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Sequence spaces defined by a modulus

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Ruckle [4] used the idea of a modulus function f (see Definition 1 below) to construct the sequence space

$$L(f) = \left\{x = (x_k): \sum_{k=1}^{\infty} f(|x_k|) < \infty\right\}. \tag{1}$$

This space is an FK space, and Ruckle proved that the intersection of all such $L(f)$ spaces is ϕ, the space of finite sequences, thereby answering negatively a question of A. Wilansky: 'Is there a smallest FK-space in which the set $\{e_1, e_2, \ldots\}$ of unit vectors is bounded?'

The space $L(f)$ is closely related to the space l_1 which is an $L(f)$ space with $f(x) = x$ for all real $x \geqslant 0$. Thus, Ruckle proved that, for any modulus f,

$$L(f) \subset l_1 \quad \text{and} \quad L(f)^\alpha = l_\infty, \tag{2}$$

where $L(f)^\alpha = \{y = (y_k): \Sigma|y_k x_k| < \infty \text{ for all } x \in L(f)\}$ is the α-dual of $L(f)$.

In the present note we introduce and examine some properties of three sequence spaces defined using a modulus f, which generalize the well-known spaces w_0, w and w_∞ of strongly summable sequences (see, for example, [2], [3], [5]).

Definition 1. Let $f: [0, \infty) \to [0, \infty)$. Then f is called a modulus if

(i) $f(x) = 0$ if and only if $x = 0$,
(ii) $f(x+y) \leqslant f(x) + f(y)$, for all $x \geqslant 0, y \geqslant 0$,
(iii) f is increasing,
(iv) f is continuous from the right at 0.

It is immediate from (ii) and (iv) that f is continuous everywhere on $[0, \infty)$.

If we take $f(x) = x^p$ $(0 < p \leqslant 1)$ then the space $L(f)$ in (1) is the familiar space l_p.

Later we shall make use of the function f given in the next proposition.

Proposition 2. *There is a modulus f such that $f(xy) \leqslant f(x) + f(y)$ for $x \geqslant 0, y \geqslant 0$.*

Proof. Define $f(x) = \log(1+x)$. Then f is a modulus, subadditivity following from the fact that $1 + x + y \leqslant (1+x)(1+y)$ for $x \geqslant 0, y \geqslant 0$. The other property of f follows from $1 + xy \leqslant (1+x)(1+y)$.

We now introduce the generalizations of the classical spaces of strongly summable sequences.

Definition 3. Let f be a modulus. Denote by s the space of all complex sequences $x = (x_k)$ and write $e = (1, 1, 1, \ldots)$. Define

$$t_n(x) = n^{-1} \sum_{k=1}^{n} f(|x_k|),$$

$$w_0(f) = \{x \in s: t_n(x) \to 0 \ (n \to \infty)\},$$

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$$w(f) = \{x \in s: x - le \in w_0(f) \text{ for some number } l\},$$
$$w_\infty(f) = \{x \in s: \sup_n t_n(x) < \infty\}.$$

If we take $f(x) = x^p$ $(0 < p \leqslant 1)$ in Definition 3 then we obtain the familiar spaces of strongly summable sequences with index p. For $0 < p < 1$, the space $w_p = w(f)$, with $f(x) = x^p$, is important in connection with Kuttner's theorem (see [1–3], [5]).

In contrast with the first inclusion of (2) above, we have:

THEOREM 4. *For any modulus f we have*

$$w_1 \subset w(f). \tag{3}$$

Proof. Let $x \in w_1$, so that

$$s_n = n^{-1} \sum_{k=1}^{n} |x_k - l| \to 0 \quad (n \to \infty).$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 \leqslant t \leqslant \delta$. Write $y_k = |x_k - l|$ and consider

$$\sum_{k=1}^{n} f(y_k) = \Sigma_1 + \Sigma_2,$$

where the first summation is over $y_k \leqslant \delta$ and the second over $y_k > \delta$. Then $\Sigma_1 \leqslant \epsilon n$, and for $y_k > \delta$ we use the fact that

$$y_k < y_k/\delta < 1 + [y_k/\delta],$$

where $[t]$ denotes the integer part of t. By Definition 1, (iii) and (ii) we have for $y_k > \delta$,

$$f(y_k) \leqslant (1 + [y_k/\delta]) f(1) \leqslant 2f(1)\, y_k/\delta.$$

Hence $\Sigma_2 \leqslant 2\delta^{-1} f(1)\, n s_n$, which together with $\Sigma_1 \leqslant \epsilon n$ yields (3).

If we write $w_0(f) = w_0$ and $w_\infty(f) = w_\infty$ in the case when $f(x) = x$, then the method of proof of Theorem 4 shows that, for any modulus f, we have

$$w_0 \subset w_0(f) \quad \text{and} \quad w_\infty \subset w_\infty(f).$$

For the space $L(f)$ it is easy to see that the space of multipliers of $L(f)$ is l_∞, i.e.

$$\{a \in s: ax = (a_k x_k) \in L(f) \text{ for all } x \in L(f)\} = l_\infty.$$

Some information on multipliers for $w_\infty(f)$ is given below. For any set E of sequences we denote by $M(E)$ the space $\{a \in s: ax \in E \text{ for all } x \in E\}$.

THEOREM 5.

(i) *For any modulus f we have $l_\infty \subset M(w_\infty(f)) \subset w_\infty(f)$.*

(ii) *If f is bounded then $M(w_\infty(f)) = w_\infty(f) = s$.*

(iii) *If f is unbounded and there is a positive constant c such that $f(xy) \geqslant cf(x)f(y)$ for all $x \geqslant 0$, $y \geqslant 0$ then $M(w_\infty(f)) = l_\infty$.*

(iv) *There exists an unbounded f with $l_\infty \neq M(w_\infty(f)) = w_\infty(f)$.*

Proof. (i) $a \in l_\infty$ implies $|a_k| < 1 + [H]$ for some $H > 0$ and all k. Hence $x \in w_\infty(f)$ implies

$$t_n(ax) \leqslant (1 + [H])\, t_n(x),$$

which gives the first inclusion. The second inclusion follows from the fact that $e = (1, 1, 1, \ldots) \in w_\infty(f)$.

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(ii) If f is bounded then, for any $x \in s$,

$$t_n(x) \leqslant \sup\{f(y): y \geqslant 0\}$$

so that $w_\infty(f) = s$.

(iii) In view of (i) we need only show that $M(w_\infty(f)) \subset l_\infty$.

In what follows we use the fact that $x \in w_\infty(f)$ if and only if

$$\sup_r \frac{1}{2^r} \Sigma_r f(|x_k|) < \infty,$$

where Σ_r denotes a sum over $2^r \leqslant k \leqslant 2^{r+1}$.

Also, we denote by $\max_r |y_k|$ the maximum taken over $2^r \leqslant k < 2^{r+1}$.

Now suppose, if possible, that there exists $a \in M(w_\infty(f))$ with $(M(r))$ unbounded, where

$$M(r) = \max_r f(|a_k|).$$

If the maximum occurs for $k = k(r)$, then there exist $r_1 < r_2 < \ldots$ with $f(|a_{k(r_i)}|) > i$ for $i \geqslant 1$. Also there exists y_i with $f(y_i) = 2^{r_i}$.

Define $x_k = y_i$ for $k = k(r_i)$ and $x_k = 0$ otherwise. Then

$$\sup_r 2^{-r} \Sigma_r f(|x_k|) = 1,$$

but for $r = r_i$,

$$2^{-r} \Sigma_r f(|a_k x_k|) \geqslant ci,$$

contrary to $a \in M(w_\infty(f))$. Hence we must have $\sup_k f(|a_k|) < \infty$ whenever $a \in M(w_\infty(f))$. But it is easy to see that, since f is unbounded, $\sup_k f(|a_k|) < \infty$ is equivalent to $a \in l_\infty$.

(iv) Define $f(x) = \log(1+x)$ as in Proposition 2 and take $a_k = k$ for $k = 2^r$ with $a_k = 0$ otherwise. Then $a \in M(w_\infty(f)) \backslash l_\infty$ since $x \in w_\infty(f)$ implies

$$\begin{aligned} 2^{-r} \Sigma_r f(|a_k x_k|) &\leqslant 2^{-r}(\log 2^r + f(|x_{2^r}|)) \\ &\leqslant o(1) + 2^{-r} \Sigma_r f(|x_k|). \end{aligned}$$

Finally, if a and x are in $w_\infty(f)$ it follows from Proposition 2 that

$$2^{-r} \Sigma_r f(|a_k x_k|) \leqslant 2^{-r} \Sigma_r f(|a_k|) + 2^{-r} \Sigma_r f(|x_k|)$$

whence $w_\infty(f) \subset M(w_\infty(f))$, which completes the proof.

We note that the restriction of Theorem 5 (iii) is not necessary for $M(w_\infty(f)) = l_\infty$. For example, if we define $g(x) = x + x^{\frac{1}{2}}$ then there is no c with $g(xy) \geqslant cg(x)g(y)$ for all $x \geqslant 0$, $y \geqslant 0$. But a slight modification of the argument of Theorem 5 (iii) shows that $M(w_\infty(g)) = l_\infty$. In fact for this particular g is it is also true that $w_\infty(g) = w_\infty$, so the result follows from Theorem 5 (iii) with $f(x) = x$.

The next theorem shows the relation between $w_\infty(f)$ and $w_\infty(g)$ for moduli f and g.

THEOREM 6. *$w_\infty(g) \subset w_\infty(f)$ if and only if*

$$\limsup_{x \to \infty} \frac{f(x)}{g(x)} < \infty. \tag{4}$$

Proof. It is trivial that (4) is sufficient for the inclusion.

Conversely, let the inclusion hold. In case g is bounded it follows from Theorem 5 (ii) that $w_\infty(g) = s$ and so $w_\infty(f) = s$, which clearly implies that f is bounded. Consequently (4) must hold.

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Now suppose that g is unbounded and that (4) fails. Then there exists $x_1 > 0$ with $f(x_1) > 2g(x_1) > 2^2$. Let $r(1)$ be the natural number such that

$$2^{r(1)-1} < g(x_1) \leqslant 2^{r(1)}.$$

Then $r(1) > 1$. Also, there exists $x_2 > 0$ with $f(x_2) > 4g(x_2) > 4 \cdot 2^{r(1)}$, so if $r(2)$ is the natural number such that $2^{r(2)-1} < g(x_2) \leqslant 2^{r(2)}$ we have $r(2) > r(1)$. Proceeding inductively, we determine x_i and $r(i)$ with $r(i) > r(i-1)$ such that

$$f(x_i) > (2i)\, g(x_i) > (2i)\, 2^{r(i-1)} \quad \text{and} \quad 2^{r(i)-1} < g(x_i) \leqslant 2^{r(i)}.$$

Let us define $y_k = x_i$ for $k = 2^{r(i)}$ and $y_k = 0$ otherwise. Then for $r = r(i)$ we have

$$2^{-r}\Sigma_r g(|y_k|) \leqslant 1 \quad \text{and} \quad 2^{-r}\Sigma_r g(|y_k|) > i,$$

contrary to the fact that $w_\infty(g) \subset w_\infty(f)$.

COROLLARY 7. *$w_\infty(g) = w_\infty(f)$ if and only if*

$$0 < \liminf_{x\to\infty} \frac{f(x)}{g(x)} \leqslant \limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty.$$

If, for example, we take $f(x) = x/(1+x)^{\frac{1}{2}}$ for $x \geqslant 0$, it is easy to check that f is an unbounded modulus such that $f(xy) \geqslant f(x)f(y)$ for all $x \geqslant 0, y \geqslant 0$. By Theorem 5(iii) it follows that $M(w_\infty(f)) = l_\infty$. But Corollary 7 shows that $w_\infty(f) = w_\infty(g)$, with $g(x) = x^{\frac{1}{2}}$, whence $w_\infty(f)$ is the space

$$\{x\colon 2^{-r}\Sigma_r |x_k|^{\frac{1}{2}} = O(1)\}$$

for which it is obvious that $M(w_\infty(g)) = l_\infty$.

Also, it is worth observing that if f is any modulus which satisfies $f(xy) = f(x)f(y)$ for all $x > 0$ and $y > 0$, then f is necessarily of the form $f(x) = x^p$ for some p with $0 < p \leqslant 1$. This follows by considering $g\colon R \to R$ given by

$$g(x) = \log f(e^x)$$

and using the fact that g is additive and continuous, which implies, as is well known, that $g(x) = xg(1)$.

It is interesting to ask whether there is a modulus f such that f satisfies the condition of Theorem 5(iii), so that $M(w_\infty(f)) = l_\infty$, but is such that $w_\infty(f)$ does not coincide with any $w_\infty(g)$, where $g(x) = x^p$ $(0 < p \leqslant 1)$.

I wish to thank Dr D. H. Armitage for supplying me with the following example of such a modulus:

$$f(x) = x/(\log(x + e^2)).$$

In this case we have $f(xy) \geqslant f(x)f(y)$, and it follows from Corollary 7 that $w_\infty(f)$ is not $w_\infty(g)$ for any $g(x) = x^p$ $(0 < p \leqslant 1)$.

In conclusion we consider the metrization of the $w(f)$ spaces – these spaces, given in Definition 3, are obiously linear subspaces of s with the usual coordinatewise operations.

Recall that a paranormed space (X, g) is a topological linear space whose metric topology is given by a paranorm g (a real subadditive function with $g(\theta) = 0$, $g(x) = g(-x)$ for all $x \in X$ and with continuous scalar multiplication).

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It is natural to metrize $w_\infty(f)$ and its subspaces $w_0(f)$ and $w(f)$ by

$$g(x) = \sup_r 2^{-r}\Sigma_r f(|x_k|), \tag{5}$$

where the sum is over $2^r \leqslant k < 2^{r+1}$.

In the next result we use the terms paranorm and metric in connection with g given by (5).

THEOREM 8.

(i) *$w_0(f)$ and $w(f)$ are paranormed FK spaces.*

(ii) *$w_\infty(f)$ is a complete metric topological group with continuous coordinates, in which the map $x \to \lambda x$ is continuous at $x = \theta$ for each fixed scalar λ and the map $(\lambda, x) \to \lambda x$ is continuous at $\lambda = 0$, $x = \theta$. In general, $w_\infty(f)$ is not paranormed by g.*

Proof. (i) The statements about $w_0(f)$ are easily proved, so we consider $w(f)$, where only completeness and the continuity of λx (x fixed, $\lambda \to 0$) are not entirely obvious.

Suppose that $(x^{(n)})$ is a Cauchy sequence in $w(f)$. Then there is a sequence $x = (x_k)$ such that $x_k^{(n)} \to x_k$ ($n \to \infty$, each k) and $g(x^{(n)} - x) \to 0$ ($n \to \infty$). Also, for each n, there exists $l^{(n)}$ with

$$2^{-r}\Sigma_r f(|x_k^{(n)} - l^{(n)}|) \to 0 \quad (r \to \infty),$$

whence $f(|l^{(n)} - l^{(m)}|) \to 0$ ($n, m \to \infty$) and so $(l^{(n)})$ converges to l, say. Consequently

$$2^{-r}\Sigma_r f(|x_k - l|) \to 0 \ (r \to \infty),$$

so that $x \in w(f)$ and the space is complete.

Now fix $x \in w(f)$, so that we have

$$a_r = 2^{-r}\Sigma_r f(|x_k - l|) \to 0 \quad (r \to \infty). \tag{6}$$

For each R, by continuity of f,

$$b(R, \lambda) = f(|\lambda l|) + \sum_{k=1}^{2^{R+1}} f(|\lambda x_k|) \to 0 \quad (\lambda \to 0). \tag{7}$$

Also, for $|\lambda| < 1$ and all $r \geqslant 0$,

$$2^{-r}\Sigma_r f(|\lambda x_k|) \leqslant a_r + f(|\lambda l|). \tag{8}$$

Let $\epsilon > 0$ and choose R such that $r > R$ implies $a_r < \epsilon/2$. Then choose $\delta < 1$ such that $|\lambda| < \delta$ implies $b(R, \lambda) < \epsilon/2$. It follows readily from (6)–(8) that $g(\lambda x) \to 0$ ($\lambda \to 0$).

(ii) We deal with the last sentence only, where we show that in order that g should be a paranorm it is necessary, but not sufficient, that f should be unbounded.

Suppose f is bounded and take $x = (1, 2, 3, 4, \ldots)$ in $w_\infty(f)$. For any $\lambda > 0$ we choose k such that $k\lambda > 1$, whence $g(\lambda x) > f(1)$, and so g is not a paranorm.

Finally, consider the unbounded modulus given by $f(x) = \log(1+x)$ and define

$$x_k = e^k \quad \text{for } k = 2^r \quad \text{and} \quad x_k = 0 \quad \text{otherwise.}$$

For any $\lambda > 0$ we find that

$$2^{-r}\Sigma_r f(\lambda|x_k|) \to 1 \quad (r \to \infty),$$

whence $g(\lambda x) \geqslant 1$, and so g is not a paranorm.

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