—Technical Note—

COMPOUND PATH INTEGRAL SOLUTION OF RESPONSE EXCEEDANCE PROBABILITIES OF AN OFFSHORE STRUCTURE

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ABSTRACT

In this technical note, a compound path integral solution (CPIS) method is utilized to calculate the response exceedance probabilities of a nonlinear compliant offshore structure subjected to slow drift wave force excitations. The structure's slow drift response exceedance probabilities have also been calculated by using the original PIS method for comparison purpose. It is found that the efficiency of the CPIS method for predicting the structure's slow drift response exceedance probabilities is higher than that of the original PIS method.

Keywords : Compound path integral solution, Offshore structure, Exceedance probabilities, Slow drift response.

1. INTRODUCTION

In order to design a compliant offshore structure with high reliability, accurate prediction of the slow drift responses (especially the exceedance probabilities) of the structure system is necessary. This technical note solely deals with the methods for predicting the slow drift response exceedance probabilities of a compliant offshore structure moored in random seas.

Methods based on the Markov diffusion process theory are frequently used for predicting the slow drift responses of a compliant offshore structure. Martinsen [1] has investigated the dynamics of slow-drift oscillations with non-linear restoring forces. The problem was formulated in terms of a three-dimensional Fokker-Planck equation in the Markov process theory and solved by using a Galerkin method. However, the results obtained were unable to predict the response probability densities at high response levels. Johnsen and Naess [2] showed that the path integral solution (PIS) method based on the Markov process theory can be used to solve the full three-dimensional Fokker-Planck equation of similar problems with high accuracy. Naess and Johnsen [3] applied the PIS method based on a cubic B-spline interpolation scheme for calculating the slow drift response statistics of a moored offshore structure. It is shown in their paper [3] that the PIS method is capable of providing response statistics that are accurate to very low probability levels. Recently Karlsen and Naess [4] applied the PIS method based on a cubic B-spline scheme for studying the slow drift response statistics of a moored large volume floating structure. However, it should be noted that cubic B-spline is a high order interpolation scheme, and the computational time for spline interpolation increases exponentially with the number of dimension. Therefore, the amount of computer time required to obtain an accurate estimate of the slow drift response statistics was found to be excessive in [3] and the PIS technique in [3,4] is therefore not well suited to the initial design stage.

Yu *et al*. [5] had proposed a PIS method based on the implicit Gauss-Legendre integration rule for improving the computational efficiency and accuracy of the path integration. Their PIS procedure based on the Gauss-Legendre scheme was applied in studying the slow drift extreme responses of moored ocean structures [6,7], and it was found that the computation efficiency can indeed be improved if the researchers have rich experience on the engineering problem being tackled so that they can make a good estimation of the initial response probability density. If a bad estimation on the initial probability density were made, the computation time for the path integral solution would have been significantly increased. To find an optimal way for choosing the initial probability density in order to further improve the efficiency of the PIS method, the author of this technical note first proposed a compound path integral solution (CPIS) method [8]. In this technical note, the motion response of a compliant offshore structure driven by slowly varying wave drift forces will be modeled as a nonlinear oscillator with Duffing

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type restoring force, and the structure response exceedance probabilities will be calculated by utilizing the CPIS method proposed in [8]. To the best of the author's knowledge, this is the first effort to apply the CPIS method based on the Gauss-Legendre scheme for predicting the response exceedance probabilities of a compliant offshore structure. The work presented here will likely to offer a valuable tool to the practicing engineers for predicting slow drift response exceedance probabilities of an offshore structure in the initial design stage.

2. THE DYNAMIC MODEL OF THE SLOW DRIFT MOTION

The example studies in this technical note are concerned with the horizontal wave drift motions of a compliant offshore structure moored in random seas. The nonlinear random differential equation of motion of the structure can be written in the form [8]:

$$M\ddot{x} + D\dot{x} + \rho C_D R\dot{x}\,|\,\dot{x}\,| + K_1 x + K_3 x^3 = cW(t) \qquad (1)$$

This equation may represent the surge motion of a moored SPAR platform or a semi-submersible. In the above equation, M is the structure total mass; x is the structure surge displacement; the symbol $(\dot{})$ denotes differentiation with respect to time; D is the wave drift damping coefficient; ρ is the sea water density; C_D is the Morrison drag coefficient; R is the structure characteristic dimension; K_1 is the linear restoring force coefficient; K_3 is the nonlinear restoring force coefficient; $W(t)$ is a unit Gausssian white noise; c is the strength of the white noise. Figure 1 shows the simulated time histories of the slow drift wave force random excitation $W(t)$.

Writing the quadratic damping term in Eq. (1) as the sum of a linear term and a cubic term as in [9] and rearranging the equation, we obtain:

$$\ddot{x} + \left\{ D/M + \left(\rho C_D R\sqrt{\frac{2}{\pi}} \right)/M \right\} \dot{x} + \left\{ \frac{1}{3M} \rho C_D R\sqrt{\frac{2}{\pi}} \right\} \dot{x}^3 + \frac{K_1}{M} x + \frac{K_3}{M} x^3 = \frac{c}{M} W(t) \qquad (2)$$

To get the surge response exceedance probabilities of the compliant offshore structure, the author of this technical note once used a path integral solution (PIS) method [7]. However, the efficiency of applying the PIS method is largely dependant on whether a good estimation on the initial response probability density values can be made or not. Recently, the author of this technical note first proposed a compound path integral solution (CPIS) method [8] for predicting the slow drift response statistics of a compliant offshore structure. The accuracy and efficiency of the method of compound path integral solution have been validated in [8]. In the following sections, the theoretical background of the CPIS method will be briefly introduced, and the response exceedance probabilities of a compliant offshore structure driven by slowly varying wave drift forces will be calculated by utilizing the compound path integral solution method.

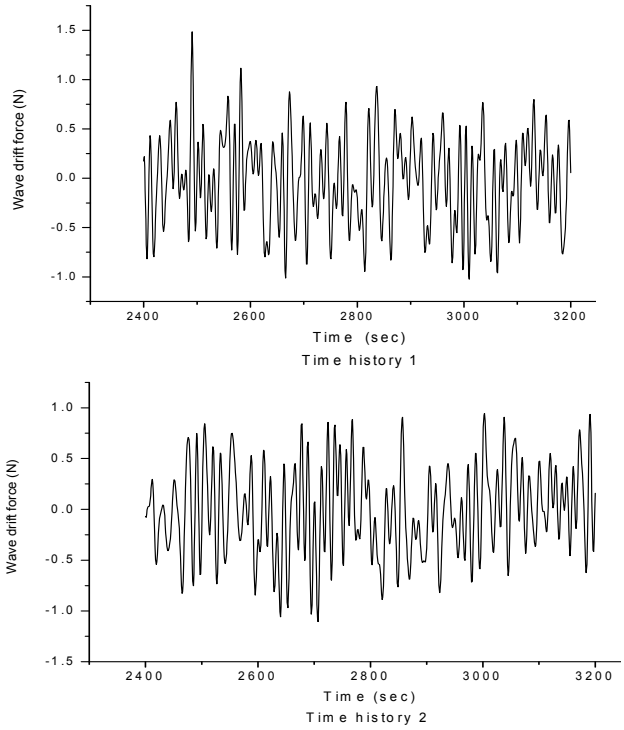


Fig. 1 Simulated time histories of the slow drift wave force $W(t)$

3. THEORETICAL BACKGROUND OF THE COMPOUND PATH INTEGRAL SOLUTION

The approach of the path integral solution is based on the path integral representation of the Fokker-Plank equation, which is the governing equation of the transition probability density for a Markov process. The numerical method involves a stepwise algorithm that computes the evolution of probability distribution $p(x, t)$ from an initial probability distribution $p(x_0, t_0)$, which is estimated by the researcher based on his (her) experience. We can express a typical evolution of a probability from $t_i - 1$ to t_i as follows:

$$p(x^{(i)}, t_i) = \int_R p(x^{(i)}, t_i \,|\, x^{(i-1)}, t_{i-1})\; p(x^{(i-1)}, t_{i-1})\, dx^{(i-1)} \qquad (3)$$

where R is the reduced integration range, provided that the probability evolution outside this range is negligible. The essential features of the Gauss-Legendre-based algorithm will be illustrated for a one-dimensional system. In this case, integral (3) can be discretized into the following composite Gauss-Legendre quadrature form:

$$p(x^{(i)}, t_i) = \sum_{k=1}^{K} \frac{d_k}{2} \sum_{l=1}^{L_k} c_{kl}\; p(x_{kl}^{(i-1)}, t_{i-1})\; p(x^{(i)}, t_i \,|\, x_{kl}^{(i-1)}, t_{i-1}) \qquad (4)$$

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where K is the number of sub-intervals, d_k is the length of the sub-interval k, L_k is the number of quadrature points in sub-interval k. Each c_{kl} is the quadrature coefficient of a Gauss quadrature point, and x_{kl} is the position of the quadrature points. Equation (4) can be used to calculate the probability density at any point $x^{(i)}$ at step i provided that the probability density at the Gauss quadrature points are known at step $i-1$. In order to obtain the probability density at any point $x^{(i+1)}$ at step $(i+1)$, the following probability densities at the Gauss quadrature points need to be calculated:

$$p(x_{mn}^{(i)}, t_i) = \sum_{k=1}^{K} \frac{d_k}{2} \sum_{l=1}^{L_k} c_{kl}\ p(x_{kl}^{(i-1)}, t_{i-1})\ p(x_{mn}^{(i)}, t_i \left| x_{kl}^{(i-1)}, t_{i-1}\right.)$$
$$m = 1,\ 2 \ldots K \text{ and } n = 1,\ 2, \ldots L_m \tag{5}$$

with the assumption that the transition probability densities $p(x^{(i)}, t_i \left| x_{kl}^{(i-1)}, t_{i-1}\right.)$ and $p(x_{mn}^{(i)}, t_i \left| x_{kl}^{(i-1)}, t_{i-1}\right.)$ are Gaussian distributed when $t_i - t_{i-1}$ is sufficiently small, computer programs can be developed to solve Eqs. (4) and (5). When using the above path integral solution method for solving the motion response statistics of a structure, the initial probability density $p(x_0, t_0)$ of the response of the structure is estimated by the researcher based on his (her) experience. Whether a good estimation is made or not will have big influence on the computer running time of the path integral solution. If a bad estimation of the initial probability density were made, the running time of the path integral solution would have been significantly increased. Here, the estimation of the initial probability density means to estimate the values of the mean and standard deviation in the probability density formula. To find an optimal way for estimating the initial probability density $p(x_0, t_0)$, the CPIS method proposed in [8] tries to first estimate a rough initial probability density using small number of Monte Carlo simulations and the 3 sigma rule of Normal Distribution [10]. The 3 sigma rule tells us that if the probability distribution of a random variable is approximately Normal, then there are about 99.7% of the values of this random variable are in the scope of 3 times standard deviation from the mean. This rule is shown in Fig. 2. In Fig. 2, σ represents standard deviation. μ represents mean.

After a small number of Monte Carlo simulations have been performed on the structure equation of motion, the initial response displacement and velocity probability density curves of the structure can be obtained. According to the 3 sigma rule of Normal Distribution, we can promptly obtain the mean and standard deviation values of these two curves. Subsequently, we can construct the formula of the initial joint probability density which will not have a big difference from the final response probability density values. The constructed initial probability density formula is then inserted into the source code of the original path integral solution program. Good prediction of the response statistics can then be achieved by running the modified computer program in a relatively short time compared to the one used by the original PIS method. We see that the above compound path integral solution procedures have taken the advantage of the method of Monte Carlo simulation, whose strength is in predicting the central part of the response probability density curve. The CPIS procedures have also taken the advantage of the method of path integral solution, whose strength is in predicting the extreme responses (the tail regions of the response probability density curve).

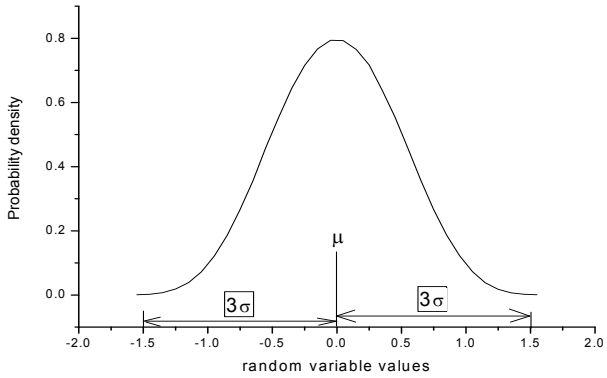


Fig. 2 Three-sigma rule of the Normal (or approximately Normal) probability distribution

4. APPLICATION OF THE COMPOUND PATH INTEGRAL SOLUTION

We will now apply the compound path integral solution method for predicting the response exceedance probabilities of a compliant offshore structure subjected to slow drift wave force random excitations. Here we first give the Rice formula [11] which can be used to calculate the mean level up-crossing rates:

$$E\left[n_+(\xi,\ t)\right] = \int_0^\infty \dot{x} p_{\{X\}\{\dot{X}\}}(\xi,\ \dot{x},\ t)\ d\dot{x} \tag{6}$$

where $E[n_+(\xi,\ t)]$ denotes the mean rate of ξ up- crossings by a stochastic process $X(t)$. If the stochastic process $X(t)$ is the response displacement process of a structure, then in the above equation, $p_{\{X\}\{\dot{X}\}}(\xi,\ \dot{x},\ t)$ denotes the joint probability density of the response displacement and velocity.

Let $P_F(T) = P_F(T,\ \xi)$ denote the probability that $X(t)$ exceeds the value ξ during a period of time of length T, then [11]:

$$P_F(T) = 1 - \exp\left\{-E\left[n_+(\xi,\ t)\right] TC(\xi)\right\} \tag{7}$$

The above equation is the formula for calculating the exceedance probabilities for a specified period of time of length T. In the above equation $C(\xi)$ is an optional function which describes the relative proportion of level ξ crossings considered statistically independent. A unitary value of $C(\xi)$ indicates statistical independence

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of successive peaks, a reasonable assumption for a high excursion level.

We now utilize the compound path integral solution method to calculate the response exceedance probabilities of a compliant offshore structure driven by slowly varying wave drift forces described by Eq. (2). Specifically we will calculate the structure response level up-crossing rates and exceedance probabilities. The specific structure we will study is the Faltinsen and Løken moored floating cylinder [3]. The total mass per unit length of the cylinder is $M = 3.21 \times 10^5$kg/m, and the radius of the cylinder is $R = 10$m. The drag coefficient C_D is equal to 0.6. The undamped natural frequency is chosen to be $\omega_0 = (K_1/M)^{0.5} = 0.06$rad/s, and the ratio of the stiffness parameter to the total mass is $K_3/M = 2.5 \times 10^{-5}\text{m}^{-2}\text{s}^{-2}$. The wave drift damping coefficient is chosen to be constant, corresponding to a damping ratio $\xi = D/2\omega_0 M = 0.05$. The c/M value is chosen to be 0.0245. Equation (2) can now be written as:

$$\ddot{x} + 0.021\dot{x} + 0.005\dot{x}^3 + 0.004x + 0.000025x^3 = 0.0245W(t) \tag{8}$$

The equations for the first and second order moments of $x(t)$ in Eq. (9) can be obtained using the well known Gaussian closure as:

$$\left.\begin{aligned}
\dot{m}_{10} &= m_{01} \\
\dot{m}_{01} &= -0.021m_{01} - 0.004m_{10} - 0.015m_{01}\, m_{02} \\
&\quad + 0.01m_{01}^3 - 0.000075m_{10}\, m_{20} + 0.00005m_{10}^3 \\
\dot{m}_{20} &= 2m_{11} \\
\dot{m}_{11} &= m_{02} - 0.021m_{11} - 0.004m_{20} - 0.015m_{11}\, m_{02} \\
&\quad + 0.01m_{10}\, m_{01}^3 - 0.000075m_{20}^2 + 0.00005m_{10}^4 \\
\dot{m}_{02} &= -0.042m_{02} - 0.008m_{11} - 0.03m_{02}^2 + 0.02m_{01}^4 \\
&\quad - 0.00015m_{20}\, m_{11} + 0.0001m_{10}^3\, m_{01} + 0.0006
\end{aligned}\right\} \tag{9}$$

where $m_{ij} = E[x^i, \dot{x}^j]$. The moments obtained from solving Eq. (9) for a short time step are used to form the two dimensional Gaussian transition probability density.

The initial probability density distribution is estimated by Monte Carlo simulation 50 times and the 3 sigma rule. According to the 3 sigma rule of Normal Distribution, we can quickly obtain that the mean value of the marginal probability density is $m_1 \approx -0.50$, and the standard deviation is $\sigma_1 \approx 6.3/3 \approx 2.1$. Similarly we can obtain the mean value and standard deviation of the response velocity marginal probability density curve: $m_2 \approx -0.1$, $\sigma_2 \approx 0.17$. Thus, we have obtained the following initial probability density via Monte Carlo simulation 50 times.

$$p(x^{(0)}, \dot{x}^{(0)}) = \frac{1}{2\pi \times 2.1 \times 0.17} \exp\left\{ -\frac{(x+0.5)^2}{2\times(2.1^2)} - \frac{(\dot{x}+0.1)^2}{2\times(0.17^2)} \right\} \tag{10}$$

Using the above formula, we can quickly modify the source code of the original path integral solution method. The evolution of the structure response probability density is then computed using the CPIS program with a time step of $t = 1$s, and within a reduced state space $[-10, 10] \times [-1, 1]$, which was divided into 40 uniform sub-intervals in the direction of response displacement with two Gauss quadrature points in each sub-interval, and was divided into 4 uniform sub-intervals in the direction of response velocity with two Gauss quadrature points in each sub-interval. Normalization procedure is adopted during the calculation process in order to avoid diffusion of the final result. Running the CPIS program 10 times, we have obtained the joint probability density values of the response displacements and velocities of the offshore structure. Wang has compared the structure's extreme response probability density values predicted via 158 minutes Monte Carlo simulation, 12 minutes PIS solution and 6 minutes CPIS solution (CPIS 7 times) when the structure system has analytical solutions in a special case, and it is found that the results obtained by applying the CPIS method are much more accurate [12] (Wang Yingguang, 2008). With this fact in mind, we can conclude that the joint probability density values of the response displacements and velocities of the offshore structure obtained from 10 times CPIS solution are sufficiently accurate.

Having obtained the above results, we have calculated the moored floating cylinder's slow drift response level up-crossing rates according to Eq. (6). Then the cylinder's slow drift response exceedance probabilities for a time period of 3 hours are calculated according to Eq. (7) in which a unitary value of $C(\xi)$ is chosen. In order to demonstrate the accuracy and efficiency of the CPIS method, the cylinder's slow drift response exceedance probabilities have also been calculated by using the original PIS method for comparison purpose. The calculation results are shown in Fig. 3.

In Fig. 3, the red curve with solid circles represents the logarithm of the cylinder's response exceedance probabilities via 10 rounds path integrations. The green curve with solid up triangles represents the logarithm of the cylinder's response exceedance probabilities via 20 rounds path integrations. The blue curve with solid down triangles represents the logarithm of the cylinder's response exceedance probabilities via 100 rounds path integrations. The 100 rounds path integrations costs about 29 minutes on a Dell Dimension 5150/E510 desktop computer. The black curve with solid squares represents the logarithm of the cylinder's response exceedance probabilities via 10 times compound path integral solutions. It takes about 7 minutes to finish the 10 rounds CPIS solutions on a Dell Dimension 5150/E510 desktop computer. We notice the tendency in Fig. 4 that as the path integration time is prolonged, the results from the PIS solutions get more close to the results from the CPIS solutions. The results from 29 minutes PIS solutions agree considerably well with the results from 7 minutes CPIS solutions. Obviously, the efficiency of the CPIS method for predicting the structure's slow drift response exceedance probabilities is higher than that of the PIS method.

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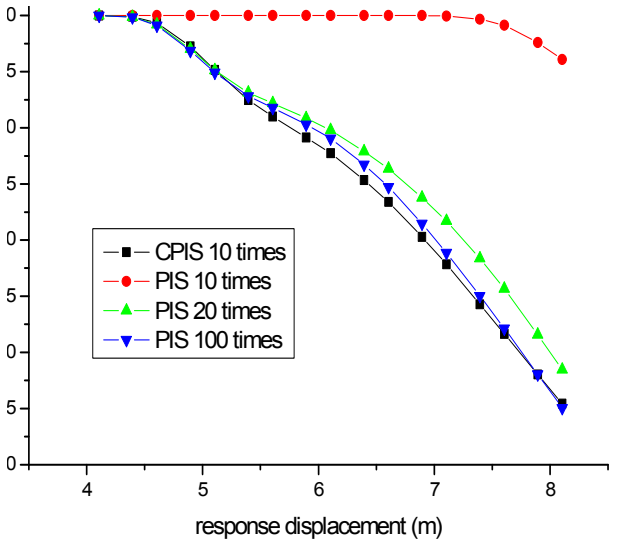


Fig. 3 Comparison of the logarithm values of the path integral solutions and the compound path integral solutions of the extreme response exceedance probabilities

5. CONCLUDING REMARKS

In this technical note, the theoretical background of the compound path integral solution method is elucidated. The compound path integral solution procedures have taken the advantage of the method of Monte Carlo simulation, whose strength is in predicting the central part of the response probability density curve. The CPIS procedures have also taken the advantage of the method of path integral solution, whose strength is in predicting the extreme responses.

To demonstrate the usefulness of the compound path integral solution (CPIS) method, the motion response level up-crossing rates and exceedance probabilities of a nonlinear compliant offshore structure driven by slowly varying wave drift forces are calculated by utilizing the CPIS method. The structure's slow drift response exceedance probabilities have also been calculated by using the original PIS method for comparison purpose. We notice the tendency that as the path integration time is prolonged, the results of response exceedance probabilities from the PIS solutions get more close to the results from the CPIS solutions. The results from 29 minutes PIS solutions agree considerably well with the results from 7 minutes CPIS solutions. Therefore, the efficiency of the CPIS method for predicting the structure's slow drift response exceedance probabilities is higher than that of the PIS method. The compound path integral solution method can be utilized as a valuable tool to the practicing engineers for predicting slow drift response exceedance probabilities of a compliant offshore structure in the initial design stage.

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REFERENCES

1. Martinsen, T., "Dynamics of Slow-Drift Oscillations with Non-Linear Restoring Forces," *IUTAM Symposium Non-linear Water Waves*, Tokyo, Japan (1987).
2. Johnsen, J. M. and Naess, A., "Time Variant Wave Drift Damping and its Effect on the Response Statistics of Moored Offshore Structures," *International Journal of Offshore and Polar Engineering*, **3**, pp. 273–279 (1993).
3. Naess, A. and Johnson, J. M., "Response Statistics of Nonlinear, Compliant Offshore Structures by the Path Integral Solution Method," *Probabilistic Engineering Mechanics*, **8**, pp. 91–106 (1993).
4. Karlsen, H. C. and Naess, A., "Statistical Response Predictions for a Nonlinearly Moored Large Volume Structure in Random Seas," *Applied Ocean Research*, **27**, pp. 273–280 (2005).
5. Yu, J. S., Cai, G. Q. and Lin, Y. K., "A New Path Integration Procedure Based on Gauss-Legendre Scheme," *International Journal of Non-Linear Mechanics*, **32**, pp. 759–768 (1997).
6. Wang, Y. G. and Tan, J. H., "Markov Modeling for Slow Drift Oscillations of Moored Vessels in Irregular Waves," *Journal of Ship Mechanics*, **39**, pp. 1493–1500 (2008).
7. Wang, Y. G., Tan, J. H. and Xue, L. P., "On the Exceedance Probabilities of Extreme Drift Motions of An Offshore Structure," *China Ocean Engineering*, **39**, pp. 1493–1500 (2008).
8. Wang, Y. G. and Tan, J. H., "Nonlinear Analysis of Slow Drift Extreme Responses of a Compliant Offshore Structure," *Acta Mechanica Sinica*, **25**, pp. 651–657 (2009).
9. Borgman, L. E., "Ocean Wave Simulation for Engineering Design," *Journal of the Waterways and Harbors Division*, *Proceedings of the American Society of Civil Engineers*, **95**, pp. 557–583 (1969).
10. Devore, J. L. *Probability and Statistics for Engineering and the Sciences*, 6th Ed., Beijing, China Machine Press (2005).
11. Grime, A. J. and Langley, R. S., "On the Efficiency of Crossing Rate Prediction Methods Used to Determine Extreme Motions of Moored Offshore Structures," *Applied Ocean Research*, **25**, pp. 127–135 (2003)
12. Wang, Y. G., "Research on Slow Drift Extreme Response and Stability of Ocean Structures in Random Seas," Ph.D. dissertation, School of Naval Architecture, Ocean and Civil Engineering, Shanghai Jiao Tong University (2008).

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