Locating parameters of interest in a conic section

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In an interesting Note [1] a method for constructing the directrices of a conic section using simple tools, viz. a pencil, a pair of compasses and a straight edge, was presented and it was wondered if the method might have been known before. From earlier days at school this author was aware that for a specific conic section (parabola, ellipse or hyperbola) displayed without reference on a piece of paper it was possible to determine not just the directrices but also the focal points and the usual axis set, again using just the simple tools. Others might also be aware of these facts, but if they are not so well known it might be of interest to present an argument in recipe format that will enable readers to appreciate a route to fathom the business, although there might well be other paths leading also to the same end-result, depending on the starting point.

The usual parameters and descriptions associated with conic sections will be taken for granted. Typically, these are the distances a and b (where appropriate) and the associated eccentricity e. It will be convenient also to recall the familiar rectangular Cartesian coordinate system description for each conic section, as follows:

I) Parabola: $y^2 = 4ax$; $e = 1$.
Focus is the point $(a, 0)$.
Directrix is the line $x = -a$.

II) Ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$; $ae = \sqrt{a^2 - b^2}$; $0 \leqslant e < 1$.
Foci are the points $(-ae, 0)$ and $(ae, 0)$.
Directrices are the lines $x = -\dfrac{a}{e}$ and $x = \dfrac{a}{e}$, respectively.

III) Hyperbola: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$; $ae = \sqrt{a^2 + b^2}$; $1 < e$.
Foci are the points $(-ae, 0)$ and $(ae, 0)$.
Directrices are the lines $x = -\dfrac{a}{e}$ and $x = \dfrac{a}{e}$, respectively.

Many of the standard geometrical propositions relating to conic sections can be established also using methods of algebraic geometry and trigonometry.

Each conic section here will be treated separately. In each case it will be necessary to establish first the reference frame for the section, and a common starting point is to use the simple tools to construct arbitrary parallel chords to the section and invoke commensurate propositions such as those expounded in [2]. Thereafter, other known properties can be called upon to locate points and lines of interest using such tools.

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I) *Parabola*

1. Draw two arbitrary, parallel chords of the parabola.
2. Find the midpoints of each chord and draw a straight line through them. This chord-midpoint line will be parallel to the axis of symmetry of the parabola (Proposition XV, [2, p. 20]) and it will meet the parabola again at some point. At this point, draw a line that is parallel to the two chords above. This is a tangent to the parabola.
3. From the above tangent point, drop a vertical line that is perpendicular to the chord-midpoint line, to meet the parabola again at a new point and form the line joining these two points. Determine this line's midpoint and then draw an orthogonal line through this midpoint (it will be parallel to the above chord-midpoint line). This new line is the axis of symmetry of the parabola – the x-axis, say – and it will meet the parabola at some point (the origin) through which a perpendicular can be constructed. This is the y-axis.

Steps 1, 2 and 3 are illustrated in Figure 1.

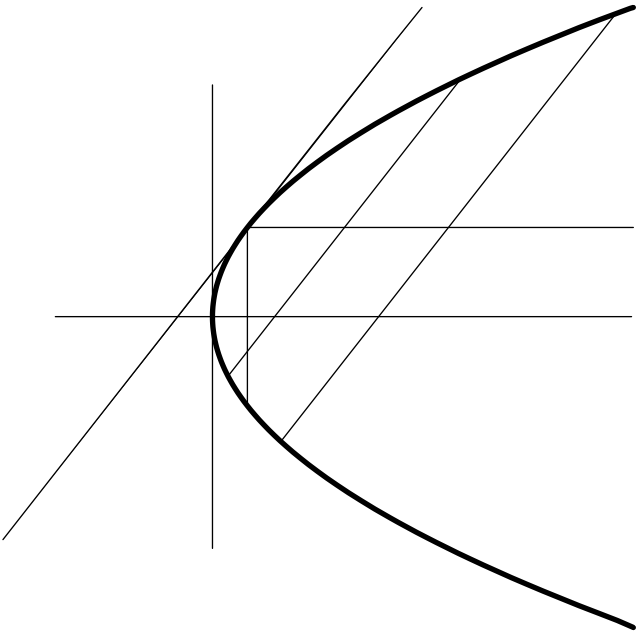


FIGURE 1: Parallel chords, midpoint line, tangent, vertical chord and axs set for the parabola

4. At the tangent point in 2, draw a line perpendicular to the tangent. This is a normal to the parabola, and it can be viewed at the point of tangency as comprising an 'inward normal' pointing into the region containing the focus and an 'outward normal' pointing in the opposite direction.

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The inward normal forms an angle with the associated chord-midpoint line. Reproduce this angle on the other side of the inward normal and mark where the line so produced crosses the axis of symmetry. This is the focal point of the parabola. Physicists will appreciate this as a manifestation of Snell's Law (angle of incidence equals the angle of reflection).

5. Mark off the focal distance from the origin and reproduce a point at the same distance on the axis of symmetry in the half space that does not contain the parabola. Draw a line through this point perpendicular to the axis of symmetry. This line is the directrix of the parabola.

Steps 4 and 5 are illustrated in Figure 2.

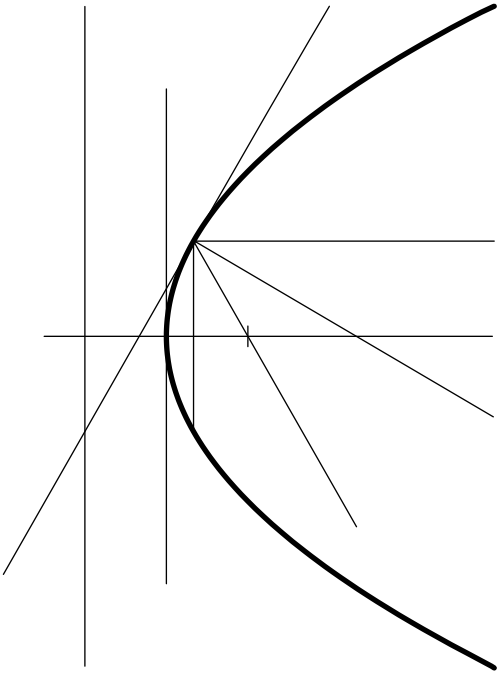


FIGURE 2: The inward normal, the line through the focal point on the axis of symmetry and the directrix of the parabola

II) *Ellipse*

1. Draw two arbitrary parallel chords to the ellipse.
2. Bisect the parallel chords and join the midpoints to form a chord-midpoint line. The line so formed passes through the centre of the ellipse (Proposition XXIV, [2, p. 62]) and meets the ellipse at two other points; the line joining them can be bisected to obtain the centre point. Typically, it is the origin of the ellipse coordinate system. Alternatively, draw another two parallel chords and form another chord-

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midpoint line. The point of intersection of the two chord-midpoint lines is also the centre of the ellipse.

Steps 1 and 2 are illustrated in Figure 3.

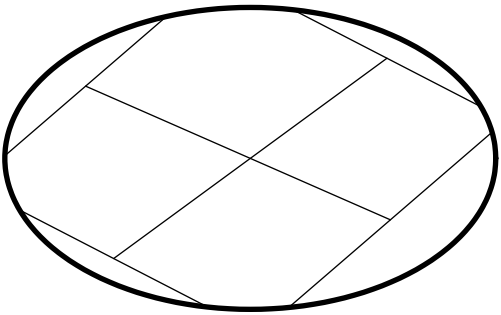


FIGURE 3: Two sets of parallel chords of an ellipse and lines joining their respective midpoints, intersecting at the ellipse centre

3. Draw a circle centred at the origin, to intersect the ellipse in four points that by symmetry form a rectangle. Opposite sides of the rectangle will be parallel to the principal axes of the ellipse. Accordingly, these axes may be constructed in turn by determining the midpoints of opposite sides of the rectangle and joining them. The points where the principal axes meet the ellipse will determine the lengths $2a$ and $2b$ of its axes (typically along the x- and y-axes).

Step 3 is illustrated in Figure 4.

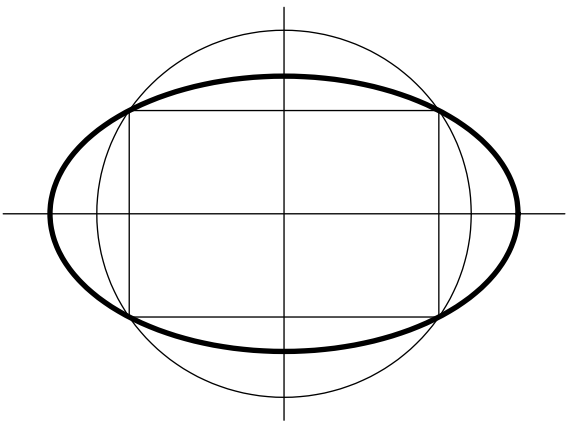


FIGURE 4: The ellipse intersecting circle and inscribed rectangle with sides parallel to the ellipse principal axes

4. Draw a circle of radius b centred at the origin and construct a tangent from one end of the major axis to meet this circle. By inspection, the distance from the end of the major axis to the point of tangency is simply $\sqrt{a^2 - b^2}$, which is also the focal distance from the origin. Mark

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off this distance from the origin in both directions along the major axis to obtain the ellipse focal point positions.

5. Extend the above tangent to meet the minor axis at some point. By simple trigonometry, the distance from this point to the end of the major axis is $\frac{a}{e}$, which is also the distance of the directrix from the origin. This distance can then be marked off from the origin along the major axis, in both directions, to determine the points where the directrix crosses the major axis. Perpendiculars through these points can be drawn to obtain the directrix lines.

Steps 4 and 5 are illustrated in Figure 5.

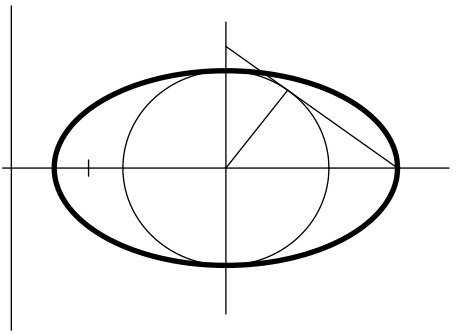


FIGURE 5: The inscribed circle on the minor axis, a tangent from one end of the major axis, one representative focal point and its associated directrix

III) *Hyperbola*

1. Draw an arbitrary chord on one branch of the hyperbola and then draw a parallel chord on the same branch or on the opposite branch.
2. Bisect each chord to determine its midpoint and join the two midpoints to form a chord-midpoint line. The line so formed passes through the centre of the hyperbola (Proposition XXV, [2, p. 107]) and meets the hyperbola at two other points; the line joining these two other points can be bisected to obtain the centre point. Typically, it is the origin of the hyperbola coordinate system. Alternatively, draw another two parallel chords and form their chord-midpoint line. The point of intersection of the two chord-midpoint lines is also the centre of the hyperbola.

Steps 1 and 2 are illustrated in Figure 6.

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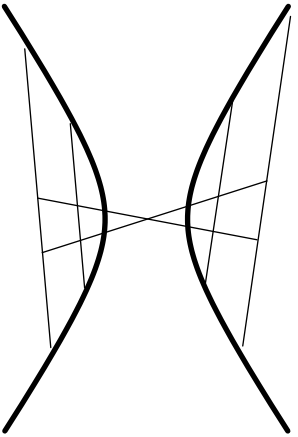


FIGURE 6: Bisectors of parallel chords meeting at the centre (origin) of the hyperbola

3. Draw a circle centred at the origin, to intersect the hyperbola in four points that by symmetry form a rectangle. Opposite sides of the rectangle will be parallel to the principal axes of the hyperbola. Accordingly, these axes may be constructed in turn by determining the midpoints of opposite sides of the rectangle and joining them. The points where one principal axis meets the hyperbola will determine the semi-length of the major axis (typically the x-axis), that is to say, the length a. The other principal axis, the minor or y-axis, does not meet the hyperbola.

Step 3 is illustrated in Figure 7.

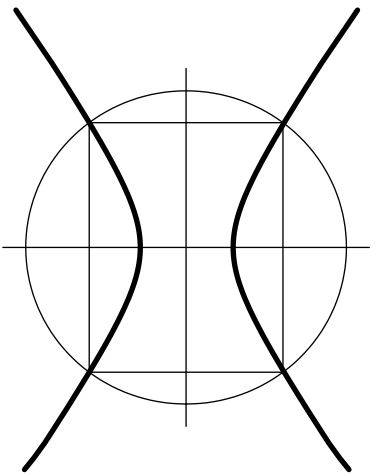


FIGURE 7: The hyperbola intersecting circle and inscribed rectangle with sides parallel to the hyperbola principal axes

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4. To appreciate the other principal distance, b, mark first a distance a from the origin along the minor axis. The distance from this point, so marked, to the point where the hyperbola met the major axis (also at the distance a from the origin) is thus of length $a\sqrt{2}$ (Pythagoras' Theorem). This length can be struck off from the origin to a point along the major axis. Draw a perpendicular through this point to meet the hyperbola at some further point. The distance between these last two points is the minor axis length b. Mark this distance off from the origin along the minor axis.
5. Join the above two points (one on the major and one on the minor axes), and join each to the origin so as to form a right-angled triangle. The hypotenuse of this triangle is $\sqrt{a^2 + b^2}$, which is also the distance ae of the focus from the origin; it can be marked off from the origin along the major axis in both directions to produce a focal point for each branch of the hyperbola.
6. Construct a perpendicular from the origin to meet the hypotenuse in 4 at some point. The distance from this point to the end of the hypotenuse on the major axis is, by simple trigonometry, $\frac{a}{e}$ which is also the distance of a directrix from the origin; this distance can be marked off from the origin along the major axis in both directions to produce points through which directrices must pass. At each of these points, construct a line perpendicular to the major axis. These lines are the directrices of the hyperbola.

Steps 4, 5 and 6 are illustrated in Figure 8.

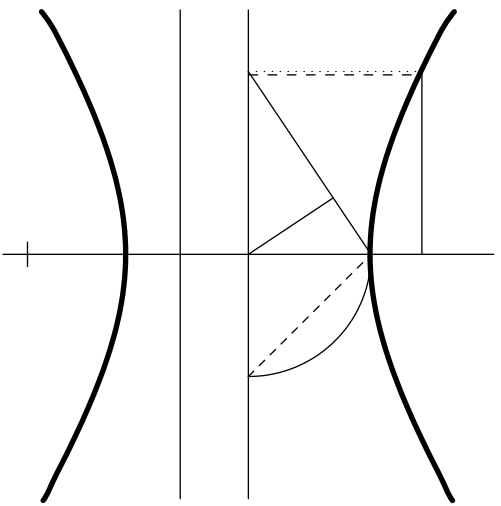


FIGURE 8: The final construction leading to the siting of one focus and one directrix for the hyperbola (symmetry can locate the other focus and directrix)

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Conclusion

Following a comment in an earlier *Gazette* article, concerning the manner in which the directrices of a conic section might be fathomed using the simple tools of school geometry, it has been shown how, if an accurate pictorial representation of the section is available, the location of both directrices and focal points can be determined using these tools.

Interestingly, the problem of locating the focal point of a parabolic reflector is one that can be faced by an RF engineer performing field site radiation pattern measurements on an antenna under test. This author encountered such a situation when a reflector-antenna (transmitter) mounted on a distant tower was illuminating an antenna under test. The transmitting antenna was a 'rear fed' paraboloid (a parabola of revolution) and energy to illuminate it was delivered via a waveguide that poked, from behind, through a hole centred near the reflector vertex and shaped like a gooseneck to terminate in a flared horn section (feed) at the focal point. Large seagulls driven inland by storms at sea had used the goose-necked end of the waveguide as a sheltering perch during continuing high buffeting winds and this resulted in a bending of the waveguide and subsequent displacement of the feed from its focal point position. The extent of the displacement could be appreciated by dismantling the reflector and taking depth measurements along a line of symmetry in its aperture to facilitate the construction of a simple card template that allowed for the determination of the focal point feed position, as described above. In the event, the feed position was re-assessed and corrected in a workshop prior to continuing with radiation pattern measurements.

References

1. Marcos Alvarez, On conics, *Math. Gaz.* **101** (November 2017), pp. 511-515.
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