\Phi(\tau-\tau_1)d\tau_1\right|^2 d\tau\\
&\quad\lesssim \int_{\mathbb{R}}\left|\int_{\mathbb{R}} \left(n^2+|\tau_1-l_{m,n}|\right)^{b}\left(1+|\tau-\tau_1|\right)^{b} U^{(2)}(\tau_1)\Phi(\tau-\tau_1)d\tau_1\right|^2 d\tau\\
&\quad\lesssim \left\{\int_{(n^2+|\tau_1-l_{m,n}|)\leqslant\delta^{-1}} \frac{d\tau_1}{(n^2+|\tau_1-l_{m,n}|)^{2(b'-b)}}\right\}\\
&\qquad\times\left\{\int_{\mathbb{R}}\int_{\mathbb{R}} |U^{(2)}(\tau_1)|^2|\Phi(\tau-\tau_1)|^2\left(1+|\tau-\tau_1|\right)^{2b}\left(n^2+|\tau_1-l_{m,n}|\right)^{2b'} d\tau_1 d\tau\right\}\\
&\quad\lesssim \delta^{2b'-2b-1}\left(\int_{\mathbb{R}}(1+\hat{\tau}^2)|\Phi(\hat{\tau})|^2 d\hat{\tau}\right)\left(\int_{\mathbb{R}} \left(n^2+|\tau_1-l_{m,n}|\right)^{2b'}\left|U^{(2)}(\tau_1)\right|^2 d\tau_1\right)\\
&\quad\lesssim \delta^{2b'-2b-1}\int_{-\delta}^{\delta}\left(|\phi(t)|^2+|\phi'(t)|^2\right)dt\left(\int_{\mathbb{R}} \left(n^2+|\tau_1-l_{m,n}|\right)^{2b'}\left|U^{(2)}(\tau_1)\right|^2 d\tau_1\right)\\
&\quad\lesssim \delta^{2b'-2b}\left(\int_{\mathbb{R}} \left(n^2+|\tau_1-l_{m,n}|\right)^{2b'}\left|Q(\tau_1)\right|^2 d\tau_1\right). \qquad \text{(A 5)}
\end{aligned}$$

Therefore, the Lemma follows by combining the two results. □

Lemma 26 *For b and b' in the interval $(\frac{1}{2},1)$ and $\tau\in\mathbb{R}$,*

$$\frac{1}{\left(k_3+|\tau-k_2|\right)^{2(1-b')}\left(k_3+|\tau-k_4|\right)^{2b}}\leqslant\frac{1}{k_3^{2b}(k_3+|k_2-k_4|)^{2(1-b')}} \qquad \text{(A 6)}$$

and

$$\frac{1}{\left(k_3+|\tau-k_2|\right)^{2b}\left(k_3+|\tau-k_4|\right)^{2(1-b')}}\leqslant\frac{1}{k_3^{2b}(k_3+|k_2-k_4|)^{2(1-b')}}. \qquad \text{(A 7)}$$

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Proof Consider the first statement. Note that $1-b'<b$. Note that we may write the left-hand side as $\frac{1}{[f(\tau)]^{2(1-b')}}$, where

$$f(\tau)=\left(k_3+|\tau-k_2|\right)\left(k_3+|\tau-k_4|\right)^{b/(1-b')}. \tag{A 8}$$

It is clear that in the intervals $\tau<\min\{k_2,k_4\}$, $\tau>\max\{k_2,k_4\}$, $f(\tau)$ is monotonically decreasing and increasing respectively with increasing τ. Therefore, the minimum of $f(\tau)$ occurs for $\tau\in[k_2,k_4]$ when $k_2\leqslant k_4$ and for $\tau\in[k_4,k_2]$ when $k_2>k_4$. Consider first the case when $k_2\leqslant k_4$; then in $\tau\in[k_2,k_4]$, $f(\tau)=f_0(\tau)$, where

$$f_0(\tau)=(k_3+\tau-k_2)\,(k_3+k_4-\tau)^{b/(1-b')}. \tag{A 9}$$

It can be checked that $f_0'(k_4)<0$ and simple calculus shows $f_0'(\tau)<0$ for $\tau\in(\tau_c,k_4]$ and $f_0'(\tau)>0$ for $\tau\in(-\infty,\tau_c)$, where

$$\tau_c=\frac{-k_3(b'+b-1)+k_4(1-b')+k_2b}{1-b'+b}<k_4.$$

We have two cases: (i) $\tau_c<k_2$ and (ii) $\tau_c\in(k_2,k_4)$. In case (i), it is clear that the minimum of $f_0(\tau)$ for $\tau\in[k_2,k_4]$ occurs at $\tau=k_4$. In case (ii), it is clear that $f_0(\tau)$ has a local maximum at $\tau=\tau_c$. Therefore, in the interval $[k_2,k_4]$ minimum of f_0 is either at k_2 or k_4. We notice that

$$\frac{f_0(k_2)}{f_0(k_4)}=\frac{k_3\,(k_3+k_4-k_2)^{b/(1-b')}}{k_3^{b/(1-b')}\,(k_3+k_4-k_2)}=\left(\frac{k_3+k_4-k_2}{k_3}\right)^{(b'+b-1)/(1-b')}>1 \tag{A 10}$$

implying that $f_0(k_2)>f_0(k_4)$. So, the minimum of f in the interval $[k_2,k_4]$ occurs at $\tau=k_4$. This implies that

$$\begin{aligned}\frac{1}{\left(k_2+|\tau-k_2|\right)^{2(1-b')}\left(k_3+|\tau-k_4|\right)^{2b}}&=\frac{1}{[f(\tau)]^{2(1-b')}}\leqslant\frac{1}{[f(k_4)]^{2(1-b')}}\\&=\frac{1}{k_3^{2b}(k_3+|k_2-k_4|)^{2(1-b')}}.\end{aligned} \tag{A 11}$$

On the other hand, if $k_2>k_4$, in the interval $\tau\in[k_4,k_2]$, $f(\tau)=f_1(\tau)$, where

$$f_1(\tau)=(k_3+k_2-\tau)\,(k_3+\tau-k_4)^{b/(1-b')}. \tag{A 12}$$

Then, $f_1'(\tau_c)=0$, where $\tau_c=k_3(b'+b-1)+k_4(1-b')+k_2b>k_4$. Also, it is to be noted that for $\tau\in(k_4,\tau_c)$, $f'>0$ and $f'<0$ for $\tau\in(\tau_c,\infty)$. Therefore, over the interval $[k_2,k_4]$ minimum can occur at either k_2 or k_4. However, since

$$\frac{f_1(k_2)}{f_1(k_4)}=\left(\frac{k_3+k_2-k_4}{k_3}\right)^{(b'+b-1)/(1-b')}>1. \tag{A 13}$$

Therefore, once again, the minimum of $f(\tau)$ is at $\tau=k_4$ as before and we have the same result and the first statement has been proved. For the second part, we simply interchange k_2 and k_4 in the previous argument and the result follows. □

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Lemma 27 (Based on Kenig, Ponce, Vega [16, Lemma 2.3]) *For* $\frac{1}{2} < b < 1$ *and* $k_3 \geqslant k_1 \geqslant 1$, *then the following hold*

$$\int_{\mathbb{R}} \frac{d\tau}{\left(k_1 + |\tau - k_2|\right)^{2b} \left(k_3 + |\tau - k_4|\right)^{2b}} \lesssim \frac{1}{k_1^{2b-1} \left(k_3 + |k_2 - k_4|\right)^{2b}}. \tag{A 14}$$

Proof It suffices to argue only the case $k_2 \leqslant k_4$; since the final result is symmetric in k_2 and k_4; if this were not true, we can simply interchange the k_2 and k_4 in the following argument. We note that

$$\frac{1}{\left(k_1 + |\tau - k_2|\right)^{2b} \left(k_3 + |\tau - k_4|\right)^{2b}} \leqslant \frac{1}{(k_3 + |\tau - k_2|)^{2b} \left(k_1 + |\tau - k_4|\right)^{2b}}, \text{for } \tau \geqslant \frac{k_2 + k_4}{2}. \tag{A 15}$$

Then, the integral becomes

$$\begin{aligned}
&\left\{ \int_{-\infty}^{(k_2+k_4)/2} + \int_{(k_2+k_4)/2}^{\infty} \right\} \frac{d\tau}{\left(k_1 + |\tau - k_2|\right)^{2b} \left(k_3 + |\tau - k_4|\right)^{2b}} \\
&\quad \leqslant \int_{-\infty}^{(k_2+k_4)/2} \frac{d\tau}{\left(k_1 + |\tau - k_2|\right)^{2b} \left(k_3 + |\tau - k_4|\right)^{2b}} \\
&\qquad + \int_{(k_2+k_4)/2}^{\infty} \frac{d\tau}{\left(k_3 + |\tau - k_2|\right)^{2b} \left(k_1 + |\tau - k_4|\right)^{2b}} \\
&\quad \leqslant \frac{1}{(k_3 + |k_2 - k_4|/2)^{2b}} \left\{ \int_{-\infty}^{(k_2+k_4)/2} \frac{d\tau}{\left(k_1 + |\tau - k_2|\right)^{2b}} + \int_{(k_2+k_4)/2}^{\infty} \frac{d\tau}{\left(k_1 + |\tau - k_4|\right)^{2b}} \right\} \\
&\quad \leqslant \frac{1}{(k_3 + |k_2 - k_4|/2)^{2b}} \left\{ \int_{-\infty}^{k_2} \frac{d\tau}{(k_1 - \tau + k_2)^{2b}} + \int_{k_2}^{(k_2+k_4)/2} \frac{d\tau}{(k_1 - k_2 + \tau)^{2b}} \right. \\
&\qquad \left. + \int_{(k_2+k_4)/2}^{k_4} \frac{d\tau}{(k_1 - \tau + k_4)^{2b}} + \int_{k_4}^{\infty} \frac{d\tau}{(k_1 - k_4 + \tau)^{2b}} \right\} \lesssim \frac{1}{k_1^{2b-1} \left(k_3 + |k_2 - k_4|\right)^{2b}}.
\end{aligned} \tag{A 16}$$

□

Lemma 28 (Based on Tao [21, Corollary 2.10]) *If* $b > \frac{1}{2}$, *then*

$$\sup_t \left| M(t) \right|^2 \lesssim \frac{1}{|n|^{4b-2}} \int_{\mathbb{R}} \left(n^2 + |\tau - l_{m,n}|\right)^{2b} \left| \mathscr{F}[M](\tau) \right|^2 d\tau. \tag{A 17}$$

Proof We note that

$$M(t) = \int_{\mathbb{R}} e^{i\tau t} \mathscr{F}[M](\tau) d\tau. \tag{A 18}$$

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With substitution $\tau = \tau_0 + l_{m,n}$,

$$M(t) = e^{il_{m,n}t} \int_{\mathbb{R}} e^{i\tau_0 t} \mathscr{F}[M](\tau_0 + l_{m,n}) d\tau_0. \tag{A 19}$$

So, for any $t \in \mathbb{R}$,

$$\begin{aligned} |M(t)|^2 &= \Big| \int_{\mathbb{R}} e^{i\tau_0 t + il_{m,n}t} \mathscr{F}[M](\tau_0 + l_{m,n}) d\tau_0 \Big|^2 \\ &\leqslant \left(\int_{\mathbb{R}} \left| \mathscr{F}[M](\tau_0 + l_{m,n}) \right|^2 \left(n^2 + |\tau_0|\right)^{2b} d\tau_0 \right) \cdot \int_{\mathbb{R}} \frac{d\tau_0}{(n^2 + |\tau_0|)^{2b}} \\ &\lesssim \frac{1}{|n|^{4b-2}} \int_{\mathbb{R}} \left| \mathscr{F}[M](\tau) \right|^2 \left(n^2 + |\tau - l_{m,n}|\right)^{2b} d\tau. \end{aligned} \tag{A 20}$$

Since the right side is independent of t, the lemma follows. □

Lemma 29 (Kenig, Ponce, Vega [16, Lemma 2.4])

$$\int_{\mathbb{R}} \frac{d\xi_1}{\left(1 + |\tau - \xi_1^3 - (\xi - \xi_1)^3|\right)^{2b}} \lesssim \frac{1}{\sqrt{|\xi|}\left(1 + |4\tau - \xi^3|\right)^{1/2}}. \tag{A 21}$$

Proof We introduce change of variables

$$\mu = \tau - \xi_1^3 - (\xi - \xi_1)^3. \tag{A 22}$$

Then, it is easily checked that

$$d\mu = 3\xi(\xi - 2\xi_1) d\xi_1, \xi_1 = \frac{1}{2}\left[\xi \pm \sqrt{\frac{4\tau - \xi^3 - 4\mu}{3\xi}}\right]$$

and so

$$|\xi(\xi - 2\xi_1)| = 3^{-1/2}\sqrt{|\xi|}\sqrt{|4\tau - \xi^3 - 4\mu|}$$

and therefore

$$\begin{aligned} \int_{\mathbb{R}} \frac{d\xi_1}{\left(1 + |\tau - \xi_1^3 - (\xi - \xi_1)^3|\right)^{2b}} &\leqslant \frac{1}{\sqrt{3|\xi|}} \int_{\mathbb{R}} \frac{d\mu}{\left(4\tau - \xi^3 - 4\mu\right)^{1/2} (1 + |\mu|)^{2b}} \\ &\lesssim \frac{1}{\sqrt{|\xi|}\left(1 + |4\tau - \xi^3|\right)^{1/2}}. \end{aligned} \tag{A 23}$$

□

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