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# HAUSDORFF DIMENSIONS OF SOME LIMINF SETS IN DIOPHANTINE APPROXIMATION

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*Abstract.* Let $Q$ be an infinite subset of $\mathbb{N}$. For any $\tau > 2$, denote $W_\tau(Q)$ (respectively $W_\tau$) to be the set of $\tau$ well-approximable points by rationals with denominators in $Q$ (respectively in $\mathbb{N}$). We consider the Hausdorff dimension of the liminf set $W_\tau \setminus W_\tau(Q)$ after Adiceam. By using the tools of continued fractions, it is shown that if $Q$ is a so-called $\mathbb{N}\setminus Q$-free set, the Hausdorff dimension of $W_\tau \setminus W_\tau(Q)$ is the same as that of $W_\tau$, i.e. $2/\tau$.

§1. *Introduction.* It is well known by Dirichlet's theorem that, for any irrational $x \in \mathbb{R}$, there exists infinitely many $(p, q) \in \mathbb{Z} \times \mathbb{N}$ such that

$$|x - p/q| < q^{-2}.$$

For any $\tau > 2$, a real number $x$ is said to be *$\tau$-well-approximable* if

$$|x - p/q| < q^{-\tau} \text{ i.m. } (p, q) \in \mathbb{Z} \times \mathbb{N}, \tag{1.1}$$

where 'i.m.' denotes *infinitely many* for brevity. A real number $x$ is said to be a Liouville number if (1.1) is satisfied for all $\tau > 2$.

Jarník [**11**] and, independently, Besicovitch [**3**] showed that, for any $\tau > 2$, the Hausdorff dimension of the set of $\tau$-well-approximable points

$$W_\tau = W_\tau(\mathbb{N}) = \{x \in \mathbb{R} : |x - p/q| < q^{-\tau} \text{ i.m. } (p, q) \in \mathbb{Z} \times \mathbb{N}\} \tag{1.2}$$

is $2/\tau$. Thus, the set $L$ of Liouville numbers is of zero Hausdorff dimension, i.e. $\dim_H L = 0$, where $\dim_H$ denotes Hausdorff dimension. For the definitions and properties of Hausdorff measure and Hausdorff dimension, we refer to [**6**].

Inspired by a question of Erdös, how many points can have the denominators of their convergents fall into a given set infinitely often, Borosh and Fraenkel [**4**] considered the set of $\tau$-well-approximable points by rationals with denominators restricted in $Q$, i.e.

$$W_\tau(Q) = \{x \in \mathbb{R} : |x - p/q| < q^{-\tau} \text{ i.m. } (p, q) \in \mathbb{N} \times Q\},$$

where $Q$ is an infinite subset of $\mathbb{N}$. It was shown in [**4**] that, for any $\tau \geqslant 1 + v(Q)$,

$$\dim_H W_\tau(Q) = \frac{1 + v(Q)}{\tau} \quad \text{with } v(Q) = \inf\left\{v : \sum_{q \in Q} q^{-v} < \infty\right\}.$$

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See [9] when the error term $q^{-\tau}$ is replaced by a general function. For more metrical results concerning the Diophantine approximation by rationals with restricted numerators or denominators, one is referred to Harman's monograph [8] and references therein.

Quite recently, Adiceam [1] initiated the study of the so-called liminf set:

$$W_\tau \backslash W_\tau(Q),$$

another type of the set of points which can be well approximated by rationals with restricted denominators. Explicitly, this is the set of all of those $x$ in $\mathbb{R}$ which admit infinitely many approximations at order $\tau$ by rational numbers $p/q$ whose denominators $q$ lie in $\mathbb{N}\backslash Q$, but only finitely many approximations by rational numbers whose denominators lie in $Q$. When $Q$ is an $\mathbb{N}\backslash Q$-free set (see Definition 1.1 below), it was shown in [1] that $W_\tau \backslash W_\tau(Q)$ contains infinitely many Liouville numbers and when $\tau > 3$, the Hausdorff dimension of $W_\tau \backslash W_\tau(Q)$ verifies [2]

$$\dim_H(W_\tau \backslash W_\tau(Q)) = \frac{1 + v(\mathbb{N}\backslash Q)}{\tau}. \tag{1.3}$$

In [1], Adiceam asked whether $W_\tau \backslash W_\tau(Q)$ contains reals other than Liouville numbers. This was considerably solved by Adiceam [2] when $\tau > 3$ in view of (1.3), since the Hausdorff dimension of the Liouville numbers is zero. So, there is still a gap when $\tau \leqslant 3$.

In view of (1.3), it would be very natural to conjecture that the formula (1.3) should be held for all $\tau > 1 + v(\mathbb{N}\backslash Q)$. In this paper, we give a partial answer to this under some divisibility on $Q$ (called $\mathbb{N}\backslash Q$-free). Moreover, an example is provided indicating that (1.3) may not hold for general $Q$ when $\tau \leqslant 2$.

First we give the formal definition of an $\mathbb{N}\backslash Q$-free set following Adiceam [1].

*Definition* 1.1. A subset $Q$ of integers is called an $\mathbb{N}\backslash Q$-free set, if

$$q \in Q \Longleftrightarrow \text{for all } \not\in Q, r \nmid q.$$

In other words, if $q \in Q$, then all of the divisors of $q$ are in $Q$ too. An example of an $\mathbb{N}\backslash Q$-free set given by Adiceam [1] is as follows: for any integer $m \geqslant 2$, let

$$Q(m) = \{n \in \mathbb{N}, (n, m) = 1\}.$$

Now we can state our main results.

THEOREM 1.2. *Let $Q$ be an $\mathbb{N}\backslash Q$-free set with $Q \neq \mathbb{N}$. Then, for any $\tau > 2$,*

$$\dim_H(W_\tau \backslash W_\tau(Q)) = 2/\tau.$$

*Consequently, the liminf set $W_\tau \backslash W_\tau(Q)$ contains reals other than Liouville numbers.*

*Remark* 1. It should be remarked that when $Q$ is an $\mathbb{N}\backslash Q$-free set with $Q \neq \mathbb{N}$, $v(\mathbb{N}\backslash Q) = 1$. More precisely, if $m \notin Q$, then $km \notin Q$ for any $k \in \mathbb{N}$. So Theorem 1.2 does not conflict with the formula (1.3).

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On the other hand, without the assumption of an $\mathbb{N}\backslash Q$-free set on $Q$, the following result indicates that the dimensional formula (1.3) is no longer valid when $\tau \leqslant 2$.

THEOREM 1.3. *For any $0 \leqslant \alpha \leqslant 1$, there exists $Q \subset \mathbb{N}$ such that $v(\mathbb{N}\backslash Q) = \alpha$ and for any $\tau$ satisfying $1 + \alpha \leqslant \tau \leqslant 2$,*

$$\dim_H(W_\tau \backslash W_\tau(Q)) = 0 \neq \frac{1+\alpha}{\tau}.$$

§2. *Preliminaries.* In this section, we recall some basic properties of continued fractions, pressure functions and some preliminary results that will be used later.

Continued fraction expansions can be induced by the Gauss transformation $T : [0, 1) \to [0, 1)$ given as

$$T(0) = 0, \ T(x) = 1/x - \lfloor 1/x \rfloor \quad \text{when } x \in (0, 1),$$

where $\lfloor \xi \rfloor$ is the integer part of $\xi \in \mathbb{R}$. Then every irrational number $x \in [0, 1)$ is expanded into an infinite continued fraction expansion

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \ddots + \cfrac{1}{a_n(x) + T^n x}}} = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}} \tag{2.1}$$

where $a_n(x) = \lfloor 1/T^{n-1}(x) \rfloor$, for each $n \geqslant 1$. For short, we write

$$x = [a_1(x), a_2(x), \ldots, a_n(x) + T^n(x)] = [a_1(x), a_2(x), \ldots].$$

When $x \in [0, 1)$ is a rational number, the expansion of $x$ is finite.

The integers $\{a_n(x)\}_{n \geqslant 1}$ are called the partial quotients of $x$. The fractions $p_n(x)/q_n(x) := [a_1(x), \ldots, a_n(x)]$ $(n \geqslant 1)$, obtained by finite truncations in (2.1), are called the convergents of $x$. With the conventions $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = 0$ and $q_0 = 1$, we have the following recursive relations [12]:

$$\begin{aligned} p_{n+1}(x) &= a_{n+1}(x)p_n(x) + p_{n-1}(x), \\ q_{n+1}(x) &= a_{n+1}(x)q_n(x) + q_{n-1}(x), \quad n \geqslant 0. \end{aligned} \tag{2.2}$$

By these recursive relations, it is direct that

$$p_{n-1}(x)q_n(x) - p_n(x)q_{n-1}(x) = (-1)^n,$$

thus, $(q_{n-1}(x), q_n(x)) = 1$, i.e. coprime.

Clearly, $q_n(x)$ is completely determined only by $a_1(x), \ldots, a_n(x)$. So we also write instead $q_n(a_1(x), \ldots, a_n(x))$. When $a_1, \ldots, a_n$ are clear from the context, we also use $q_n$ to denote $q_n(a_1, \ldots, a_n)$ for simplicity.

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For each $n \geqslant 1$ and $(a_1, \ldots, a_n) \in \mathbb{N}^n$, we call

$$I_n(a_1, \ldots, a_n) = \{x \in [0, 1) : a_k(x) = a_k, 1 \leqslant k \leqslant n\}$$

a cylinder of order $n$. We use $I_n(x)$ to denote the $n$th-order cylinder containing $x \in [0, 1)$ and $|I_n(x)|$ for the length of the interval $I_n(x)$.

The first proposition collects basic properties of continued fractions that will be used later. For more properties, we refer to the monographs [10, 12].

PROPOSITION 2.1 [12]. *Let $a_1, \ldots, a_n \in \mathbb{N}$. One has:*

(1) $q_n(a_1, \ldots, a_n) \geqslant 2^{n/2-1}$ *and, for any $1 \leqslant k < n$,*

$$1 \leqslant \frac{q_n(a_1, \ldots, a_n)}{q_k(a_1, \ldots, a_k)q_{n-k}(a_{k+1}, \ldots, a_n)} \leqslant 2,$$
$$\prod_{k=1}^{n} a_k \leqslant q_n(a_1, \ldots, a_n) \leqslant \prod_{k=1}^{n}(a_k + 1);$$

(2) *for any $x \in [0, 1)$,*

$$\frac{1}{(a_{n+1}(x) + 2)q_n^2(x)} \leqslant \left|x - \frac{p_n(x)}{q_n(x)}\right| \leqslant \frac{1}{a_{n+1}(x)q_n^2(x)}; \tag{2.3}$$

(3) *the length of $I_n(a_1, \ldots, a_n)$ is given as*

$$|I_n(a_1, \ldots, a_n)| = \frac{1}{q_n(a_1, \ldots, a_n)(q_n(a_1, \ldots, a_n) + q_{n-1}(a_1, \ldots, a_{n-1}))}.$$

By Proposition 2.1(3) and the recursive relationship on $q_n$, it can be checked directly that

$$\frac{1}{3a_n^2} \leqslant \frac{|I_n(a_1, \ldots, a_n)|}{|I_{n-1}(a_1, \ldots, a_{n-1})|} \leqslant \frac{2}{a_n^2}. \tag{2.4}$$

A well-known result describing the connections between balls and cylinders is the following *three-regular property* [5] (see Wu [16] for an elementary proof).

PROPOSITION 2.2 [16]. *For any irrational $z \in [0, 1)$ and $n \geqslant 2$, if $|I_{n+1}(z)| \leqslant r < |I_n(z)|$, then*

$$I_{n+1}(z) \subset B(z, r) \subset I_{n-2}(z). \tag{2.5}$$

Now we state Legendre's theorem which connects continued fractions tightly with Diophantine approximation.

THEOREM 2.3. *Let $p/q$ be a rational number. Then*

$$\left|x - \frac{p}{q}\right| < \frac{1}{2q^2} \Longrightarrow \frac{p}{q} = \frac{p_n(x)}{q_n(x)} \quad \text{for some } n \geqslant 1.$$

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Legendre's theorem tells us that if a real number $x$ can be well approximated by some rational, this rational must be a convergent of $x$. So to find good rational approximations of an irrational, we only need focus on its convergents. Meanwhile, the estimate (2.3) tells us that $x$ can be well approximated by its convergent $p_n/q_n$, if and only if its $(n+1)$th partial quotient $a_{n+1}$ is sufficiently large.

Next we recall some basic properties of the pressure function. Let $A$ be a finite or infinite subset of $\mathbb{N}$. Define

$$X_A = \{x \in [0,1) : a_n(x) \in A, \text{ for all } n \geqslant 1\}.$$

Then $(X_A, T)$ is a subsystem of $([0,1), T)$. Let $\phi : [0,1) \to \mathbb{R}$ be a real function. The pressure function restricted to the system $(X_A, T)$ is defined by

$$\mathrm{P}_A(T,\phi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{(a_1,\ldots,a_n)\in A^n} \sup_{x\in X_A} e^{S_n\phi([a_1,\ldots,a_n+x])}, \tag{2.6}$$

where $S_n\phi(x)$ denotes the ergodic sum $\phi(x) + \cdots + \phi(T^{n-1}x)$. When $A = \mathbb{N}$, also denote $\mathrm{P}_{\mathbb{N}}(T,\phi)$ by $\mathrm{P}(T,\phi)$. When $A$ is an infinite subset of $\mathbb{N}$, it is possible that $\mathrm{P}_A(T,\phi)$ is infinity.

Use the notation

$$\mathrm{Var}_n(\phi) := \sup\{|\phi(x) - \phi(y)| : I_n(x) = I_n(y)\}$$

for the $n$th variation of $\phi$.

The following proposition states that in the system of continued fractions the pressure function has a continuity property when the system $([0,1), T)$ is approximated by its subsystems $(X_A, T)$ [7]. For more thorough results on pressure function in infinite conformal iterated function systems, see Hanus, Mauldin, Urbański [7], Mauldin and Urbański [13] or their monograph [14].

PROPOSITION 2.4 [7, Proposition 2]. *Let $\phi : [0,1) \to \mathbb{R}$ be a real function with $\mathrm{Var}_1(\phi) < \infty$ and $\mathrm{Var}_n(\phi) \to 0$ as $n \to \infty$. We have*

$$\mathrm{P}_{\mathbb{N}}(T,\phi) = \sup\{\mathrm{P}_A(T,\phi) : A \text{ is a finite subset of } \mathbb{N}\}.$$

*As a consequence, we have*

$$s(\mathbb{N},\phi) = \sup\{s(A,\phi) : A \text{ is a finite subset of } \mathbb{N}\}, \tag{2.7}$$

*where $s(A,\phi)$ is defined as $\inf\{s \geqslant 0 : \mathrm{P}_A(T, s\phi) \leqslant 0\}$ for $A \subset \mathbb{N}$.*

From now on we focus on a specific potential $\phi = -(\tau/2)\log|T'|$, where $T'$ is the derivative of $T$, and define a sequence of real numbers $\{s_n\}_{n\geqslant 1}$, called dimensional numbers. Fix an integer $M$ and let $A = \{1,\ldots,M\}$. For each $n \geqslant 1$, let $s_n(M)$ be the solution $s$ to the equation

$$\sum_{1\leqslant a_1,\ldots,a_n\leqslant M} \left(\frac{1}{q_n^{\tau}(a_1,\ldots,a_n)}\right)^s = 1. \tag{2.8}$$

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and define

$$s(M) = \inf\left\{s \geqslant 0 : \mathrm{P}_A\left(T, -s \cdot \frac{\tau}{2}\log|T'|\right) \leqslant 0\right\},$$
$$s(\mathbb{N}) = \inf\left\{s \geqslant 0 : \mathrm{P}\left(T, -s \cdot \frac{\tau}{2}\log|T'|\right) \leqslant 0\right\}.$$

From the definition of $s_n(M)$ and Proposition 2.4, we have the following result.

COROLLARY 2.5 [**13**]. *For any integer $M \in \mathbb{N}$, we have $s_n(M)$ and $s(M)$ are increasing with respect to $M$ and*

$$\lim_{n\to\infty} s_n(M) = s(M), \qquad \lim_{M\to\infty} s(M) = s(\mathbb{N}) = 2/\tau. \tag{2.9}$$

At last, we give a simple fact about an $\mathbb{N}\backslash Q$-free set $Q$ and present a way to construct $q_n$ lying out of $Q$.

LEMMA 2.6. *Let $Q$ be an $\mathbb{N}\backslash Q$-free set with $m \notin Q$. Then no multiples of $m$ are in $Q$.*

*Proof.* This follows just from the definition of an $\mathbb{N}\backslash Q$-free set. □

At last we show a proposition, which presents a way to construct $q_n$ lying out of an $\mathbb{N}\backslash Q$-free set $Q$.

PROPOSITION 2.7. *Let $m \geqslant 2$ be an integer and $a_1, \ldots, a_n \in \mathbb{N}$. Let $q_{n-1}(a_1, \ldots, a_{n-1})$ and $q_n(a_1, \ldots, a_n)$ be recursively given by (2.2). Then one of the following three cases will occur:*

(i) $q_n(a_1, \ldots, a_n) \equiv 0 \pmod m$;

(ii) *there exists an integer $0 < z_1 \leqslant m$ such that*

$$q_{n+1}(a_1, \ldots, a_n, z_1) \equiv 0 \pmod m;$$

(iii) *there exist integers $0 < z_1, z_2 \leqslant m$ such that*

$$q_{n+2}(a_1, \ldots, a_n, z_1, z_2) \equiv 0 \pmod m.$$

*Proof.* Assume that

$$\begin{cases} q_{n-1} := q_{n-1}(a_1, \ldots, a_{n-1}) \equiv b_1 \pmod m, \\ q_n := q_n(a_1, \ldots, a_n) \equiv b_2 \pmod m. \end{cases}$$

If case (i) is not true, i.e. $b_2 \neq 0$, we show one of the other two will hold. This is argued by distinguishing three cases.

*Case 1:* $(b_2, m) = 1$, *i.e. $b_2$ and $m$ are coprime*. Then there exists an integer $0 < z_1 \leqslant m$ such that

$$z_1 b_2 \equiv m - b_1 \pmod m.$$

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Then

$$q_{n+1}(a_1,\dots,a_n,z_1) \equiv z_1 q_n + q_{n-1} \pmod m$$
$$\equiv z_1 b_2 + b_1 \pmod m \equiv 0 \pmod m.$$

Thus, case (ii) occurs.

*Case 2:* $(b_1, m) = 1$. Take $z_1 = m$. Then

$$q_{n+1}(a_1,\dots,a_n,z_1) \equiv m q_n + q_{n-1} \equiv b_1 \pmod m.$$

Then replace the roles of $q_{n-1}, q_n$ by $q_n, q_{n+1}$. We can proceed with the same argument as in case (1) to find $0 < z_2 \leqslant m$ such that

$$q_{n+2}(a_1,\dots,a_n,z_1,z_2) \equiv 0 \pmod m.$$

Thus, case (iii) occurs.

*Case 3:* $(b_1, m) > 1$, $(b_2, m) > 1$. Since $(q_{n-1}, q_n) = 1$, it follows that $(b_1, b_2, m) = 1$. Hence, we can write

$$(b_1, m) = p_{11}^{i_1} \cdots p_{1t}^{i_t}, \qquad (b_2, m) = p_{21}^{j_1} \cdots p_{2r}^{j_r},$$

and

$$m = (p_{11}^{\ell_1} \cdots p_{1t}^{\ell_t})(p_{21}^{k_1} \cdots p_{2r}^{k_r})(p_{31}^{s_1} \cdots p_{3u}^{s_u}),$$

where

$$\{p_{11},\dots,p_{1t},p_{21},\dots,p_{2r},p_{31},\dots,p_{3u}\}$$

are distinct primes. Let $z_1 = p_{31} \cdots p_{3u}$. Then

$$q_{n+1}(a_1,\dots,a_n,z_1) \equiv z_1 b_2 + b_1 \pmod m.$$

It is clear (if not trivial) that $(z_1 b_2 + b_1, m) = 1$. More precisely, let $p$ be a prime with $p|m$. By the choice of $z_1$, every prime divisor of $m$ must be a divisor of one and only one of $z_1, b_2, b_1$. Then we have

$$(p|z_1 b_2,\ p \nmid b_1) \quad \text{or} \quad (p \nmid z_1 b_2,\ p|b_1).$$

If $p|(z_1 b_2 + b_1)$, then $p|z_1 b_2$ and $p|b_1$, which arrives at a contradiction. This shows that

$$q_{n+1}(a_1,\dots,a_n,z_1) :\equiv b_3 \pmod m \quad \text{with } (b_3, m) = 1.$$

Then we can proceed with the same argument as the case (1) to find $0 < z_2 \leqslant m$ such that

$$q_{n+2}(a_1,\dots,a_n,z_1,z_2) \equiv 0 \pmod m.$$

Thus, case (iii) occurs. □

To end, we list some notation.

- When $w = (a_1, a_2, \dots,)$ is a word of length $\geqslant n$, we use $I_n(w)$ to denote the cylinder $I_n(a_1,\dots,a_n)$ and $q_n(w)$ to denote $q_n(a_1,\dots,a_n)$.
- We use $I_n(x)$ to denote the $n$th-order cylinder containing $x$.
- We say that two positive numbers $a$ and $b$ are equivalent, if there exist absolute constants $c_1, c_2 > 0$ independent of $a$ and $b$ such that $a \leqslant c_1 b \leqslant c_2 a$.

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§3. *Proof of Theorem 1.2.* The upper bound of $\dim_H(W_\tau\backslash W_\tau(Q))$ is clear since $\dim_H W_\tau = 2/\tau$. So, in the following, we focus on the lower bound of $\dim_H(W_\tau\backslash W_\tau(Q))$. This is done by constructing a suitable Cantor set inside $W_\tau\backslash W_\tau(Q)$ and then use the following mass distribution principle to estimate the dimension of this Cantor set.

PROPOSITION 3.1 [**6**, Proposition 4.2]. *Let $E$ be a Borel subset of $\mathbb{R}^d$ and $\mu$ be a Borel measure with $\mu(E) > 0$. Assume that there exist $\delta, c > 0$ such that for any $U \subset \mathbb{R}^d$ with diameter $|U| \leqslant \delta$, $\mu(U) \leqslant c|U|^s$, then* $\dim_H E \geqslant s$.

*Main idea.* Let us give a brief outline of the ideas to construct such a Cantor subset. Recall the definitions of the sets $W_\tau$ and $W_\tau(Q)$. We begin with two trivial facts that have been mentioned before.

- Let $m \notin Q$. Since $Q$ is $\mathbb{N}\backslash Q$-free, no multiples of $m$ are in $Q$.
- Legendre's theorem states that $|x - p/q|$ is sufficiently small (compared with $1/2q^2$) if and only if $p/q = p_n/q_n$ for some $n \in \mathbb{N}$ and the $(n+1)$th partial quotient $a_{n+1}(x)$ assumes a large value.
- Choose those $x$ for which $a_{n+1}(x)$ is large only when $q_n(x) \notin Q$. This ensures that $|x - p/q| < 1/q^\tau$ cannot happen when $q \in Q$.

For any $(a_1, \ldots, a_n)$, after at most two steps by a suitable choice of $(a_{n+1})$ or $(a_{n+1}, a_{n+2})$, we can ensure that $q_{n+1}$ or $q_{n+2}$ will be a multiple of $m$ (Proposition 2.7), so not in $Q$. Then once $q_{n+i}$ is a multiple of $m$ for some $1 \leqslant i \leqslant 2$, we choose $a_{n+i+1}$ as large as we want to ensure that $x - p_{n+i}/q_{n+i}$ sufficiently small. More precisely, for any $\tau > 2$,

$$\left|x - \frac{p_{n+i}(x)}{q_{n+i}(x)}\right| < \frac{1}{q_{n+i}^{\tau}(x)} \Longleftarrow a_{n+i+1}(x) \geqslant q_{n+i}^{\tau-2}(x)$$

by Proposition 2.1(2). This will guide us on the choice of large partial quotients below. Then the points in the cylinder $I_{n+i}(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+i})$ will be the possible candidates in $W_\tau\backslash W_\tau(Q)$.

3.1. *Cantor subset.* Assume that $m \notin Q$, then all of the multiples of $m$ lie out of $Q$. Let $\epsilon > 0$. Fix two large integers $M \geqslant m$ and $N \geqslant \max\{4/\epsilon, 4/(1-2/\tau)\}$. We choose a sequence of integers $\{n_k\}_{k\geqslant 0}$ such that $n_0 = 0$ and for each $k \geqslant 1$:

- $n_k \geqslant k(n_1 + \cdots + n_{k-1})$;
- $n_k - n_{k-1} - 3 := m_k$ is a multiple of $N$, so write $n_k - n_{k-1} - 3 = t_k N$ for some integer $t_k \in \mathbb{N}$.

Let

$$\mathcal{M} = \{w = (i_1, \ldots, i_N) : 1 \leqslant i_k \leqslant M,\ 1 \leqslant k \leqslant N\}.$$

In the following, when the symbol $w_i^{(k)}$ is used to denote a word, it is always an element from $\mathcal{M}$, i.e. a word of length $N$ with all its components being bounded by $M$.

*Level 1 of the Cantor set.* For any $w_1^{(1)}, \ldots, w_{t_1}^{(1)} \in \mathcal{M}$, apply Proposition 2.7 to

$$q_{m_1-1}(w_1^{(1)}, \ldots, w_{t_1}^{(1)}), \qquad q_{m_1}(w_1^{(1)}, \ldots, w_{t_1}^{(1)}).$$

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If case (i) in Proposition 2.7 occurs, define

$$\begin{aligned}\mathbb{F}_1(w_1^{(1)},\dots,w_{t_1}^{(1)})&\\ &= \{I_{n_1}(w_1^{(1)},\dots,w_{t_1}^{(1)},a_{n_1-2},2,2) : q_{n_1-3}^{\tau-2}(w_1^{(1)},\dots,w_{t_1}^{(1)})\\ &\quad \leqslant a_{n_1-2} < 2q_{n_1-3}^{\tau-2}(w_1^{(1)},\dots,w_{t_1}^{(1)})\}.\end{aligned}$$

If case (ii) in Proposition 2.7 occurs, define

$$\begin{aligned}\mathbb{F}_1(w_1^{(1)},\dots,w_{t_1}^{(1)})&\\ &= \{I_{n_1}(w_1^{(1)},\dots,w_{t_1}^{(1)},z_1,a_{n_1-1},2) : q_{n_1-2}^{\tau-2}(w_1^{(1)},\dots,w_{t_1}^{(1)},z_1) \leqslant a_{n_1-1}\\ &\quad < q_{n_1-2}^{\tau-2}(w_1^{(1)},\dots,w_{t_1}^{(1)},z_1) + q_{n_1-3}^{\tau-2}(w_1^{(1)},\dots,w_{t_1}^{(1)})\}.\end{aligned}$$

If case (iii) in Proposition 2.7 occurs, define

$$\begin{aligned}\mathbb{F}_1(w_1^{(1)},\dots,w_{t_1}^{(1)})&\\ &= \{I_{n_1}(w_1^{(1)},\dots,w_{t_1}^{(1)},z_1,z_2,a_{n_1}) : q_{n_1-1}^{\tau-2}(w_1^{(1)},\dots,w_{t_1}^{(1)},z_1,z_2) \leqslant a_{n_1}\\ &\quad < q_{n_1-1}^{\tau-2}(w_1^{(1)},\dots,w_{t_1}^{(1)},z_1,z_2) + q_{n_1-3}^{\tau-2}(w_1^{(1)},\dots,w_{t_1}^{(1)})\}.\end{aligned}$$

Then the first level of the Cantor set is defined as

$$\mathbb{F}_1 = \bigcup_{w_1^{(1)},\dots,w_{t_1}^{(1)}\in\mathcal{M}} \mathbb{F}_1(w_1^{(1)},\dots,w_{t_1}^{(1)}).$$

To save notation, we use $I_{n_1}(\epsilon^{(1)})$ to denote a general cylinder in $\mathbb{F}_1$, where $\epsilon^{(1)}$ is a word of length $n_1$ representing the vector of the partial quotients defining the cylinder $I_{n_1}(\epsilon^{(1)})$.

*Remark* 2. The integers $z_1$ and $z_2$ are chosen according to Proposition 2.7. So for different choice of $(w_1^{(1)},\dots,w_{t_1}^{(1)})$, the integers $z_1, z_2$ may be different. But for simplifying the notation, we hide this dependence, which will not cause any trouble in the argument later. The integer 2 occurring in the cylinder $I_{n_1}(\epsilon^{(1)})$ above does not have any specific role other than an integer larger than 1. It can be replaced by any fixed integer $b \geqslant 2$.

*From level $k-1$ to level $k$.* General levels of the Cantor set will be constructed similarly. Assume that the $(k-1)$th level $\mathbb{F}_{k-1}$ of the Cantor set, which is a family of cylinders of order $n_{k-1}$, has been well constructed. For each $I_{n_{k-1}}(\epsilon^{(k-1)}) \in \mathbb{F}_{k-1}$ and $w_1^{(k)},\dots,w_{t_k}^{(k)} \in \mathcal{M}$, we define a subfamily $\mathbb{F}_k(\epsilon^{(k-1)}, w_1^{(k)},\dots,w_{t_k}^{(k)})$ of the $k$th level $\mathbb{F}_k$ as follows. Apply Proposition 2.7 to

$$q_{n_{k-1}+m_k-1}(\epsilon^{(k-1)}, w_1^{(k)},\dots,w_{t_k}^{(k)}), \qquad q_{n_{k-1}+m_k}(\epsilon^{(k-1)}, w_1^{(k)},\dots,w_{t_k}^{(k)}).$$

(Note that the length of the word $(w_1^{(k)},\dots,w_{t_k}^{(k)})$ is $t_kN = m_k = n_k - n_{k-1} - 3$ and that $n_{k-1} + m_k = n_k - 3$.)

If case (i) in Proposition 2.7 occurs, then

$$q_{n_k-3}(\epsilon^{(k-1)}, w_1^{(k)},\dots,w_{t_k}^{(k)}) \equiv 0 \pmod m, \tag{3.1}$$

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so not in $Q$. Define

$$\begin{aligned}&\mathbb{F}_k(\epsilon^{(k-1)}, w_1^{(k)}, \dots, w_{t_k}^{(k)})\\&\quad = \{I_{n_k}(\epsilon^{(k-1)}, w_1^{(k)}, \dots, w_{t_k}^{(k)}, a_{n_k-2}, 2, 2) : q_{n_k-3}^{\tau-2}(\epsilon^{(k-1)}, w_1^{(k)}, \dots, w_{t_k}^{(k)})\\&\quad \leqslant a_{n_k-2} < q_{n_k-3}^{\tau-2}(\epsilon^{(k-1)}, w_1^{(k)}, \dots, w_{t_k}^{(k)}) + q_{m_k}^{\tau-2}(w_1^{(k)}, \dots, w_{t_k}^{(k)})\}.\end{aligned}$$

So, for any $x \in I_{n_k}(\epsilon^{(k)}) \in \mathbb{F}_k(\epsilon^{(k-1)}, w_1^{(k)}, \dots, w_{t_k}^{(k)})$,

$$\left|x - \frac{p_{n_k-3}(x)}{q_{n_k-3}(x)}\right| < \frac{1}{a_{n_k-2}(x)q_{n_k-3}^2(x)} \leqslant \frac{1}{q_{n_k-3}^{\tau}(x)}. \tag{3.2}$$

If case (ii) in Proposition 2.7 occurs, we can find an integer $1 \leqslant z_1 \leqslant m$ such that

$$q_{n_k-2}(\epsilon^{(k-1)}, w_1^{(k)}, \dots, w_{t_k}^{(k)}, z_1) \equiv 0 \pmod m, \tag{3.3}$$

so not in $Q$. Define

$$\begin{aligned}&\mathbb{F}_k(\epsilon^{(k-1)}, w_1^{(k)}, \dots, w_{t_k}^{(k)})\\&= \{I_{n_k}(\epsilon^{(k-1)}, w_1^{(k)}, \dots, w_{t_k}^{(k)}, z_1, a_{n_k-1}, 2) : q_{n_k-2}^{\tau-2}(\epsilon^{(k-1)}, w_1^{(k)}, \dots, w_{t_k}^{(k)}, z_1)\\&\leqslant a_{n_k-1} < q_{n_k-2}^{\tau-2}(\epsilon^{(k-1)}, w_1^{(k)}, \dots, w_{t_k}^{(k)}, z_1) + q_{m_k}^{\tau-2}(w_1^{(k)}, \dots, w_{t_k}^{(k)})\}.\end{aligned}$$

So, for any $x \in I_{n_k}(\epsilon^{(k)}) \in \mathbb{F}_k(\epsilon^{(k-1)}, w_1^{(k)}, \dots, w_{t_k}^{(k)})$,

$$\left|x - \frac{p_{n_k-2}(x)}{q_{n_k-2}(x)}\right| < \frac{1}{a_{n_k-1}(x)q_{n_k-2}^2(x)} \leqslant \frac{1}{q_{n_k-2}^{\tau}(x)}. \tag{3.4}$$

If case (iii) in Proposition 2.7 occurs, we can find integers $1 \leqslant z_1, z_2 \leqslant m$ such that

$$q_{n_k-1}(\epsilon^{(k-1)}, w_1^{(k)}, \dots, w_{t_k}^{(k)}, z_1, z_2) \equiv 0 \pmod m, \tag{3.5}$$

so not in $Q$. Define

$$\begin{aligned}&\mathbb{F}_k(\epsilon^{(k-1)}, w_1^{(k)}, \dots, w_{t_k}^{(k)})\\&= \{I_{n_k}(\epsilon^{(k-1)}, w_1^{(k)}, \dots, w_{t_k}^{(k)}, z_1, z_2, a_{n_k}) : q_{n_k-1}^{\tau-2}(\epsilon^{(k-1)}, w_1^{(k)}, \dots, w_{t_k}^{(k)}, z_1, z_2)\\&\leqslant a_{n_k} < q_{n_k-1}^{\tau-2}(\epsilon^{(k-1)}, w_1^{(k)}, \dots, w_{t_k}^{(k)}, z_1, z_2) + q_{m_k}^{\tau-2}(w_1^{(k)}, \dots, w_{t_k}^{(k)})\}.\end{aligned}$$

So, for any $x \in I_{n_k}(\epsilon^{(k)}) \in \mathbb{F}_k(\epsilon^{(k-1)}, w_1^{(k)}, \dots, w_{t_k}^{(k)})$,

$$\left|x - \frac{p_{n_k-1}(x)}{q_{n_k-1}(x)}\right| < \frac{1}{a_{n_k}(x)q_{n_k-1}^2(x)} \leqslant \frac{1}{q_{n_k-1}^{\tau}(x)}. \tag{3.6}$$

It should be noted that in each of the above three cases, $\mathbb{F}_k(\epsilon^{(k-1)}, w_1^{(k)}, \dots, w_{t_k}^{(k)})$ contains $q_{m_k}^{\tau-2}(w_1^{(k)}, \dots, w_{t_k}^{(k)})$ many elements. In other words, from the cylinder $I_{n_k-3}(\epsilon^{(k-1)}, w_1^{(k)}, \dots, w_{t_k}^{(k)})$, we only select $q_{m_k}^{\tau-2}(w_1^{(k)}, \dots, w_{t_k}^{(k)})$ many sub-cylinders of order $n_k$ to constitute the Cantor set.

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Then the $k$th level of the Cantor set is defined as

$$\mathbb{F}_k = \bigcup_{I_{n_{k-1}}(\epsilon^{(k-1)})\in\mathbb{F}_{k-1}} \bigcup_{w_1^{(k)},\dots,w_{t_k}^{(k)}\in\mathcal{M}} \mathbb{F}_k(\epsilon^{(k-1)}, w_1^{(k)}, \dots, w_{t_k}^{(k)}).$$

In this way, we define a sequence of families $\{\mathbb{F}_k\}_{k\geqslant 1}$ composed of cylinders. Finally, define the desired Cantor set as

$$\mathbb{F}_\infty = \bigcap_{k\geqslant 1} \bigcup_{I_{n_k}(\epsilon^{(k)})\in\mathbb{F}_k} I_{n_k}(\epsilon^{(k)}).$$

For each $x \in \mathbb{F}_\infty$, call $I_n(x)$ a *basic cylinder* of order $n$.

LEMMA 3.2. *We have* $\mathbb{F}_\infty \subset W_\tau \backslash W_\tau(Q)$.

*Proof.* Let $x \in \mathbb{F}_\infty$. By (3.2), (3.4) and (3.6), we know that $x \in W_\tau$. Assume that $|x - p/q| < q^\tau$ for some $q \in Q$. Then by Legendre's theorem, we know that $p/q = p_n(x)/q_n(x)$ for some $n \geqslant 1$. Since $(p_n(x), q_n(x)) = 1$, we have $q_n(x)$ is a divisor of $q$, so in $Q$ by the definition of an $\mathbb{N}\backslash Q$-free set. At the same time, by (2.3), we also have that

$$a_{n+1}(x) \geqslant q_n^{\tau-2}(x)/2 > M.$$

However, in the construction of $\mathbb{F}_\infty$, $a_{n+1}(x) > M$ occurs only when $q_n(x)$ lies out of $Q$. Then we get a contradiction. □

3.2. *Supporting measure on* $\mathbb{F}_\infty$. The method to construct a suitable probability measure supported on $\mathbb{F}_\infty$ is similar to that used in [**15**], where we considered the Hausdorff dimension of the set

$$\{x \in [0, 1] : a_n(x) \geqslant B^n, \text{for infinitely many } n \in \mathbb{N}\}.$$

Recall the definition of the dimensional number $s_N(M)$ given by (2.8). Now we show $\dim_H \mathbb{F}_\infty \geqslant s_N(M)$, which is sufficient to conclude the main result by Corollary 2.5. In the following, only the dimensional number $s_N(M)$ will be used, so we denote it by $s$. Thus, we have

$$\sum_{w\in\mathcal{M}} \left(\frac{1}{q_N^\tau(w)}\right)^s = 1.$$

Now we define a probability measure supported on $\mathbb{F}_\infty$. This is done by distributing the mass to each of the cylinders $\{I_n(x)\}_{n\geqslant 1}$ for all $x \in \mathbb{F}_\infty$ in the following way.

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Write the continued fraction expansion of $x$ by

$$x = [w_1^{(1)}, \dots, w_{t_1}^{(1)}, a_{n_1-2}, a_{n_1-1}, a_{n_1}, \dots, w_1^{(k)}, \dots, w_{t_k}^{(k)}, a_{n_k-2}, a_{n_k-1}, a_{n_k}, \dots]. \tag{3.7}$$

In the following steps (a1) to (a3), we define the measure of basic cylinders of order at most $n_1$. Then in steps (b1) to (b3), the measure of cylinders of arbitrary orders is given.

(a1) When $n = tN$ for some $1 \leqslant t \leqslant t_1$, we define

$$\mu(I_{tN}(x)) = \mu(I_{tN}(w_1^{(1)}, \dots, w_t^{(1)})) = \prod_{i=1}^{t} \left( \frac{1}{q_N^{\tau}(w_i^{(1)})} \right)^s. \tag{3.8}$$

Since for each $w \in \mathcal{M}$, the components of $w$ are all bounded by $M$, it follows from (3.8) and Proposition 2.1(1) that

$$1 \leqslant \frac{\mu(I_{tN}(x))}{\mu(I_{(t+1)N})(x)} \leqslant (M+1)^{\tau N} := C_1 \quad \text{for all } 0 \leqslant t < t_1.$$

(a2) When $m_1 < n \leqslant n_1$. Since the basic sub-cylinders of $I_{m_1}(x)$ of order $n_1$ are of comparable length, we distribute the mass of $I_{m_1}(x)$ evenly among its offspring:

$$\begin{aligned} \mu(I_n(x)) = \mu(I_{n_1}(x)) &= \frac{1}{q_{m_1}^{\tau-2}(w_1^{(1)}, \dots, w_{t_1}^{(1)})} \mu(I_{m_1}(x)) \\ &= \frac{1}{q_{m_1}^{\tau-2}(w_1^{(1)}, \dots, w_{t_1}^{(1)})} \prod_{i=1}^{t_1} \left( \frac{1}{q_N^{\tau}(w_i^{(1)})} \right)^s. \end{aligned}$$

(a3) To make sure that $\mu$ is a measure supported on $\mathbb{F}_\infty$, the measure on $I_n(x)$ with $n < m_1$ is defined to be the total mass of its offspring. Clearly when $tN < n < (t+1)N$ for some $0 \leqslant t < t_1$,

$$\mu(I_{(t+1)N}(x)) \leqslant \mu(I_n(x)) \leqslant \mu(I_{tN}(x)).$$

Now we define the measure $\mu$ inductively. Assume that $\mu(I_{n_{k-1}}(x))$ has been defined. Let $n_{k-1} < n \leqslant n_k$.

(b1) When $n = n_{k-1} + tN$ for some $1 \leqslant t \leqslant t_k$, we define

$$\begin{aligned} \mu(I_{n_{k-1}+tN}(x)) &= \mu(I_{n_{k-1}}(x)) \cdot \prod_{i=1}^{t} \left( \frac{1}{q_N^{\tau}(w_i^{(k)})} \right)^s \\ &= \prod_{\ell=1}^{k-1} \left( \prod_{i=1}^{t_\ell} \left( \frac{1}{q_N^{\tau}(w_i^{(\ell)})} \right)^s \cdot \frac{1}{q_{m_\ell}^{\tau-2}(w_1^{(\ell)}, \dots, w_{t_\ell}^{\ell})} \right) \\ &\quad \cdot \prod_{i=1}^{t} \left( \frac{1}{q_N^{\tau}(w_i^{(k)})} \right)^s. \end{aligned} \tag{3.9}$$

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Still by the fact that for each $w \in \mathcal{M}$, the components of $w$ are all bounded by $M$, it follows from (3.2) and Proposition 2.1(1) that

$$1 \leqslant \frac{\mu(I_{n_{k-1}+tN}(x))}{\mu(I_{n_{k-1}+(t+1)N})(x)} \leqslant (M+1)^{\tau N} := C_1 \quad \text{for all } 0 \leqslant t < t_k. \tag{3.10}$$

(b2) When $n_{k-1} + m_k < n \leqslant n_k$, the mass of $I_{n_{k-1}+m_k}(x)$ is equally distributed among its offspring, i.e.

$$\mu(I_n(x)) = \mu(I_{n_k}(x)) = \mu(I_{n_{k-1}+m_k}(x)) \frac{1}{q_{m_k}^{\tau-2}(w_1^{(k)}, \dots, w_{t_k}^{(k)})} \tag{3.11}$$

$$= \prod_{\ell=1}^{k} \left( \prod_{i=1}^{t_\ell} \left( \frac{1}{q_N^{\tau}(w_i^{(\ell)})} \right)^s \cdot \frac{1}{q_{m_\ell}^{\tau-2}(w_1^{(\ell)}, \dots, w_{t_\ell}^{(\ell)})} \right). \tag{3.12}$$

(b3) Similarly, the measure on the remaining cylinders $I_n(x)$ with $n_{k-1} < n < n_{k-1} + m_k$ is defined as the total measure of its offspring. So, clearly when $n_{k-1} + tN < n < n_{k-1} + (t+1)N$ for some $0 \leqslant t < t_k$,

$$\mu(I_{n_{k-1}+(t+1)N}(x)) \leqslant \mu(I_n(x)) \leqslant \mu(I_{n_{k-1}+tN}(x)). \tag{3.13}$$

Finally, we give an additional fact that will be used later. To refer to it easily we state it as a lemma.

LEMMA 3.3. *Given two basic cylinders* $I_{n_{k-1}+m_k}(\epsilon^{(k-1)}, w_1, \dots, w_{t_k-1}, w_{t_k})$ *and* $I_{n_{k-1}+m_k}(\epsilon^{(k-1)}, w_1, \dots, w_{t_k-1}, w'_{t_k})$ *which may differ only at the last N partial quotients. We have*

$$C_2 \leqslant \frac{\mu(I_{n_{k-1}+m_k}(\epsilon^{(k-1)}, w_1, \dots, w_{t_k}))}{\mu(I_{n_{k-1}+m_k}(\epsilon^{(k-1)}, w_1, \dots, w'_{t_k}))} \leqslant C_3$$

*for some absolute constants* $C_2, C_3$ *depending only on* $\tau$, $N$ *and* $M$.

*Proof.* Note that the components of $w_{t_k}$ and $w'_{t_k}$ are all bounded by $M$. So,

$$C_2 := (M+1)^{-N\tau} \leqslant \left( \frac{q_N(w'_{t_k})}{q_N(w_{t_k})} \right)^{s\tau} \leqslant (M+1)^{N\tau} := C_1.$$

Then the desired assertion follows directly from (3.9). □

By the definition of $s$ and also noticing that $I_{n_{k-1}+m_k}(x)$ contains $q_{m_k}^{\tau-2}(w_1^{(k)}, \dots, w_{t_k}^{(k)})$ basic sub-cylinders of order $n_k$, the set function $\mu$ defined in this way satisfies Kolmogorov's consistency condition. So it can be uniquely extended into a probability measure supported on $\mathbb{F}_\infty$.

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3.3. *Lengths of basic cylinders.* By the choice of $\{n_k\}$ at the beginning of this section, a sequence of integers increasing to infinity in a very fast speed, for any $x \in \mathbb{F}_\infty$ with its continued fraction expansion given by (3.7), we have

$$4^2 \leqslant 2^{(N/2-1)\epsilon}, \qquad 4^{t_k} m^4 q_{n_{k-1}}^{2\tau+2}(x) \leqslant q_{m_k}(w_1^{(k)}, \ldots, w_{t_k}^{(k)})^\epsilon \quad \text{for all } k \geqslant 1. \tag{3.14}$$

The following simple facts will be used frequently (Proposition 2.1):

$$q_n \geqslant 2^{n/2-1}, \qquad 1 \leqslant \frac{q_{n+m}(a_1, \ldots, a_n, b_1, \ldots, b_m)}{q_n(a_1, \ldots, a_n) q_m(b_1, \ldots, b_m)} \leqslant 2. \tag{3.15}$$

Moreover, since $s = s_N(M) < 2/\tau$ and $N \geqslant 4/(1-2/\tau)$, then $N \geqslant 4/(1-s)$. Recall that $m_k = N \cdot t_k$. Thus, by (3.15), we have for any $k \geqslant 1$,

$$\left(\prod_{i=1}^{t_k} q_N(w_i^{(k)})\right)^s \geqslant (2^{-t_k} q_{m_k}(w_1^{(k)}, \ldots, w_{t_k}^{(k)}))^s \geqslant (q_{m_k}(w_1^{(k)}, \ldots, w_{t_k}^{(k)}))^{2s-1}. \tag{3.16}$$

In the following steps (c1) to (c3), we estimate the lengths of cylinders of order at most $n_1$. Then in steps (d1) to (d3), the lengths of cylinders of arbitrary orders are given.

(c1) When $n = tN$ for some $1 \leqslant t \leqslant t_1$, by (3.15) and the first item in (3.14),

$$|I_{tN}(x)| \geqslant \frac{1}{2q_{tN}^2(w_1^{(1)}, \ldots, w_t^{(1)})} \geqslant \frac{1}{4^{t+1}} \prod_{i=1}^{t} \frac{1}{q_N^2(w_i^{(1)})} \geqslant \left(\prod_{i=1}^{t} \frac{1}{q_N^2(w_i^{(1)})}\right)^{1+\epsilon}.$$

(c2) When $m_1 < n \leqslant n_1$,

$$|I_n(x)| \geqslant |I_{n_1}(x)| \geqslant \frac{1}{8q_{m_1}^2(x)} \frac{1}{q_3^2(a_{n_1-2}, a_{n_1-1}, a_{n_1})}.$$

Note that

$$q_3(a_{n_1-2}, a_{n_1-1}, a_{n_1}) \leqslant 2^3 a_{n_1-2} a_{n_1-1} a_{n_1} \leqslant 2^3 \cdot m^2 \cdot 2q_{m_1}^{\tau-2}(x).$$

Thus, it follows that

$$\begin{aligned} |I_n(x)| \geqslant |I_{n_1}(x)| &\geqslant \frac{1}{q_{m_1}^2(x)} \left(\frac{1}{q_{m_1}^{2(\tau-2)}(x)}\right)^{1+\epsilon} \\ &\geqslant \left(\prod_{i=1}^{t_1} \frac{1}{q_N^2(w_i^{(1)})} \cdot \frac{1}{q_{m_1}^{2(\tau-2)}(x)}\right)^{1+\epsilon}. \end{aligned}$$

(c3) When $n = tN + i$ for some $1 \leqslant i < N$ and $0 \leqslant t < t_1$, since $a_{tN+1}(x), \ldots, a_n(x)$ are all bounded by $M$, we have

$$|I_n(x)| \geqslant \frac{1}{8(M+1)^{2N}} |I_{tN}(x)| := C_3 |I_{tN}(x)|.$$

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Now we estimate the lengths of basic cylinders inductively. Assume that $|I_{n_{k-1}}(x)|$ has been estimated, saying that

$$|I_{n_{k-1}}(x)| \geqslant \left(\prod_{\ell=1}^{k-1}\left(\prod_{i=1}^{t_\ell}\frac{1}{q_N^2(w_i^{(\ell)})}\cdot\frac{1}{q_{m_\ell}^{2(\tau-2)}(w_1^{(\ell)},\dots,w_{t_\ell}^{(\ell)})}\right)\right)^{1+\epsilon}. \tag{3.17}$$

Let $n_{k-1} < n \leqslant n_k$ for some $k \geqslant 1$.

(d1) When $n = n_{k-1} + tN$ for some $1 \leqslant t \leqslant t_k$,

$$\begin{aligned}|I_{n_{k-1}+tN}(x)| &\geqslant |I_{n_{k-1}}(x)|\cdot\frac{1}{8q_{tN}^2(w_1^{(k)},\dots,w_t^{(k)})}\\ &\geqslant |I_{n_{k-1}}(x)|\left(\prod_{i=1}^{t}\frac{1}{q_N^2(w_i^{(k)})}\right)^{1+\epsilon}.\end{aligned}$$

So, by (3.17), we have

$$\begin{aligned}|I_{n_{k-1}+tN}(x)| \geqslant &\left(\prod_{\ell=1}^{k-1}\left(\prod_{i=1}^{t_\ell}\frac{1}{q_N^2(w_i^{(\ell)})}\cdot\frac{1}{q_{m_\ell}^{2(\tau-2)}(w_1^{(\ell)},\dots,w_{t_\ell}^{(\ell)})}\right)\right)^{1+\epsilon}\\ &\cdot\left(\prod_{i=1}^{t}\frac{1}{q_N^2(w_i^{(k)})}\right)^{1+\epsilon}.\end{aligned} \tag{3.18}$$

(d2) When $n_{k-1} + m_k < n \leqslant n_k$, similar to the estimation in step (c2),

$$\begin{aligned}|I_n(x)| \geqslant |I_{n_k}(x)| &\geqslant |I_{n_{k-1}}(x)|\cdot\frac{1}{q_{m_k}^2(w_1^{(k)},\dots,w_{t_k}^{(k)})}\\ &\quad\times\left(\frac{1}{q_{m_k}^{2(\tau-2)}(w_1^{(k)},\dots,w_{t_k}^{(k)})}\right)^{1+\epsilon}\\ &\geqslant |I_{n_{k-1}}(x)|\left(\prod_{i=1}^{t_k}\frac{1}{q_N^2(w_i^{(k)})}\cdot\frac{1}{q_{m_k}^{2(\tau-2)}(w_1^{(k)},\dots,w_{t_k}^{(k)})}\right)^{1+\epsilon}.\end{aligned} \tag{3.19}$$

So by (3.17), we have

$$|I_n(x)| \geqslant \left(\prod_{\ell=1}^{k}\left(\prod_{i=1}^{t_\ell}\frac{1}{q_N^2(w_i^{(\ell)})}\cdot\frac{1}{q_{m_\ell}^{2(\tau-2)}(w_1^{(\ell)},\dots,w_{t_\ell}^{(\ell)})}\right)\right)^{1+\epsilon}. \tag{3.20}$$

(d3) When $n = n_{k-1} + tN + i$ for some $1 \leqslant i < N$ and $0 \leqslant t < t_k$, since $a_{n_{k-1}+tN+1},\dots,a_n(x)$ are all bounded by $M$,

$$|I_n(x)| \geqslant \frac{1}{8(M+1)^{2N}}|I_{n_{k-1}+tN}(x)|. \tag{3.21}$$

Similar to Lemma 3.3, we give another simple fact for later use.

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LEMMA 3.4. *Given two basic cylinders* $I_{n_k}(\epsilon^{(k-1)}, w_1, \ldots, w_{t_k-1}, w_{t_k}, a_{n_k-2}, a_{n_k-1}, a_{n_k})$ *and* $I_{n_k}(\epsilon^{(k-1)}, w_1, \ldots, w_{t_k-1}, w'_{t_k}, a'_{n_k-2}, a'_{n_k-1}, a'_{n_k})$ *which may differ only at the last* $N+3$ *partial quotients, we have*

$$C_4 \leqslant \frac{|I_{n_k}(\epsilon^{(k-1)}, w_1, \ldots, w_{t_k}, a_{n_k-2}, a_{n_k-1}, a_{n_k})|}{|I_{n_k}(\epsilon^{(k-1)}, w_1, \ldots, w'_{t_k}, a'_{n_k-2}, a'_{n_k-1}, a'_{n_k})|} \leqslant C_5 \tag{3.22}$$

*for some absolute constants* $C_4, C_5$ *depending only on* $N$ *and* $M$.

*Proof.* This can be seen by two facts: from the construction of $\mathbb{F}_k$ and by (2.4), we have that:

- the components of $w_{t_k}$ and $w'_{t_k}$ are all bounded by $M$;
- among the last 3 words $a_{n_k-2}, a_{n_k-1}, a_{n_k}$, two of them are bounded by $m$, while the other one is equivalent to $q^{\tau-2}_{n_{k-1}+m_k}(\epsilon^{(k-1)}, w_1, \ldots, w_{t_k-1}, w_{t_k})$, so is equivalent to $q^{\tau-2}_{n_{k-1}+m_k-N}(\epsilon^{(k-1)}, w_1, \ldots, w_{t_k-1})$. □

3.4. *Mass distribution principle.* In this subsection, we will compare the measure of basic cylinders with their lengths and then move to arbitrary balls.

LEMMA 3.5. *There exists an absolute constant* $c_1$ *depending only on* $m, N$ *and* $M$ *such that, for any* $x \in \mathbb{F}_\infty$,

$$\mu(I_n(x)) \leqslant c_1 \cdot |I_n(x)|^{s/(1+\epsilon)}. \tag{3.23}$$

*Proof.* By (3.10), (3.13) and (3.11) on the $\mu$-measure of a cylinder and by (3.19) and (3.21) on the length of a cylinder, it suffices to show the desired result when $n = n_{k-1} + tN$ and $n = n_k$ for all $1 \leqslant t \leqslant t_k$ and $k \geqslant 1$.

When $n = n_{k-1} + tN$, recall (3.9) and (3.18). Look at the inner terms of the first product in (3.9) and (3.18). By moving same terms to one side, the formula (3.16) is nothing but to say

$$\prod_{i=1}^{t_\ell} \left(\frac{1}{q^\tau_N(w_i^{(\ell)})}\right)^s \cdot \frac{1}{q^{\tau-2}_{m_\ell}(w_1^{(\ell)}, \ldots, w_{t_\ell}^{\ell})} \leqslant \left(\prod_{i=1}^{t_\ell} \frac{1}{q^2_N(w_i^{(\ell)})} \cdot \frac{1}{q^{2(\tau-2)}_{m_\ell}(w_1^{(\ell)}, \ldots, w_{t_\ell}^{(\ell)})}\right)^s.$$

At the same time, since $\tau > 2$, it is clear that

$$\left(\prod_{i=1}^{t} \frac{1}{q^\tau_N(w_i^{(k)})}\right)^s \leqslant \left(\prod_{i=1}^{t} \frac{1}{q^2_N(w_i^{(k)})}\right)^s.$$

Combining these two estimations together, we arrive at

$$\mu(I_n(x)) \leqslant |I_n(x)|^{s/(1+\epsilon)}.$$

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When $n = n_k$, by (3.12), (3.20) and (3.16), the estimation can be carried out similarly. We highlight the estimation when $n = n_{k-1} + m_k$, which will be more accurate than (3.23). Recall (3.9) on the measure of $I_{n_{k-1}+m_k}(x)$. By (3.14), we have

$$\mu(I_{n_{k-1}+m_k}(x)) \leqslant \left(\prod_{i=1}^{t_k} \frac{1}{q_N^\tau(w_i^{(k)})}\right)^s \leqslant \left(\frac{1}{q_{n_k-3}^\tau(x)}\right)^{s/(1+\epsilon)}. \tag{3.24}$$

□

LEMMA 3.6. *Let* $s_0 = \min\{s/(1+\epsilon), (\tau s/(1+\epsilon))/(2\tau - 2 - (\tau - 2)/(1+\epsilon))\}$. *There exists an absolute constant* $c_2$ *depending only on* $m, N$ *and* $M$ *such that, for any* $x \in \mathbb{F}_\infty$,

$$\mu(B(x,r)) \leqslant c_2 \cdot |I_n(x)|^{s_0}.$$

*Proof.* Let $x \in \mathbb{F}_\infty$ and $r > 0$ small enough. Let $k \geqslant 1$ be the integer such that $|I_{n_k}(x)| \leqslant r < |I_{n_{k-1}}(x)|$. We will distinguish three cases according to the range of $r$.

*Case i:* $|I_{n_{k-1}+N}(x)| \leqslant r < |I_{n_{k-1}}(x)|$. By Proposition 2.2, we have

$$\begin{aligned}\mu(B(x,r)) &\leqslant \mu(I_{n_{k-1}-2}(x)) = \mu(I_{n_{k-1}}(x))\\ &\leqslant c_1 |I_{n_{k-1}}(x)|^{s/(1+\epsilon)}\\ &\leqslant c_1 \cdot 8(M+1)^{2N} \cdot |I_{n_{k-1}+N}(x)|^{s/(1+\epsilon)},\end{aligned}$$

where the last estimation follows from (3.21). Hence, we get

$$\mu(B(x,r)) \leqslant c_2 r^{s/(1+\epsilon)}.$$

*Case ii:* $|I_{n_{k-1}+(t+1)N}(x)| \leqslant r < |I_{n_{k-1}+tN}(x)|$ *for some* $1 \leqslant t < t_k$. Similarly,

$$\begin{aligned}\mu(B(x,r)) &\leqslant \mu(I_{n_{k-1}+(t-1)N}(x))\\ &\leqslant c_1 |I_{n_{k-1}+(t-1)N}(x)|^{s/(1+\epsilon)}\\ &\leqslant c_1 \cdot 64(M+1)^{4N} \cdot |I_{n_{k-1}+(t+1)N}(x)|^{s/(1+\epsilon)},\end{aligned}$$

where the last estimation follows from (3.21). Hence, we get

$$\mu(B(x,r)) \leqslant c_2 r^{s/(1+\epsilon)}.$$

*Case iii:* $|I_{n_k}(x)| \leqslant r < |I_{n_k-3}(x)|$. On the one hand,

$$\mu(B(x,r)) \leqslant \mu(I_{n_k-3-N}(x)) \leqslant (M+1)^{2N} \mu(I_{n_k-3}(x)).$$

On the other hand, since the lengths of sub-basic cylinders of $I_{n_k-3-N}(x)$ of order $n_k$ are equivalent (Lemma 3.4), the ball $B(x,r)$ can contain at most

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$cr/|I_{n_k}(x)|$ basic cylinders of order $n_k$ contained in $I_{n_k-3-N}(x)$. Moreover, since all of these basic cylinders are of equivalent $\mu$-measure (Lemma 3.3), we have

$$\begin{aligned}\mu(B(x,r)) &\leqslant 2cr q_{n_k}^2(x)\cdot C_1\mu(I_{n_k}(x))\\ &= 2cr q_{n_k}^2(x)\frac{C_1}{q_{m_k}^{\tau-2}(w_1^{(k)},\ldots,w_{t_k}^{(k)})}\mu(I_{n_k-3}(x)).\end{aligned}$$

Note that

$$q_{n_k}(x)\leqslant 2q_{n_k-3}(x)q_3(a_{n_k-2},a_{n_k-1},a_{n_k})\leqslant 4(m+1)^2 q_{n_k-3}(x)^{\tau-1}.$$

At the same time, by (3.14),

$$q_{n_k-3}(x)\leqslant 2q_{n_{k-1}}(x)\cdot q_{m_k}(w_1^{(k)},\ldots,w_{t_k}^{(k)})\leqslant (q_{m_k}(w_1^{(k)},\ldots,w_{t_k}^{(k)}))^{1+\epsilon}.$$

As a result, we get

$$\mu(B(x,r))\leqslant c'r\cdot(q_{n_k-3}(x))^{2\tau-2-(\tau-2)/(1+\epsilon)}\cdot\mu(I_{n_k-3}(x)).$$

Combining these two facts together, we get

$$\mu(B(x,r))\leqslant c_3\mu(I_{n_k-3}(x))\min\{1, r(q_{n_k-3})^{2\tau-2-(\tau-2)/(1+\epsilon)}\}. \tag{3.25}$$

Note that (see (3.24))

$$\mu(I_{n_k-3}(x))\leqslant\left(\frac{1}{q_{n_k-3}(x)}\right)^{\tau s/(1+\epsilon)}.$$

Recall a simple inequality that $\min\{a,b\}\leqslant a^{1-\eta}b^{\eta}$ for any $0\leqslant\eta\leqslant 1$, $a,b>0$. Now apply this inequality to (3.25) with $\eta=(\tau s/(1+\epsilon))(2\tau-2-(\tau-2)/(1+\epsilon))$, we get

$$\mu(B(x,r))\leqslant c_3 r^{\eta}.$$

Applying the mass distribution principle, we arrive at

$$\dim_H(W_\tau\backslash W_\tau(Q))\geqslant\dim_H\mathbb{F}_\infty\geqslant\min\left\{\frac{s}{1+\epsilon},\frac{\tau s/(1+\epsilon)}{2\tau-2-(\tau-2)/(1+\epsilon)}\right\}.$$

Letting $\epsilon\to 0$, we get $\dim_H(W_\tau\backslash W_\tau(Q))\geqslant s$. This finishes the proof. □

§4. *Proof of Theorem 1.3.* For any irrational $x\in[0,1]$, recall that $\{p_n(x)/q_n(x)\}_{n\geqslant 1}$ are the convergents of $x$ and that for any $n\geqslant 1$

$$\left|x-\frac{p_n(x)}{q_n(x)}\right|<\frac{1}{q_n(x)^2}.$$

So, once $x\notin W_\tau(Q)$ with $\tau\leqslant 2$, one must have that all but finitely many of the denominators of the convergents of $x$ will lie out of $Q$. Now we give an infinite subset $Q$ of $\mathbb{N}$ as follows.

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Let $\{t_k\}_{k\geqslant 1}$ be a sequence of integers such that $t_{k+1} \geqslant k^2 t_k$. Now for any $0 \leqslant \alpha \leqslant 1$, let

$$\mathbb{N}\backslash Q = \{2^{t_k}+1, \ldots, 2^{t_k}+\lfloor 2^{\alpha t_k}\rfloor, k \geqslant 1\}.$$

It can be claimed that $v(\mathbb{N}\backslash Q) = \alpha$. Recall the definition of $v(\mathbb{N}\backslash Q)$:

$$v(\mathbb{N}\backslash Q) = \inf\left\{v \geqslant 0 : \sum_{q\in\mathbb{N}\backslash Q} q^{-v} < \infty\right\}.$$

For any $v \geqslant 0$ and $k \geqslant 1$, one has

$$\frac{1}{2^{1+v}}\cdot\frac{2^{t_k\alpha}}{2^{t_k v}} \leqslant \frac{\lfloor 2^{t_k\alpha}\rfloor}{(2^{t_k}+\lfloor 2^{t_k\alpha}\rfloor)^v} \leqslant \sum_{2^{t_k}<q\leqslant 2^{t_k}+\lfloor 2^{\alpha t_k}\rfloor} q^{-v} \leqslant \frac{\lfloor 2^{t_k\alpha}\rfloor}{2^{t_k v}} \leqslant \left(\frac{2^\alpha}{2^v}\right)^{t_k}.$$

Thus, when $v = \alpha$,

$$\sum_{q\in\mathbb{N}\backslash Q} q^{-v} = \sum_{k=1}^{\infty}\sum_{2^{t_k}<q\leqslant 2^{t_k}+\lfloor 2^{\alpha t_k}\rfloor} q^{-v} \geqslant \sum_{k=1}^{\infty}\frac{1}{2^{1+v}} = \infty,$$

while when $v > \alpha$

$$\sum_{q\in\mathbb{N}\backslash Q} q^{-v} = \sum_{k=1}^{\infty}\sum_{2^{t_k}<q\leqslant 2^{t_k}+\lfloor 2^{\alpha t_k}\rfloor} q^{-v} \leqslant \sum_{k=1}^{\infty}\left(\frac{1}{2^{(v-\alpha)}}\right)^{t_k} < \infty.$$

This shows the claim.

Now we show that for any $\tau \leqslant 2$, the complement of $W_\tau(Q)$ contains only Liouville numbers. Let $x \notin W_\tau(Q)$. Then all but finitely many of the denominators of the convergents of $x$ lie out of $Q$. Let $q_{n_k}(x)$ be the largest term such that

$$q_{n_k(x)} \leqslant 2^{t_k} + \lfloor 2^{t_k\alpha}\rfloor.$$

Such an term $q_{n_k}(x)$ exists once $2^{t_k} \geqslant a_1(x)$. Since $q_{n_k+1}(x) \notin Q$ when $k$ is large and $q_{n_k+1}(x) > q_{n_k}(x)$, by the maximality of $n_k$, we have

$$q_{n_k+1}(x) > 2^{t_{k+1}} \geqslant 2^{k^2 t_k} \geqslant q_{n_k}(x)^k.$$

This shows that $x$ is a Liouville number. As a result,

$$\dim_H(W_\tau\backslash W_\tau(Q)) = 0.$$

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