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RESEARCH ARTICLE

On the study of bivariate and conditional dynamic negative cumulative extropy

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Failure extropy, introduced by Nair and Sathar Nair [(2020). On dynamic failure extropy. J. Indian Soc. Probab. Stat. 21: 287—313], provides a complementary perspective to entropy for quantifying uncertainty in lifetime distributions. However, it becomes mathematically invalid for distributions with unbounded support. To overcome this limitation, Tahmasebi and Toomaj [(2022). On negative cumulative extropy with applications. Commun. Stat. Theory Methods 51(15): 5025—5047] proposed the concept of negative cumulative extropy (NCEx), offering a bounded and interpretable alternative. In this paper, we extend the notion of NCEx to the bivariate dynamic setting, where uncertainty is assessed for systems whose components have failed at specified times. The proposed formulation effectively captures the uncertainty associated with past lifetimes under dependence, which the existing NCEx cannot address. The measure is further generalized to a vector-valued form, and its fundamental properties are established, including monotonicity, invariance, bounds expressed in terms of the expected inactivity time, and key characterizations. A new stochastic ordering based on the proposed measure is also established. To facilitate practical implementation, a nonparametric estimator is developed and its performance evaluated through extensive Monte Carlo simulations. The practical relevance of the proposed measure is demonstrated using a real dataset, and its superiority over existing entropy-based approaches is shown on an additional dataset.

1. Introduction

Shannon [29] presented a pivotal measure of information (uncertainty) called Shannon entropy, which has since become a foundational concept in various disciplines. Let X denote a nonnegative random variable characterized by probability density function (PDF) f and cumulative distribution function (CDF) F. Then the mathematical representation of Shannon entropy for this variable is given by

$$\mathcal{H}(X) = -\int_0^\infty f(x)\ln f(x)\,dx. \tag{1.1}$$

Shannon entropy quantifies the anticipated quantity of information present in a dataset or message. A higher entropy value indicates greater uncertainty or unpredictability in the data. This measure has been extensively applied across a wide range of domains, including information theory, where it is used for coding and data compression, as well as in machine learning, statistical physics, and other areas requiring uncertainty quantification and data analysis. Its broad applicability has made it a fundamental tool in understanding information flow and complexity in various systems.

Lad *et al*. [16] presented the notion of extropy, which is required to complement entropy, offering a dual perspective on the order and uncertainty of distributions. The article addresses long-standing

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inquiries regarding the axiomatization of information, thereby improving comprehension of probability measures. The introduction of extropy as a unique measure, its mathematical properties, and its applications in statistical scoring criteria, particularly in forecasting, are among the developments. For a random variable X, its extropy is expressed as

$$J(X) = -\frac{1}{2}\int_0^\infty f^2(x)dx. \tag{1.2}$$

For further recent developments on extropy, see [17, 23–25, 36] and [22]. Since the density function required for extropy calculation is often unavailable in practical situations, [11] proposed the cumulative residual extropy (CREx), which relies on the survival function (SF) instead of the PDF. It is defined as

$$\mathcal{J}(X) = -\frac{1}{2}\int_0^\infty \overline{F}^2(x)dx, \tag{1.3}$$

where $\overline{F}$ denotes the SF of the random variable X.

Subsequently, Nair and Sathar [20] introduced the failure extropy (FEx), which depends on the CDF and is given by

$$\mathcal{J}(X) = -\frac{1}{2}\int_A F^2(x)dx, \tag{1.4}$$

where A denotes the support set of X. Kayal [12] also studied DFEx and its weighted versions, and proposed nonparametric estimators based on the empirical distribution function.

However, both CREx and FEx are nonpositive, and FEx diverges for random variables with unbounded support. This limits its practical applicability and weakens its usefulness in risk and reliability analysis. This divergence motivates the need for a bounded and interpretable alternative. To overcome these drawbacks, Tahmasebi and Toomaz [33] proposed the negative cumulative extropy (NCEx), a finite and positive-valued measure suitable for risk and information analysis, defined as

$$\mathcal{CJ}(X) = \frac{1}{2}\int_0^\infty \left(1 - F^2(x)\right)dx. \tag{1.5}$$

Unlike FEx, the NCEx formulation remains finite even for unbounded distributions and offers a direct interpretation in terms of reliability characteristics. Tahmasebi and Toomaz [33] obtained several of its properties and demonstrated stronger connections with reliability quantities such as mean residual and mean past lifetimes compared to entropy or extropy-based analogues. They extended NCEx to the bivariate case for the random vector (RV) $\mathbf{X} = (X_1, X_2)$, termed as bivariate NCEx (BNCEx), defined as

$$\mathcal{CJ}(\mathbf{X}) = \frac{1}{2}\int_0^\infty \int_0^\infty (1 - F^2(x_1, x_2))dx_2 dx_1. \tag{1.6}$$

They also established its relationship with the Gini mean difference under independence. Almaspoor *et al.* {[2] studied the negative NCEx for concomitants of m-generalized order statistics within the Morgenstern family, and developed empirical estimators for NCEx. Noughabi [21] utilized NCEx for testing uniformity on $[0, 1]$, showing that NCEx lies between 0 and 1/2 for all densities on this interval and proposed a simple test statistic based on this property. Chakraborty *et al.* [4] introduced the weighted NCEx measure, which is related to reliability measures such as weighted mean residual and

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past lifetimes. They proposed a nonparametric estimator based on the empirical CDF and developed a consistent uniformity test that outperforms existing methods, particularly when alternatives have points near zero. For additional details on NCEx, see [5, 10, 32] and [30].

In many practical situations, uncertainty may also relate to the past of a system. Consider a bivariate lifetime vector $\mathbf{X} = (X_1, X_2)$ representing the lifetimes of two dependent components, systems, or organs. For fixed $t_1 > 0$ and $t_2 > 0$, suppose that both components have already failed before times t_1 and t_2, respectively; that is, the event $(X_1 < t_1,\ X_2 < t_2)$ has occurred. The bivariate past lifetime at (t_1, t_2) is then defined as the conditional RV $[(X_1, X_2) \mid (X_1 < t_1,\ X_2 < t_2)]$, which describes the joint distribution of the two lifetimes given that failure in both components has already taken place before (t_1, t_2). We denote this conditional vector by $\mathbf{X^t} = [(X_1, X_2) \mid (X_1 < t_1,\ X_2 < t_2)]$, and refer to it as the bivariate past lifetime. The associated marginal past lifetimes are given by the conditional random variables $\overline{X}_i = (X_i \mid X_1 < t_1,\ X_2 < t_2),\ i = 1, 2$, which represent the individual component lifetimes restricted to the past region.

The BNCEx defined in (1.6) is constructed for the RV $\mathbf{X}$ and therefore does not properly capture the uncertainty associated with the conditional RV $\mathbf{X^t}$, which describes the joint past lifetimes. Since conditioning on $(X_1 < t_1, X_2 < t_2)$ alters the dependence structure and the distributional behavior of the components, the BNCEx measure becomes inadequate for assessing uncertainty in this transformed setting. Motivated by this limitation and by the substantial developments in past-lifetime analysis, see [1, 6, 7, 14, 15, 26, 31] and [34, 35], we extend the NCEx framework to the bivariate past lifetime setting. This allows us to develop a measure that accurately reflects the evolving uncertainty inherent in joint past lifetimes.

The subsequent sections of the paper are organized as follows: In Section 2, we study the bivariate dynamic NCEx and its behavior under monotonic transformations. Section 3 introduces a vector-valued definition of bivariate dynamic NCEx, explores the relationships of its components with various established reliability measures, and examines characterizations and stochastic orders. In Section 4, we develop an empirical estimator for the vector-valued bivariate dynamic NCEx, demonstrate its performance using simulated data, and apply it to two real data sets for further analysis. Finally, we summarize the findings and outline future works in Section 5.

2. Bivariate dynamic negative cumulative extropy

This section defines the NCEx for the bivariate RV $\mathbf{X^t}$, which is conditioned on the occurrence of both components falling below predetermined thresholds. The localized behavior of the joint distribution in the lower tail region can be captured by this conditional formulation, which is especially pertinent for applications in tail dependence analysis and risk assessment.

If $\mathbf{X} = (X_1, X_2)$ denotes the lifetimes of two components in a system and both components have failed before times t_1 and t_2, respectively, then, provided that $F(t_1, t_2) > 0$, the uncertainty associated with the past lifetime of the system is quantified by the bivariate dynamic negative cumulative extropy (BDNCEx) of the conditional RV $(X_1, X_2) \mid (X_1 < t_1, X_2 < t_2)$, whose CDF is given by $F(x_1, x_2; t_1, t_2) = \frac{F(x_1, x_2)}{F(t_1, t_2)}$. The BDNCEx is defined as

$$\mathcal{CJ}^{\mathbf{X}}(t_1, t_2) = \frac{1}{2} \int_0^{t_1} \int_0^{t_2} \Big(1 - F^2(x_1, x_2; t_1, t_2)\Big)\, dx_2\, dx_1. \tag{2.7}$$

As $t_1, t_2 \to \infty$, the conditioning becomes non-informative, and the measure in (2.7) reduces to the BNCEx defined in (1.6).

Now, we consider the following examples to illustrate the BDNCEx.

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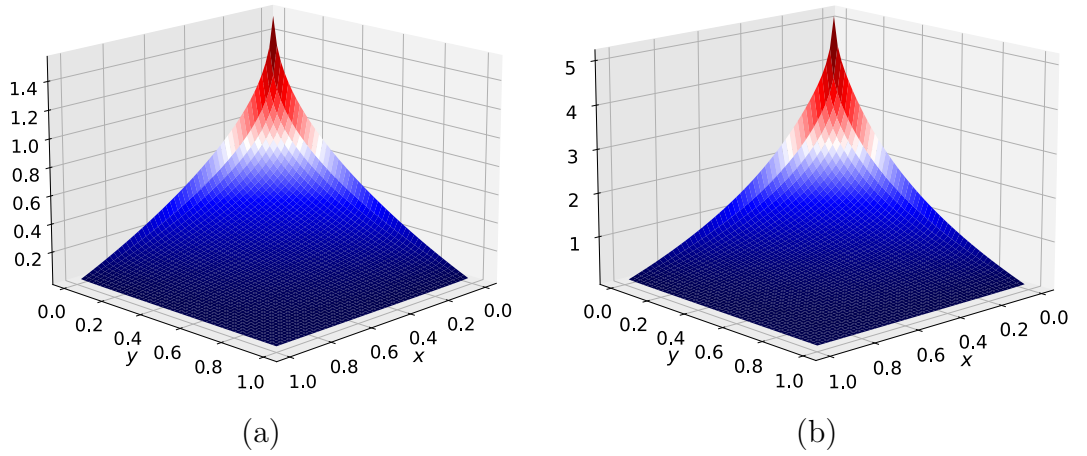


Figure 1. *Plot of BDNCEx for BGE distribution (left) and BGR distribution (right).*

Example 2.1. Consider the bivariate generalized exponential (BGE) distribution with CDF

$$F(x_1, x_2) = \left(1 - e^{-\beta_1 x_1}\right)^{\theta} + \left(1 - e^{-\beta_2 x_2}\right)^{\theta} - \left(1 - e^{-(\beta_1 x_1 + \beta_2 x_2)}\right)^{\theta}, \quad \beta_1, \beta_2 > 0, \theta > 0.$$

Using Eq. (2.7), we obtain the BDNCEx measure numerically for parameters $\beta_1 = 1.5$, $\beta_2 = 1.2$, and $\theta = 0.8$, since deriving an explicit closed-form expression is analytically complex. The resulting plot is shown in Figure 1(a).

Example 2.2. Let $\mathbf{X}$ be a nonnegative RV distributed as a bivariate generalized Rayleigh (BGR) distribution with CDF

$$F(x_1, x_2) = \left(1 - e^{-\frac{1}{2}\beta_1 x_1^2}\right)^{\theta} + \left(1 - e^{-\frac{1}{2}\beta_2 x_2^2}\right)^{\theta} - \left(1 - e^{-\frac{1}{2}(\beta_1 x_1^2 + \beta_2 x_2^2)}\right)^{\theta}, \quad \beta_1, \beta_2 > 0, \theta > 0.$$

Due to the analytical complexity involved in calculating BDNCEx, a closed-form solution is intractable. Therefore, we compute its values numerically for the parameter set $\beta_1 = 1.8$, $\beta_2 = 2.2$, and $\theta = 0.9$. The resulting behavior is presented graphically in Figure 1(b). It is worth mentioning that, for the purpose of plotting the curves, the transformations $t_1 = -\ln x$ and $t_2 = -\ln y$ have been employed.

In the following theorem, we investigate the behavior of BDNCEx under monotonic transformations.

Theorem 2.1 *Assume that* $\mathbf{X} = (X_1, X_2)$ *and* $\mathbf{Y} = (Y_1, Y_2)$ *be nonnegative bivariate RVs. Let* $Y_i = \Phi_i(X_i)$, $i = 1, 2$, *where* Φ_i *is a strictly monotone differentiable function. Then BDNCEx of* $\mathbf{Y} = (Y_1, Y_2)$

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is given by

$$\mathcal{CJ}^{\mathbf{Y}}(t_1,t_2)=\begin{cases}\frac{1}{2}\int_{\Phi_1^{-1}(0)}^{\Phi_1^{-1}(t_1)}\int_{\Phi_2^{-1}(0)}^{\Phi_2^{-1}(t_2)}\left(1-\left(\frac{F(x_1,x_2)}{F(\Phi_1^{-1}(t_1),\Phi_2^{-1}(t_2))}\right)^2\right)|J|\,dx_2\,dx_1,\\ \quad \textit{if } \Phi_i \textit{ is strictly increasing,}\\ \frac{1}{2}\int_{\Phi_1^{-1}(t_1)}^{\Phi_1^{-1}(0)}\int_{\Phi_2^{-1}(t_2)}^{\Phi_2^{-1}(0)}\left(1-\left(\frac{\overline{F}(x_1,x_2)}{\overline{F}(\Phi_1^{-1}(t_1),\Phi_2^{-1}(t_2))}\right)^2\right)|J|\,dx_2\,dx_1,\\ \quad \textit{if } \Phi_i \textit{ is strictly decreasing,}\end{cases}$$

where $J=\frac{\partial}{\partial x_1}\Phi_1(x_1)\frac{\partial}{\partial x_2}\Phi(x_2)$ *is the Jacobian transformation.*

Proof. Let $\mathbf{X}=(X_1,X_2)$ be a nonnegative bivariate RV with joint distribution function $F_{\mathbf{X}}(x_1,x_2)=P(X_1\le x_1,X_2\le x_2)$. Assume that Φ_i is strictly increasing. Then, for any $y_1,y_2>0$,

$$F_{\mathbf{Y}}(y_1,y_2)=P(Y_1\le y_1,Y_2\le y_2)=F_{\mathbf{X}}(\Phi_1^{-1}(y_1),\Phi_2^{-1}(y_2))=F_{\mathbf{X}}(x_1,x_2).$$

In particular,

$$F_{\mathbf{Y}}(t_1,t_2)=F_{\mathbf{X}}(\Phi_1^{-1}(t_1),\Phi_2^{-1}(t_2)).$$

Using the definition,

$$\mathcal{CJ}^{\mathbf{Y}}(t_1,t_2)=\frac{1}{2}\int_0^{t_1}\int_0^{t_2}\left(1-F^2(y_1,y_2;t_1,t_2)\right)dy_2\,dy_1. \tag{2.8}$$

Utilizing the transformation $x_i=\Phi_i^{-1}(y_i)$, we have

$$dy_2\,dy_1=\left|\Phi_1'(x_1)\Phi_2'(x_2)\right|dx_2\,dx_1=|J|\,dx_2\,dx_1,$$

where $J=\Phi_1'(x_1)\Phi_2'(x_2)$ is the Jacobian of transformation. Substituting into (2.8), we obtain

$$\mathcal{CJ}^{\mathbf{Y}}(t_1,t_2)=\frac{1}{2}\int_{\Phi_1^{-1}(0)}^{\Phi_1^{-1}(t_1)}\int_{\Phi_2^{-1}(0)}^{\Phi_2^{-1}(t_2)}\left(1-\left(\frac{F_{\mathbf{X}}(x_1,x_2)}{F_{\mathbf{X}}\left(\Phi_1^{-1}(t_1),\Phi_2^{-1}(t_2)\right)}\right)^2\right)|J|\,dx_2\,dx_1.$$

If Φ_i is strictly decreasing, then $F_{\mathbf{Y}}(y_1,y_2)=\overline{F}_{\mathbf{X}}(x_1,x_2)$ and $F_{\mathbf{Y}}(t_1,t_2)=\overline{F}_{\mathbf{X}}(\Phi_1^{-1}(t_1),\Phi_2^{-1}(t_2))$. Therefore,

$$\mathcal{CJ}^{\mathbf{Y}}(t_1,t_2)=\frac{1}{2}\int_{\Phi_1^{-1}(t_1)}^{\Phi_1^{-1}(0)}\int_{\Phi_2^{-1}(t_2)}^{\Phi_2^{-1}(0)}\left(1-\left(\frac{\overline{F}_{\mathbf{X}}(x_1,x_2)}{\overline{F}_{\mathbf{X}}(\Phi_1^{-1}(t_1),\Phi_2^{-1}(t_2))}\right)^2\right)|J|\,dx_2\,dx_1.$$

This completes the proof. □

Next, we investigate BDNCEx under an affine transformation.

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Theorem 2.2 *Assume that* $\mathbf{X} = (X_1, X_2)$ *and* $\mathbf{Y} = (Y_1, Y_2)$ *be nonnegative bivariate RVs. Let* $Y_i = \beta_i + \theta_i X_i,\ \theta_i > 0$ *and* $\beta_i \geq 0$ *for* $i = 1, 2$. *Then*

$$\mathcal{CJ}^{\mathbf{Y}}(t_1, t_2) = \theta_1\theta_2\, \mathcal{CJ}^{\mathbf{X}}\left(\frac{t_1 - \beta_1}{\theta_1}, \frac{t_2 - \beta_2}{\theta_2}\right) + \frac{t_1\beta_2 + t_2\beta_1 - \beta_1\beta_2}{2},\ t_i \geq \beta_i.$$

Proof. Let $Y_i = \theta_i X_i + \beta_i,\ \ \theta_i > 0,\ \beta_i \geq 0,\ i = 1, 2$. Then, for any t_1, t_2 satisfying $t_i \geq \beta_i$, the joint CDF of $\mathbf{Y}$ is given by

$$F_{\mathbf{Y}}(y_1, y_2) = P\left(X_1 \leq \frac{y_1 - \beta_1}{\theta_1}, X_2 \leq \frac{y_2 - \beta_2}{\theta_2}\right) = F_{\mathbf{X}}\left(\frac{y_1 - \beta_1}{\theta_1}, \frac{y_2 - \beta_2}{\theta_2}\right).$$

Define $\tau_i = \frac{t_i - \beta_i}{\theta_i},\ i = 1, 2$. Then, from (2.7), we have

$$\begin{aligned}\mathcal{CJ}^{\mathbf{Y}}(t_1, t_2) &= \frac{1}{2}\int_0^{t_1}\int_0^{t_2}\left(1 - F^2_{\mathbf{Y}}(y_1, y_2; t_1, t_2)\right) dy_2\, dy_1 \\ &= \frac{1}{2}\int_0^{t_1}\int_0^{t_2}\left(1 - \left(\frac{F_{\mathbf{X}}\left(\frac{y_1 - \beta_1}{\theta_1}, \frac{y_2 - \beta_2}{\theta_2}\right)}{F_{\mathbf{X}}(\tau_1, \tau_2)}\right)^2\right) dy_2\, dy_1.\end{aligned}$$

Applying the change of variables $y_1 = \theta_1 x_1 + \beta_1$ and $y_2 = \theta_2 x_2 + \beta_2$, with corresponding Jacobian determinant $dy_1\, dy_2 = \theta_1\theta_2\, dx_1\, dx_2$, the limits of integration become $x_i \in \left[-\frac{\beta_i}{\theta_i}, \tau_i\right]$. Hence,

$$\mathcal{CJ}^{\mathbf{Y}}(t_1, t_2) = \frac{\theta_1\theta_2}{2}\int_{-\frac{\beta_1}{\theta_1}}^{\tau_1}\int_{-\frac{\beta_2}{\theta_2}}^{\tau_2}\left(1 - \left(\frac{F_{\mathbf{X}}(x_1, x_2)}{F_{\mathbf{X}}(\tau_1, \tau_2)}\right)^2\right) dx_2\, dx_1.$$

Since $\mathbf{X}$ is nonnegative, it follows that $F_{\mathbf{X}}(x_1, x_2) = 0$ whenever $x_1 < 0$ or $x_2 < 0$. Therefore, we can decompose the above double integral over the rectangular domain $\left[-\frac{\beta_1}{\theta_1}, \tau_1\right] \times \left[-\frac{\beta_2}{\theta_2}, \tau_2\right]$ into subregions as follows:

$$\begin{aligned}\mathcal{CJ}^{\mathbf{Y}}(t_1, t_2) = \frac{\theta_1\theta_2}{2}\Bigg[&\int_0^{\tau_1}\int_0^{\tau_2}\left(1 - \left(\frac{F_{\mathbf{X}}(x_1, x_2)}{F_{\mathbf{X}}(\tau_1, \tau_2)}\right)^2\right) dx_2\, dx_1 + \int_0^{\tau_1}\int_{-\frac{\beta_2}{\theta_2}}^{0} 1\, dx_2\, dx_1 \\ &+ \int_{-\frac{\beta_1}{\theta_1}}^{0}\int_0^{\tau_2} 1\, dx_2\, dx_1 + \int_{-\frac{\beta_1}{\theta_1}}^{0}\int_{-\frac{\beta_2}{\theta_2}}^{0} 1\, dx_2\, dx_1\Bigg].\end{aligned}$$

Evaluating each term, we obtain

$$\mathcal{CJ}^{\mathbf{Y}}(t_1, t_2) = \theta_1\theta_2\, \mathcal{CJ}^{\mathbf{X}}(\tau_1, \tau_2) + \frac{\theta_1\theta_2}{2}\left(\tau_1\frac{\beta_2}{\theta_2} + \tau_2\frac{\beta_1}{\theta_1} + \frac{\beta_1\beta_2}{\theta_1\theta_2}\right).$$

Substituting $\tau_i = (t_i - \beta_i)/\theta_i$ and simplifying yields

$$\mathcal{CJ}^{\mathbf{Y}}(t_1, t_2) = \theta_1\theta_2\, \mathcal{J}^{\mathbf{X}}\left(\frac{t_1 - \beta_1}{\theta_1}, \frac{t_2 - \beta_2}{\theta_2}\right) + \frac{t_1\beta_2 + t_2\beta_1 - \beta_1\beta_2}{2}.$$

This completes the proof. □

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3. Conditional dynamic negative cumulative extropy

This section examines BDNCEx via a component-wise analysis. Based on this examination, we introduce an alternative formulation known as the conditional dynamic negative cumulative extropy (CDNCEx). Notably, the BDNCEx defined in (2.7) does not uniquely determine the distribution function. To overcome this limitation, we propose a vector-valued version of BDNCEx that captures more detailed information by incorporating its marginal components. Consider the conditional failure model $\overline{X}_i = (X_i \mid X_1 < t_1, X_2 < t_2)$, $i = 1, 2$, and assume that $F(t_1, t_2) > 0$. The corresponding conditional distribution functions are defined by $F(x_1; t_1, t_2) = \frac{F(x_1,t_2)}{F(t_1,t_2)}$, $0 < x_1 < t_1$, and $F(x_2; t_1, t_2) = \frac{F(t_1,x_2)}{F(t_1,t_2)}$, $0 < x_2 < t_2$. In the bivariate setting, vector-valued BDNCEx is defined as

$$\mathcal{CJ}^{\mathbf{X}}(t_1, t_2) = \left(\mathcal{CJ}_1^{\mathbf{X}}(t_1, t_2), \mathcal{CJ}_2^{\mathbf{X}}(t_1, t_2)\right), \tag{3.9}$$

where

$$\mathcal{CJ}_1^{\mathbf{X}}(t_1, t_2) = \frac{1}{2}\int_0^{t_1}\left(1 - (F(x_1; t_1 t_2))^2\right) dx_1 = \frac{1}{2}\int_0^{t_1}\left(1 - \left(\frac{F(x_1, t_2)}{F(t_1, t_2)}\right)^2\right) dx_1 \tag{3.10}$$

and

$$\mathcal{CJ}_2^{\mathbf{X}}(t_1, t_2) = \frac{1}{2}\int_0^{t_1}\left(1 - (F(x_2; t_1 t_2))^2\right) dx_2 = \frac{1}{2}\int_0^{t_2}\left(1 - \left(\frac{F(t_1, x_2)}{F(t_1, t_2)}\right)^2\right) dx_2 \tag{3.11}$$

represent the NCEx of $(X_i | X_1 < t_1, X_2 < t_2)$, $i = 1, 2$. The components of (3.9) are known as CDNCEx. If the RV $\mathbf{X}$ represents the lifetimes of a two-component system, then expressions (3.10) and (3.11) quantify the level of uncertainty associated with the conditional distributions of X_i, given that the first component has failed at sometime within the interval $(0, t_1)$ and the second within $(0, t_2)$.

If we assume $t_2 \to \infty$ and $t_1 \to \infty$ in (3.10) and (3.11), respectively. Then CDNCEx reduces to univariate DNCEx, defined as

$$\mathcal{CJ}_i(\mathbf{X}; t_i) = \begin{cases} \frac{1}{2}\int_0^{t_1}\left(1 - \left(\frac{F_1(x_1)}{F_1(t_1)}\right)^2\right) dx_1 & \text{for } i = 1 \\ \frac{1}{2}\int_0^{t_2}\left(1 - \left(\frac{F_2(x_2)}{F_2(t_2)}\right)^2\right) dx_2 & \text{for } i = 2. \end{cases} \tag{3.12}$$

{ All the subsequent results for CDNCEx are consistent with, and equivalent to, the corresponding univariate case.

Now, we evaluate CDNCEx for BGE distribution.

Example 3.1. Suppose $\mathbf{X} = (X_1, X_2)$ is a nonnegative RV that follows a BGE distribution, as described in Example 2.1. Since obtaining a closed-form expression for the CDNCEx surface under the BGE distribution is analytically intractable, we compute it numerically and present the resulting 3-D plot in Figure 2. For convenience in graphical representation, the change of variables $t_1 = -\ln x$ and $t_2 = -\ln y$ has been applied.

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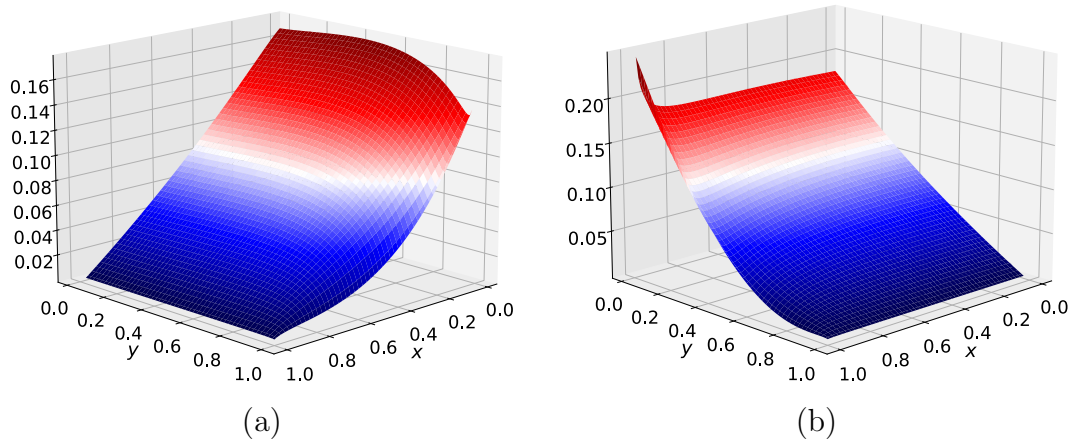


Figure 2. *Numerical plot of $\mathcal{CJ}_1^{\mathbf{X}}(t_1,t_2)$ (left) and $\mathcal{CJ}_2^{\mathbf{X}}(t_1,t_2)$ (right) for BGE distribution for parameters $\beta_1 = 1.5, \beta_2 = 1.4$ and $\theta = 0.2$ (Example 3.1).*

Now we define the bivariate reversed hazard rate (BRHR) and the bivariate expected inactivity time (BEIT) functions, proposed by Roy [27] and Nair and Asha [19], respectively.

Definition 3.1. *Let $\mathbf{X} = (X_1, X_2)$ be a nonnegative bivariate RV,*

(1) the BRHR is defined as the vector $h(t_1,t_2) = (h_1(t_1,t_2), h_2(t_1,t_2))$, where $h_i(t_1,t_2) = \frac{\partial}{\partial t_i} \log F(t_1,t_2), \quad i = 1,2$. The components $h_1(t_1,t_2)$ and $h_2(t_1,t_2)$ represent the instantaneous reversed failure rates corresponding to X_1 and X_2, respectively, conditional on the event $\{X_1 < t_1, X_2 < t_2\}$;

(2) the BEIT is defined as the vector $\overline{m}(t_1,t_2) = (\overline{m}_1(t_1,t_2), \overline{m}_2(t_1,t_2))$, where $\overline{m}_i(t_1,t_2) = \mathbb{E}(t_i - X_i \mid X_1 < t_1, X_2 < t_2), \quad i = 1,2$. That is, $\overline{m}_i(t_1,t_2)$ represents the expected waiting time elapsed before failure of the ith component, given that both components have failed before (t_1,t_2). In particular, for $i = 1$, $\overline{m}_1(t_1,t_2) = \frac{1}{F(t_1,t_2)} \int_0^{t_1} F(x_1,t_2)\, dx_1$, and similarly for $\overline{m}_2(t_1,t_2)$.

Differentiation of (3.10) with respect to t_1 yields

$$\frac{\partial}{\partial t_1} \mathcal{CJ}_1^{\mathbf{X}}(t_1,t_2) = h_1(t_1,t_2) \left[t_1 - 2\mathcal{CJ}_1^{\mathbf{X}}(t_1,t_2)\right]. \tag{3.13}$$

Similarly, differentiating (3.11) with respect to t_2 gives

$$\frac{\partial}{\partial t_2} \mathcal{CJ}_2^{\mathbf{X}}(t_1,t_2) = h_2(t_1,t_2) \left[t_1 - 2\mathcal{CJ}_2^{\mathbf{X}}(t_1,t_2)\right]. \tag{3.14}$$

Thus, in general, (3.13) and (3.14) can be written as

$$\frac{\partial}{\partial t_i} \mathcal{CJ}_i^{\mathbf{X}}(t_1,t_2) = h_i(t_1,t_2) \left[t_i - 2\mathcal{CJ}_i^{\mathbf{X}}(t_1,t_2)\right], \quad i = 1,2. \tag{3.15}$$

The subsequent theorem establishes a lower bound of CDNCEx.

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Theorem 3.1 *Assume a nonnegative RV* $\mathbf{X}$ *having BEIT* $\overline{m}_i(t_1,t_2)$. *Then for* $t_1,t_2>0$,

$$\mathcal{CJ}_i^{\mathbf{X}}(t_1,t_2) \geq \frac{1}{2}\left(t_i - \overline{m}_i(t_1,t_2)\right),\ i=1,2.$$

Proof. Since $0 \leq \frac{F(x_1,t_2)}{F(t_1,t_2)} \leq 1\ \forall\ x_1 \leq t_1$ and for all $t_1,t_2 \geq 0$. This yields

$$\begin{aligned}\frac{1}{2}\int_0^{t_1}\left(1-\left(\frac{F(x_1,t_2)}{F(t_1,t_2)}\right)^2\right)dx_1 &\geq \frac{1}{2}\int_0^{t_1}\left(1-\frac{F(x_1,t_2)}{F(t_1,t_2)}\right)dx_1\\ &= \frac{1}{2}\left(t_1-\overline{m}_1(t_1,t_2)\right).\end{aligned}$$

A similar result holds for $i=2$. □

For a rigorous probabilistic characterization of CDNCEx, consider the following definition

$$\omega_1^{(2)}(c,d;t_2) = \int_c^d F(x_1,t_2)dx_1, \tag{3.16}$$

where $c,d \in \mathbb{R}$ and $0<c\leq d$. It can be seen that $\frac{\partial}{\partial t_1}\omega_1^{(2)}(c;t_1,t_2) = F(t_1,t_2)$.

Now, we evaluate the CDNCEx of some distributions. The importance of $\omega_1^{(2)}(c,d;t_2)$ stems from the relationship between its partial derivative and the CDF of $\mathbf{X}$. Analogously, we may define

$$\omega_2^{(2)}(c,d;t_1) = \int_c^d F(t_1,x_2)dx_2.$$

For $i=1,2$, the following result establishes a fundamental relationship between $\mathcal{J}_i^{\mathbf{X}}(t_1,t_2)$ and $\omega_i^{(2)}(c,d;t_i)$.

Theorem 3.2 *Let* $\mathbf{X}$ *be a nonnegative bivariate RV. For all* $t_i,t_j \geq 0$ $(i,j\in\{1,2\})$, *we obtain*

$$E\left[\omega_i^{(2)}(X_i;t_1,t_2)|X_1<t_1,X_2<t_2\right] = 2\left(\frac{t_i}{2}-\mathcal{CJ}_i^{X}(t_1,t_2)\right)F(t_1,t_2).$$

Proof. We begin by establishing the case for $i=1$. From (3.10), we derive

$$\begin{aligned}\mathcal{CJ}_1^{\mathbf{X}}(t_1,t_2) &= \frac{t_1}{2} - \frac{1}{2F^2(t_1,t_2)}\int_0^{t_1}F^2(x_1,t_2)dx_1\\ &= \frac{t_1}{2} - \frac{1}{2F^2(t_1,t_2)}\int_0^{t_1}\left(\int_0^{x_1}\frac{\partial}{\partial u}F(u,t_2)du\right)F(x_1,t_2)dx_1\\ &= \frac{t_1}{2} - \frac{1}{2F^2(t_1,t_2)}\int_0^{t_1}\frac{\partial F(u,t_2)}{\partial u}\left(\int_u^{t_1}F(x_1,t_2)dx_1\right)du\\ &= \frac{t_1}{2} - \frac{1}{2F^2(t_1,t_2)}F(t_1,t_2)E\left[\int_{X_1}^{t_1}F(u,t_2)du|X_1<t_1,X_2<t_2\right]\\ &= \frac{t_1}{2} - \frac{1}{2F(t_1,t_2)}E\left[\omega_1^{(2)}(X_1,t_1;t_2)|X_1<t_1,X_2<t_2\right],\end{aligned}$$

proving the result. The last equality is obtained using (3.16). The proof for $i=2$ follows similarly. □

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Definition 3.2. *The bivariate RV* $\mathbf{X}$ *is classified as increasing (decreasing) in CDNCEx if* $\mathcal{CJ}_i^{\mathbf{X}}(t_1, t_2)$ *is an increasing (decreasing) function of* t_i*, where* $i = 1, 2$.

The following theorem establishes that, under specific conditions on $\mathcal{CJ}_i^{\mathbf{X}}(t_1, t_2)$, the components of the BRHR function for $\mathbf{X}$ exhibit an increasing (or decreasing) trend.

Theorem 3.3 *A nonnegative RV* $\mathbf{X}$ *is said to have increasing (decreasing) in CDNCEx if and only if*

$$\mathcal{CJ}_i^{\mathbf{X}}(t_1, t_2) \leq (\geq) \frac{t_i}{2},\ i = 1, 2.$$

Proof. From (3.15), we have

$$\frac{\partial}{\partial t_i} \mathcal{CJ}_i^{\mathbf{X}}(t_1, t_2) = h_i(t_1, t_2) \left[t_i - 2\mathcal{CJ}_i^{\mathbf{X}}(t_1, t_2) \right].$$

If $\mathcal{CJ}_i^{\mathbf{X}}(t_1, t_2)$ is increasing (decreasing) in t_i, then $\frac{\partial}{\partial t_i} \mathcal{CJ}_i^{\mathbf{X}}(t_1, t_2) \geq (\leq) 0$. It follows that

$$h_i(t_1, t_2) \left[t_i - 2\mathcal{CJ}_i^{\mathbf{X}}(t_1, t_2) \right] \geq (\leq) 0,$$

but $h_i(t_1, t_2) \geq 0$, this yields

$$\mathcal{CJ}_i^{\mathbf{X}}(t_1, t_2) \leq (\geq) \frac{t_i}{2}.$$

The converse is straightforward and is therefore omitted. □

Theorem 3.4 *Let* $\mathbf{X} = (X_1, X_2)$ *be a nonnegative absolutely continuous RV with joint PDF* $f(x_1, x_2) > 0$ *on* $(0, t_1) \times (0, t_2)$*. Then*

$$\mathcal{CJ}_i^{\mathbf{X}}(t_1, t_2) < \frac{t_i}{2}, \quad i = 1, 2.$$

Proof. Since $\mathbf{X}$ is nonnegative and absolutely continuous with joint density f, for any $0 < x_1 < t_1$ and $t_2 > 0$,

$$F(x_1, t_2) = \int_0^{x_1} \int_0^{t_2} f(u, v)\, dv\, du > 0,$$

and

$$F(t_1, t_2) - F(x_1, t_2) = \int_{x_1}^{t_1} \int_0^{t_2} f(u, v)\, dv\, du > 0.$$

Hence,

$$0 < F(x_1, t_2) < F(t_1, t_2) \implies 0 < F(x_1; t_1, t_2) < 1.$$

Integrating both sides over $(0, t_1)$ gives

$$\int_0^{t_1} \left(1 - F^2(x_1; t_1, t_2)\right) dx_1 < t_1.$$

The argument for $i = 2$ is identical. This completes the proof. □

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The following theorem establishes that the CDNCEx fails to remain invariant under non-singular transformations.

Theorem 3.5 *Consider a nonnegative RV* $\boldsymbol{X}$ *with joint CDF* $F(x_1, x_2)$. *Let* $Y_i = \Phi_i(X_i),\ i = 1, 2$, *where* Φ_i *is an injective and differentiable function.*

Then, for $i = 1$,

$$\mathcal{CJ}_1^{(\Phi(X_1),\Phi(X_2))}(t_1, t_2) = \begin{cases} \frac{1}{2}\int_{\Phi_1^{-1}(0)}^{\Phi_1^{-1}(t_1)} \left(1 - \left(\frac{F(x_1, \Phi_2^{-1}(t_2))}{F(\Phi_1^{-1}(t_1), \Phi_2^{-1}(t_2))}\right)^2\right) \Phi_1'(x_1) dx_1, \\ \quad \textit{if } \Phi_1 \textit{ is strictly increasing.} \\ \frac{1}{2}\int_{\Phi_1^{-1}(t_1)}^{\Phi_1^{-1}(0)} \left(1 - \left(\frac{\bar{F}(x_1, \Phi_1^{-1}(t_2))}{\bar{F}(\Phi_1^{-1}(t_1), \Phi_1^{-1}(t_2))}\right)^2\right) \Phi_1'(x_1) dx_1, \\ \quad \textit{if } \Phi_1 \textit{ is strictly decreasing.} \end{cases}$$

Proof. Assume that Φ_i is strictly increasing. Then, by monotonicity of the transformation, we have

$$F_{\mathbf{Y}}(y_1, t_2) = P(Y_1 < y_1, Y_2 < t_2) = P\big(X_1 < \Phi_1^{-1}(y_1),\ X_2 < \Phi_2^{-1}(t_2)\big) = F_{\mathbf{X}}\big(\Phi_1^{-1}(y_1), \Phi_2^{-1}(t_2)\big),$$

and $F_{\mathbf{Y}}(t_1, t_2) = F_{\mathbf{X}}\big(\Phi_1^{-1}(t_1), \Phi_2^{-1}(t_2)\big)$.

Using the transformation $y_i = \Phi_i(x_i)$, we obtain $dy_1 = \Phi_1'(x_1)\, dx_1$. Hence,

$$\mathcal{CJ}_1^{(\Phi(X_1),\Phi(X_2))}(t_1, t_2) = \frac{1}{2}\int_{\Phi_1^{-1}(0)}^{\Phi_1^{-1}(t_1)} \left(1 - \left(\frac{F(x_1, \Phi_2^{-1}(t_2))}{F(\Phi_1^{-1}(t_1), \Phi_2^{-1}(t_2))}\right)^2\right) \Phi_1'(x_1)\, dx_1.$$

A similar argument holds when Φ_i is strictly decreasing. □

The following result describes the affine transformation property of the CDNCEx measure.

Theorem 3.6 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be nonnegative RVs. If* $Y_i = \mu_i X_i + \eta_i,\ \mu_i > 0,\ \eta_i \geq 0$ *for* $i = 1, 2$, *then*

$$\mathcal{CJ}_i^{Y}(t_1, t_2) = \mu_i\, \mathcal{CJ}_i^{\mathbf{X}}\left(\frac{t_1 - \eta_1}{\mu_1}, \frac{t_2 - \eta_2}{\mu_2}\right) + \frac{\eta_i}{2}.$$

Proof. From (3.10), we have

$$\mathcal{CJ}_1^{\mathbf{Y}}(t_1, t_2) = \frac{1}{2}\int_0^{t_1} \left(1 - \left(\frac{F_{\mathbf{Y}}(y_1, t_2)}{F_{\mathbf{Y}}(t_1, t_2)}\right)^2\right) dy_1.$$

Applying the transformation $Y_i = \mu_i X_i + \eta_i$ yields

$$F_{\mathbf{Y}}(y_1, t_2) = F_{\mathbf{X}}\left(\frac{y_1 - \eta_1}{\mu_1}, \frac{t_2 - \eta_2}{\mu_2}\right).$$

Set $x_1 = (y_1 - \eta_1)/\mu_1$, so $y_1 = \mu_1 x_1 + \eta_1$ and $dy_1 = \mu_1\, dx_1$. Hence

$$\mathcal{CJ}_1^{\mathbf{Y}}(t_1, t_2) = \frac{\mu_1}{2}\int_{-\eta_1/\mu_1}^{(t_1-\eta_1)/\mu_1} \left(1 - \left(\frac{F_{\mathbf{X}}\big(x_1, (t_2 - \eta_2)/\mu_2\big)}{F_{\mathbf{X}}\big((t_1 - \eta_1)/\mu_1, (t_2 - \eta_2)/\mu_2\big)}\right)^2\right) dx_1.$$

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It follows that

$$\mathcal{CJ}_1^{\mathbf{Y}}(t_1,t_2)=\mu_1\,\mathcal{CJ}_1^{\mathbf{X}}\Big(\frac{t_1-\eta_1}{\mu_1},\frac{t_2-\eta_2}{\mu_2}\Big)+\frac{\eta_1}{2}\,.$$

Following the same steps used for $i=1$, we can also establish the case for $i=2$. □

We now demonstrate that the affine transformation preserves the monotonicity of $\mathcal{CJ}_i^{\mathbf{X}}(t_1,t_2)$, as formalized in the result below.

Theorem 3.7 *Assume two nonnegative bivariate RVs* $\mathbf{X}$ *and* $\mathbf{Y}$*, where* $Y_i=\mu_iX_i+\eta_i$ *with* $\mu_i>0$ *and* $\eta_i\geq 0$ *for* $i=1,2$*. Then* $\mathcal{CJ}_i^{\mathbf{Y}}(t_1,t_2)$ *is increasing in* t_i *if and only if* $\mathcal{CJ}_i^{\mathbf{X}}(t_1,t_2)$ *is increasing in* t_i*.*

Proof. Let $Y_i=\mu_iX_i+\eta_i$ with $\mu_i>0$ and $\eta_i\geq 0$. For $t_i\geq\eta_i$, the joint CDF of $\mathbf{Y}$ satisfies

$$F_{\mathbf{Y}}(t_1,t_2)=P(Y_1\leq t_1,Y_2\leq t_2)=F_{\mathbf{X}}\bigg(\frac{t_1-\eta_1}{\mu_1},\frac{t_2-\eta_2}{\mu_2}\bigg).$$

Hence, from Theorem 3.5, we have

$$\mathcal{CJ}_i^{\mathbf{Y}}(t_1,t_2)=\mu_i\,\mathcal{CJ}_i^{\mathbf{X}}\bigg(\frac{t_1-\eta_1}{\mu_1},\frac{t_2-\eta_2}{\mu_2}\bigg),\qquad t_i\geq\eta_i. \tag{3.17}$$

Let $\phi_i(t_i)=\dfrac{t_i-\eta_i}{\mu_i}$, so that $\phi_i'(t_i)=1/\mu_i>0$ and, let $s_i=\phi_i(t_i)$. Differentiating (3.17) with respect to t_i gives

$$\frac{\partial}{\partial t_i}\mathcal{CJ}_i^{\mathbf{Y}}(t_1,t_2)=\mu_i\frac{\partial}{\partial s_i}\mathcal{CJ}_i^{\mathbf{X}}\left(\phi_1(t_1),\phi_2(t_2)\right)\phi_i'(t_i)=\frac{\partial}{\partial s_i}\mathcal{CJ}_i^{\mathbf{X}}\left(s_1,s_2\right).\tag{3.18}$$

Hence, $\mathcal{CJ}_i^{\mathbf{Y}}(t_1,t_2)$ is increasing in t_i if and only if $\mathcal{CJ}_i^{\mathbf{X}}(t_1,t_2)$ is increasing in t_i. □

The following theorem establishes that CDNCEx uniquely determines the distribution function.

Theorem 3.8 *Let* $\mathbf{X}$ *be a nonnegative absolutely continuous bivariate RV with PDF* $f(x_1,x_2)>0$ *on* $(0,t_1)\times(0,t_2)$*. Then, CDNCEx uniquely determines the underlying CDF.*

Proof. Assume that $\mathbf{X}$ and $\mathbf{Y}$ are two RVs with joint CDFs F and G, respectively, and let $h_i^{\mathbf{X}}(t_1,t_2)>0$ and $h_i^{\mathbf{Y}}(t_1,t_2)>0$ denote the components of their respective BRHR. Suppose that

$$\mathcal{CJ}_i^{\mathbf{X}}(t_1,t_2)=\mathcal{CJ}_i^{\mathbf{Y}}(t_1,t_2).\tag{3.19}$$

Differentiating (3.19) with respect to t_1 and simplifying, we obtain

$$h_i^{\mathbf{X}}(t_1,t_2)\left[t_i-2\,\mathcal{CJ}_i^{\mathbf{X}}(t_1,t_2)\right]=h_i^{\mathbf{Y}}(t_1,t_2)\left[t_i-2\,\mathcal{CJ}_i^{\mathbf{Y}}(t_1,t_2)\right].$$

Now, using Theorem 3.4, we obtain,

$$h_i^{\mathbf{X}}(t_1,t_2)=h_i^{\mathbf{Y}}(t_1,t_2).$$

Consequently, the result follows from the fundamental principle that the vector-valued BRHR uniquely characterizes the underlying CDF (see [27]). □

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The following result establishes that CDNCEx reduces to univariate NCEx when the components are independent.

Theorem 3.9 *Let* $\mathbf{X}$ *be a nonnegative bivariate RV* $\mathbf{X}$ *with PDF* $f(x_1, x_2) > 0$ *on* $(0, t_1) \times (0, t_2)$. *Then*

$$\mathcal{CJ}_i^{\mathbf{X}}(t_1, t_2) = \mathcal{CJ}(X_i; t_i), \quad i = 1, 2, \tag{3.20}$$

if and only if X_1 *and* X_2 *are independent.*

Proof. Assume that (3.20) holds. Then we have

$$\frac{1}{2}\int_0^{t_1}\left(1 - \left(\frac{F(x_1, t_2)}{F(t_1, t_2)}\right)^2\right) dx_1 = \frac{1}{2}\int_0^{t_1}\left(1 - \left(\frac{F_1(x_1)}{F_1(t_1)}\right)^2\right) dx_1.$$

Differentiating both sides with respect to t_1, we obtain

$$h_1(t_1, t_2)\left[t_1 - 2\,\mathcal{CJ}_1^{\mathbf{X}}(t_1, t_2)\right] = h_1(t_1)\left[t_1 - 2\,\mathcal{CJ}(X_1; t_1)\right], \tag{3.21}$$

where $h_i(t_1, t_2)$ denotes the component of the BRHR of $\mathbf{X}$, and $h_1(t_1) = \frac{d}{dt_1}\log F_1(t_1)$ represents the univariate reversed hazard rate of X_1. Therefore, from Theorem 3.4 and (3.21), we get

$$\frac{\partial}{\partial t_1} h_1(t_1, t_2) = \frac{\partial}{\partial t_1} h_1(t_1).$$

Hence

$$\frac{\partial}{\partial t_1}\log F(t_1, t_2) = \frac{\partial}{\partial t_1}\log F_1(t_1). \tag{3.22}$$

Integrating both sides of (3.22) with respect to t_1 (for fixed t_2), we obtain

$$\log F(t_1, t_2) = \log F_1(t_1) + \varphi(t_2), \tag{3.23}$$

where $\varphi(t_2)$ is a function depending only on t_2. Exponentiating both sides of (3.23) yields

$$F(t_1, t_2) = F_1(t_1)\, e^{\varphi(t_2)}. \tag{3.24}$$

Let $g(t_2) = e^{\varphi(t_2)} > 0$, so that $F(t_1, t_2) = F_1(t_1)\, g(t_2)$. Using the definition of the second marginal, we have

$$F_2(t_2) = \lim_{t_1 \to \infty} F(t_1, t_2) = \lim_{t_1 \to \infty} F_1(t_1)\, g(t_2) = g(t_2),$$

since $\lim_{t_1 \to \infty} F_1(t_1) = 1$. Substituting this result into (3.24) gives

$$F(t_1, t_2) = F_1(t_1) F_2(t_2).$$

The converse part is straightforward and is therefore omitted. □

Next, we define a new stochastic ordering derived from CDNCEx.

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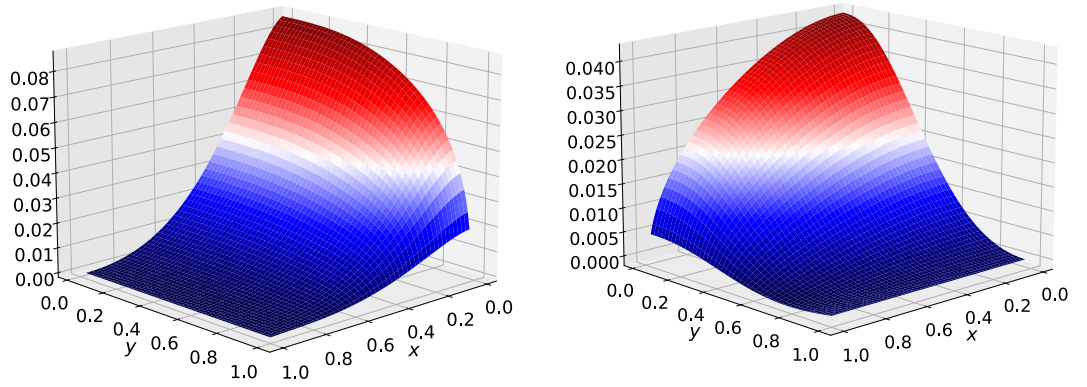


Figure 3. *Numerical plots of $\varphi_1(x, y)$ (left) and $\varphi_2(x, y)$ (right) for the BGE distribution. The observed positivity of both functions confirms Theorem 3.10.*

Definition 3.3. *Consider two nonnegative bivariate RVs* $\mathbf{X}$ *and* $\mathbf{Y}$. *We say that* $\mathbf{X}$ *is greater (or less) than* $\mathbf{Y}$ *in CDNCEx, denoted as* $\mathbf{X} \geq_{CDNCEx} (\leq_{CDNCEx}) \mathbf{Y}$, *if for all* $t_1, t_2 > 0$ *and for* $i = 1, 2$, *the following inequality holds*

$$\mathcal{CJ}_i^{\mathbf{X}}(t_1, t_2) \geq (\leq) \mathcal{CJ}_i^{\mathbf{Y}}(t_1, t_2).$$

Subsequently, we show that the usual stochastic order preserves the stochastic order based on CDNCEx.

Theorem 3.10 *Assume two nonnegative RVs* $\mathbf{X}$ *and* $\mathbf{Y}$. *Define the conditional random variables* $\overline{X}_i = (X_i \mid X_1 < t_1, X_2 < t_2)$ *and* $\overline{Y}_i = (Y_i \mid Y_1 < t_1, Y_2 < t_2)$. *If* $\overline{X}_i \geq_{st} (\leq_{st}) \overline{Y}_i$, *then* $\mathbf{X} \geq_{CDNCEx} (\leq_{CDNCEx}) \mathbf{Y}$, $i = 1, 2$.

Proof. *For* $i = 1$, *if* $\overline{X}_1 \geq_{st} (\leq_{st}) \overline{Y}_1$, *then* $\frac{F(x_1,t_2)}{F(t_1,t_2)} \leq (\geq) \frac{G(x_1,t_2)}{G(t_1,t_2)}$. *Therefore, we have* $\mathcal{CJ}_1^{\mathbf{X}}(t_1, t_2) \geq (\leq) \mathcal{CJ}_1^{\mathbf{Y}}(t_1, t_2)$. *Similarly, the rest of the part follows.* □

In support of Theorem 3.10, we consider the following example.

Example 3.2. Consider two nonnegative bivariate RVs $\mathbf{X}$ and $\mathbf{Y}$ have respective CDFs

$$F_{\mathbf{X}}(x_1, x_2) = (1 - e^{-\mu_1 x_1})^{0.5} + (1 - e^{-\mu_2 x_2})^{0.5} - (1 - e^{-\mu_1 x_1 - \mu_2 x_2})^{0.5}, x_1, x_2 \geq 0,\ \mu_1, \mu_2 > 0, \text{ and}$$

$$F_{\mathbf{Y}}(x_1, x_2) = (1 - e^{-\mu_3 x_1})^{0.5} + (1 - e^{-\mu_4 x_2})^{0.5} - (1 - e^{-\mu_3 x_1 - \mu_4 x_2})^{0.5}, x_1, x_2 \geq 0,\ \mu_3, \mu_4 > 0,$$

with corresponding marginal CDFs $F_{X_1}(x_1) = (1 - e^{-\mu_1 x_1})^{0.5}$, $F_{X_2}(x_2) = (1 - e^{-\mu_2 x_2})^{0.5}$, $G_{Y_1}(x_1) = (1 - e^{-\mu_3 x_1})^{0.5}$, and $G_{Y_2}(x_2) = (1 - e^{-\mu_4 x_2})^{0.5}$. Let $\mu_1 = 2.1, \mu_2 = 3.2, \mu_3 = 3.3$, and $\mu_4 = 4.4$. Both $\mathbf{X}$ and $\mathbf{Y}$ share a common copula of the form $C(u, v) = u + v - (u + v - (uv)^{1/\beta})$. Since $X_1 \geq_{st} Y_1$ and $X_2 \geq_{st} Y_2$, it follows from Theorem 3.2.6 of Belzunce *et al.* [3, p. 118] that $\mathbf{X} \geq_{st} \mathbf{Y}$. We have plotted $\mathcal{CJ}_i^{\mathbf{X}}(t_1, t_2) - \mathcal{CJ}_i^{\mathbf{Y}}(t_1, t_2) = \varphi_i(t_1, t_2)$ (say) in Figure 3. From Figure 3(a) and (b), it is evident that $\mathbf{X} \geq_{CDNCEx} \mathbf{Y}$. In plotting the curves, we adopt the substitutions $t_1 = -\ln x$ and $t_2 = -\ln y$.

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The following theorem establishes that the component-wise ordering of BRHR functions between two RVs implies an ordering of their CDNCEx measures.

Theorem 3.11 *Consider two nonnegative bivariate RVs* $\mathbf{X}$ *and* $\mathbf{Y}$ *with their respective CDFs* $F(x_1, x_2)$ *and* $G(x_1, x_2)$. *For* $i = 1, 2$, *let* $h_i^{\mathbf{X}}(t_1, t_2)$ *and* $h_i^{\mathbf{Y}}(t_1, t_2)$ *represent the components of the BRHR functions corresponding to* $\mathbf{X}$ *and* $\mathbf{Y}$. *If the inequality* $h_i^{\mathbf{X}}(t_1, t_2) \le h_i^{\mathbf{Y}}(t_1, t_2)$ *holds for all* $t_1, t_2 \ge 0$, *then it follows that* $\mathbf{X} \le_{CDNCEx} \mathbf{Y}$.

Proof. Let $i = 1$. Assume that

$$h_1^{\mathbf{X}}(t_1, t_2) \le h_1^{\mathbf{Y}}(t_1, t_2).$$

Then the ratio $\dfrac{F(t_1, t_2)}{G(t_1, t_2)}$ decreases with $t_1 \ge 0$. Hence, for all $0 \le x_1 \le t_1$,

$$\frac{G(x_1, t_2)}{G(t_1, t_2)} \le \frac{F(x_1, t_2)}{F(t_1, t_2)}. \tag{3.25}$$

For any $t_1, t_2 \ge 0$,

$$1 - \left(\frac{G(x_1, t_2)}{G(t_1, t_2)}\right)^2 \ge 1 - \left(\frac{F(x_1, t_2)}{F(t_1, t_2)}\right)^2. \tag{3.26}$$

Integrating both sides of (3.26) with respect to x_1 over the interval $(0, t_1)$ and multiplying by 0.5 gives the desired result. The case $i = 2$ is established similarly by interchanging the roles of t_1 and t_2. □

Recall the definition of the conditional proportional reversed hazard rate (CPRHR) model, introduced by Gupta *et al*. [8]. This model characterizes the dependence structure of bivariate RVs by relating their reversed hazard rate functions through a proportionality factor. Assume two bivariate RVs $\mathbf{X}$ and $\mathbf{Y}$ with CDFs $F(x_1, x_2)$ and $G(x_1, x_2)$, respectively. The pair $\mathbf{X}$ and $\mathbf{Y}$ is said to satisfy the CPRHR model if the reversed hazard rate functions of the conditional random variables $\overline{X}_i = (X_i \mid X_1 < t_1, X_2 < t_2)$ and $\overline{Y}_i = (Y_i \mid Y_1 < t_1, Y_2 < t_2)$ satisfy the proportionality relation $h_i^{\mathbf{Y}}(t_1, t_2) = \theta_i(t_j) h_i^{\mathbf{X}}(t_1, t_2)$, or equivalently, $G(t_1, t_2) = (F(t_1, t_2))^{\theta_i(t_j)}$, for $i = 1, 2; j = 3 - i$ and $t_1, t_2 \ge 0$, where $\theta_1(t_2)$ and $\theta_2(t_1)$ are positive functions of t_2 and t_1, respectively. Based on this, we present the following theorem.

Theorem 3.12 *If the RVs* $\mathbf{X}$ *and* $\mathbf{Y}$ *satisfy the CPRHR model, then*

$$\mathbf{X} \le_{CDNCEx} (\ge_{CDNCEx}) \mathbf{Y} \text{ if } \theta_i(t_j) > 1 (0 < \theta_i(t_j) < 1) \text{ for } i, j = 1, 2; j \ne i.$$

Proof. Let $i = 1$. Suppose that $\theta_1(t_2) > 1$ $(0 < \theta_1(t_2) < 1)$. Then

$$\left(\frac{F(x_1, t_2)}{F(t_1, t_2)}\right)^2 \ge (\le) \left(\frac{F(x_1, t_2)}{F(t_1, t_2)}\right)^{2\theta_1(t_2)}.$$

This relation can be expressed equivalently as

$$\left(\frac{F(x_1, t_2)}{F(t_1, t_2)}\right)^2 \ge (\le) \left(\frac{G(x_1, t_2)}{G(t_1, t_2)}\right)^2.$$

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Integrating both sides over the interval $(0, t_1)$ yields

$$\int_0^{t_1} \left[1 - \left(\frac{F(x_1, t_2)}{F(t_1, t_2)}\right)^2\right] dx_1 \leq (\geq) \int_0^{t_1} \left[1 - \left(\frac{G(x_1, t_2)}{G(t_1, t_2)}\right)^2\right] dx_1.$$

Hence, $\mathbf{X} \leq_{CDNCEx} (\geq_{CDNCEx}) \mathbf{Y}$. A similar result holds for $i = 2$. This completes the proof. □

Definition 3.4. *The random variable Y is considered larger than X in dispersive ordering, denoted $Y \geq^D X$, if and only if $Y = \psi(X)$, where ψ is a dilation function satisfying $\psi(x) - \psi(x^*) \geq x - x^*$. This implies $\psi'(x) \geq 1$, $\psi(x) \geq x$, and $x \geq \psi^{-1}(x)$. For additional detail, see [28].*

Based on Definition 3.4, we have the following result.

Theorem 3.13 *Let $\mathbf{X}$ and $\mathbf{Y}$ be two nonnegative bivariate RVs with CDFs $F(x_1, x_2)$ and $G(x_1, x_2)$, respectively. If $Y_i \geq^D X_i$ and $\mathcal{CJ}_i^{\mathbf{X}}(t_1, t_2)$ is increasing in t_i, $i = 1, 2$, then $\mathbf{X} \geq_{CDNCEx} \mathbf{Y}$.*

Proof. Suppose $Y_i \geq^D X_i$. Then there exist increasing functions ψ_1, ψ_2 such that $G(y_1, y_2) = F(\psi_1^{-1}(y_1), \psi_2^{-1}(y_2))$. For $i = 1$,

$$\begin{aligned}
\mathcal{CJ}_1^{\mathbf{Y}}(t_1, t_2) &= \frac{1}{2} \int_0^{t_1} \left(1 - \left(\frac{G(y_1, t_2)}{G(t_1, t_2)}\right)^2\right) dy_1 \\
&= \frac{1}{2} \int_0^{\psi_1^{-1}(t_1)} \left(1 - \left(\frac{F(x_1, \psi_2^{-1}(t_2))}{F(\psi_1^{-1}(t_1), \psi_2^{-1}(t_2))}\right)^2\right) \psi_1'(x_1) dx_1 \\
&\leq \mathcal{CJ}_1^{\mathbf{X}}(\psi_1^{-1}(t_1), \psi_2^{-1}(t_2)) \leq \mathcal{CJ}_1^{\mathbf{X}}(t_1, t_2).
\end{aligned}$$

A similar argument holds for $i = 2$. Hence the result follows. □

4. Nonparametric estimation

When the underlying distribution from which the data are derived is unknown, nonparametric estimators become essential. In this section, we investigate nonparametric approaches for estimating CDNCEx by employing an empirical plug-in estimator. Let (X_{1l}, X_{2l}), $l = 1, 2, \ldots, n$, denote a random sample of size n from a bivariate distribution with joint CDF $F(x_1, x_2)$. The empirical estimator of CDNCEx is defined component-wise as

$$\widehat{\mathcal{CJ}_i^{\mathbf{X}}}(t_1, t_2) = \begin{cases} \dfrac{1}{2} \displaystyle\int_0^{t_1} \left[1 - \left(\frac{\widehat{F}(x_1, t_2)}{\widehat{F}(t_1, t_2)}\right)^2\right] dx_1, & i = 1 \\ \dfrac{1}{2} \displaystyle\int_0^{t_2} \left[1 - \left(\frac{\widehat{F}(t_1, x_2)}{\widehat{F}(t_1, t_2)}\right)^2\right] dx_2, & i = 2, \end{cases} \tag{4.27}$$

where

$$\widehat{F}(t_1, t_2) = \frac{1}{n} \sum_{l=1}^{n} \mathbb{I}(X_{1l} \leq t_1, X_{2l} \leq t_2) \tag{4.28}$$

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Table 1. *The AB and MSE values associated with* $\widehat{\mathcal{CJ}_1}^{\mathbf{X}}(t_1, t_2)$ *for different time points* t_1 *and* t_2.

	n				
(t_1, t_2)	100	150	200	300	500
(2.5, 3.5)	0.002418	0.001025	0.000892	0.000864	0.000695
	(0.002902)	(0.001473)	(0.000978)	(0.000741)	(0.000577)
(4.5, 3.5)	0.001412	0.002284	0.000762	0.000451	0.000536
	(0.005692)	(0.002284)	(0.001864)	(0.001397)	(0.001170)
(6.5, 7.4)	0.001171	0.003091	0.000169	0.000790	0.000710
	(0.006886)	(0.003400)	(0.002273)	(0.001767)	(0.001388)
(8.7, 9.6)	0.002156	0.001719	0.000761	0.000142	0.000579
	(0.007309)	(0.007302)	(0.002419)	(0.001837)	(0.001463)
(11.8, 10.9)	0.002407	0.001739	0.001400	0.000345	0.000822
	(0.007343)	(0.003767)	(0.002456)	(0.001864)	(0.001503)

is the empirical distribution function and

$$\mathbb{I}(X_{1l} \le t_1, X_{2l} \le t_2) = \begin{cases} 1, & X_{1l} \le t_1,\ X_{2l} \le t_2 \\ 0, & \text{otherwise.} \end{cases}$$

Using the Glivenko–Cantelli theorem, one can show that the estimators in (4.27) are consistent and converge weakly to their theoretical counterparts.

4.1. Simulation study

To evaluate the performance of the proposed estimator, a Monte Carlo simulation study is conducted. All computations are carried out in R software (version 4.1.1). We generate $R = 10{,}000$ replications of random samples of sizes $n = 100, 150, 200, 300$, and 500 from a bivariate distribution with joint CDF

$$F(x_1, x_2) = \frac{1}{1 + e^{-x_1} + e^{-x_2}}, \quad -\infty < x_1, x_2 < \infty,$$

using the inverse transform method described by Mardia [18]. For each replication, the empirical estimators $\widehat{\mathcal{CJ}_i}^{\mathbf{X}}(t_1, t_2)$, $i = 1, 2$, are evaluated at selected time points t_1 and t_2. For these estimators, the absolute bias (AB) and mean squared error (MSE) are computed as

$$\mathrm{AB}\big(\mathcal{CJ}_i^{\mathbf{X}}(t_1, t_2)\big) = \left| \frac{1}{R} \sum_{r=1}^{R} \widehat{\mathcal{CJ}_{i,r}^{\mathbf{X}}}(t_1, t_2) - \mathcal{CJ}_i^{\mathbf{X}}(t_1, t_2) \right| \quad \text{and}$$

$$\mathrm{MSE}\big(\mathcal{CJ}_i^{\mathbf{X}}(t_1, t_2)\big) = \frac{1}{R} \sum_{r=1}^{R} \left(\widehat{\mathcal{CJ}_{i,r}^{\mathbf{X}}}(t_1, t_2) - \mathcal{CJ}_i^{\mathbf{X}}(t_1, t_2) \right)^2.$$

The computed AB and MSE values (the latter shown in parentheses) for $\widehat{\mathcal{CJ}_i}^{\mathbf{X}}(t_1, t_2)$ are summarized in Tables 1 and 2 for $i = 1$ and $i = 2$, respectively. To visualize the variation of estimation error, the MSE values reported in Tables 1 and 2 are plotted in Figure 4(a) and (b), respectively. We observe from Figure 4(a) and (b) that the MSE values decrease consistently with increasing sample size, confirming the consistency and efficiency of the proposed empirical estimator.

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Table 2. *The AB and MSE values associated with* $\widehat{\mathcal{CJ}_2}^{\mathbf{X}}(t_1, t_2)$ *for different time points* t_1 *and* t_2.

	n				
(t_1, t_2)	100	150	200	300	500
(2.5,3.5)	0.001598	0.000787	0.001205	0.000973	0.000200
	(0.004534)	(0.002313)	(0.001584)	(0.001166)	(0.000919)
(4.5,3.5)	0.001837	0.000621	0.000390	0.000796	0.000421
	(0.004345)	(0.002111)	(0.00144)	(0.001113)	(0.000881)
(6.5,7.4)	0.001954	0.001345	0.001055	0.000244	0.000323
	(0.007095)	(0.003625)	(0.002420)	(0.001806)	(0.001434)
(8.7,9.6)	0.003547	0.001662	0.001055	0.000244	0.000323
	(0.007523)	(0.003771)	(0.002477)	(0.001857)	(0.001477)
(11.8,10.9)	0.002384	0.001506	0.000602	0.000629	0.000995
	(0.007431)	(0.003692)	(0.002485)	(0.001890)	(0.001482)

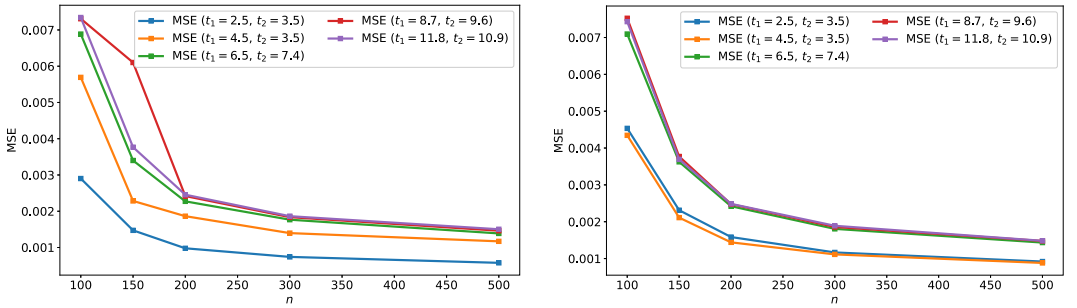


Figure 4. *MSE plots of* $\widehat{\mathcal{CJ}_1}^{\mathbf{X}}(t_1, t_2)$ *(left) and* $\widehat{\mathcal{CJ}_2}^{\mathbf{X}}(t_1, t_2)$ *(right) for different sample sizes, showing a steady decrease in MSE with increasing sample size.*

4.2. Real data analysis

This subsection presents two examples to demonstrate our method. The first example includes a data analysis based on CDNCEx. The second example provides a comparison between the CDNCEx and a traditional entropy-based approach.

Example 4.1. Diabetic retinopathy represents one of the leading causes of blindness and visual impairment among diabetic patients. The National Eye Institute provided a clinical dataset of diabetic patients with vision loss; comprehensive details can be found in [9]. A study was conducted to determine whether laser treatment reduces blindness. One eye was randomly selected for laser photocoagulation in each patient, and the number of months it took for both eyes to go blind was noted. The main objective of this study was to determine whether laser treatment can postpone blindness in any way. From an initial cohort of 197 high-risk patients in the DRS, we selected a subset of 38 patients to evaluate the efficacy of the proposed model. The information is shown in Table 3.

This dataset provides an appropriate setting for evaluating the proposed extropy-based uncertainty measure because each patient contributes a natural bivariate lifetime pair (X_1, X_2), representing the blindness times of the untreated and treated eyes. Since the measure is defined for the conditional region $(X_1 < t_1, X_2 < t_2)$, the analysis must focus on observations falling within this range. Restricting the

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Table 3. *Observed vision impairment times in individuals with diabetic retinopathy.*

X_1:	30.83	20.17	10.27	5.67	5.77	5.90	25.63	33.90	1.73	30.20	25.80
	5.73	9.90	1.70	1.77	8.30	18.70	42.17	14.30	13.33	14.27	34.57
	4.10	21.57	13.77	33.63	63.33	38.47	10.33	13.83	11.07	2.10	12.93
	24.43	13.97	13.80	13.57	42.43						
X_2:	38.57	6.90	1.63	13.83	1.33	35.53	21.90	14.80	6.20	22.00	13.87
	48.30	9.90	1.70	43.03	8.30	6.53	42.17	48.43	9.60	7.60	1.80
	12.20	9.90	13.77	33.63	27.60	1.63	0.83	1.57	1.97	11.30	4.97
	9.87	30.40	19.00	5.43	46.63						

Table 4. *The values of $\widehat{\mathcal{CJ}}_1^{\boldsymbol{X}}(t_1, t_2)$ for different combinations of t_1 and t_2.*

$t_1 \setminus t_2$	10	15	20	25	30	40	50
10	3.659960	3.258493	3.258493	3.258493	3.258493	3.272933	3.127280
15	6.003452	5.737931	5.865333	5.865333	5.865333	5.891986	5.811201
20	6.462381	6.125817	6.220278	6.220278	6.220278	6.211257	6.098251
25	7.947286	7.437185	7.451822	7.451822	7.451822	7.348820	7.139556
30	7.946398	7.922065	7.914829	8.315008	8.315008	8.163637	7.901156
40	10.009150	10.162819	10.075875	10.673633	10.673633	11.123369	10.711569
50	10.009975	10.164464	10.077237	10.674801	10.674801	11.124528	11.809664

data accordingly ensures consistency with the measure's definition and allows us to assess uncertainty specifically in early joint blindness.

We computed the CDNCEx for the data given in Table 3 using the estimation defined in (4.27) and reported in Table 4. From Table 4, it is evident that for any fixed t_2, the uncertainty measure increases as t_1 becomes larger, indicating that the method captures greater uncertainty in predicting blindness in the untreated eye at later times. For any fixed t_1, the values vary only slightly across t_2, suggesting that blindness in the treated eye exhibits more stable and consistent behavior. This contrast highlights that the untreated eye contributes more to long-term variability, whereas the treated eye remains relatively predictable over time, reinforcing the effectiveness of the treatment.

Example 4.2. In this example, we focus on Sample 1 from the simulated dataset provided by Kim and Kvam [13]. This sample contains 20 failure times from two systems, each consisting of three components. We consider the first and third components of Sample 1, denoted by Y_1 and Y_2, respectively, as reported in Table 5. We compute both the CDNCEx and the conditional dynamic cumulative past entropy (CDCPE) proposed by Kundu and Kundu [14], which is defined as

$$\overline{H}_i^{\mathbf{Y}}(t_1, t_2) = \begin{cases} -\displaystyle\int_0^{t_1} \frac{F(x_1, t_2)}{F(t_1, t_2)} \log\left(\frac{F(x_1, t_2)}{F(t_1, t_2)}\right) dx_1, & i = 1 \\ -\displaystyle\int_0^{t_2} \frac{F(t_1, x_2)}{F(t_1, t_2)} \log\left(\frac{F(t_1, x_2)}{F(t_1, t_2)}\right) dx_2, & i = 2. \end{cases}$$

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Table 5. *Load sharing data of a three-component system.*

Y_1 :	1.94	7.44	0.14	2.14	1.91	8.23	1.40	0.79	0.92	0.73
	2.78	0.85	8.50	12.93	4.46	3.50	19.59	4.98	10.29	2.22
Y_2 :	6.93	2.42	0.20	2.34	1.90	4.60	0.09	7.27	0.06	8.61
	1.38	5.05	0.52	1.11	3.54	2.38	1.89	1.54	8.61	1.22

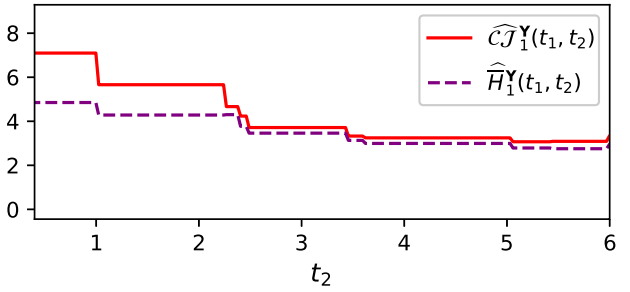


Figure 5. *Graphical representation of the empirical CDNCEx and CDCPE for load sharing data, demonstrating that CDNCEx captures more information about the system than CDCPE.*

Both measures are estimated empirically using the corresponding plug-in estimators defined in (4.28). The empirical estimate of the CDCPE is denoted by $\widehat{H}_i^{\mathbf{Y}}(t_1, t_2)$ for $i = 1, 2$. For the random variable $\overline{Y}_1 = (Y_1 \mid Y_1 < t_1, Y_2 < t_2)$, we fix $t_1 = \max(Y_1)$ and vary t_2. We then compute CDNCEx and CDCPE for $\overline{Y}_1$, and the corresponding results are presented in Figure 5. It is evident from Figure 5 that as t_2 increases, the values of both CDCPE and CDNCEx decrease, indicating a reduction in uncertainty. Moreover, the values of CDNCEx are consistently greater than those of CDCPE, suggesting that CDNCEx captures a larger amount of information. Hence, it can be concluded that CDNCEx performs better than CDCPE in distinguishing the degree of uncertainty between the component pairs, making it a more informative and sensitive measure in this context.

5. Conclusion

This study introduced BDNCEx as a new bivariate uncertainty measure. It is not invariant under non-singular monotonic transformations and is both scale- and shift-dependent under affine maps. Additionally, BDNCEx does not uniquely determine the distribution function; distinct distributions can yield identical BDNCEx values, which highlights a limitation of the measure. To address this shortcoming, the measure was extended to a vector-valued form, termed CDNCEx. The proposed CDNCEx uniquely determines the distribution function and establishes a meaningful connection with BEIT. A bound for CDNCEx under monotonicity was established, and it was shown that CDNCEx exhibits similar behavior to BDNCEx under non-singular monotonic transformations. Moreover, CDNCEx reduces to the univariate dynamic NCEx in the case of independent components. A new stochastic order based on CDNCEx was proposed, and it was shown that the usual stochastic order implies the CDNCEx-based stochastic order. Further, a nonparametric estimation was also developed and validated via simulation studies and real data analysis. Despite these promising findings, certain limitations remain. The empirical estimation of CDNCEx may be sensitive to sample size and boundary bias, and its theoretical properties under model misspecification merit further investigation. Future research could focus on developing a kernel-smoothed version of CDNCEx to enhance estimation stability and exploring its application in dependence modeling, goodness-of-fit testing, and survival prediction. Extending

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the framework to higher-dimensional and censored data would further reinforce its theoretical and practical relevance. Future research could also include an extensive comparison of this measure with other information measures to better understand its strengths and limitations.

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