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COMPLETE MOMENT CONVERGENCE FOR ARRAYS OF ROWWISE NEGATIVELY ASSOCIATED RANDOM VARIABLES AND ITS APPLICATION IN NON-PARAMETRIC REGRESSION MODEL

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In this paper, some results on the complete moment convergence for arrays of rowwise negatively associated (NA, for short) random variables are established. The results obtained in this paper correct the corresponding one obtained in Ko [13] and also improve and generalize the corresponding ones of Kuczmaszewska [14] and Ko [13]. As an application of the main results, we present a result on complete consistency for the estimator in a non-parametric regression model based on NA errors. Finally, we provide a numerical simulation to verify the validity of our result.

Keywords: complete consistency, complete convergence, complete moment convergence, negatively associated random variables, non-parametric regression model

1. INTRODUCTION

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{X_n, n \geq 1\}$ be a sequence of random variables defined on this space. The concept of complete convergence was first introduced by Hsu and Robbins [10] as follows: a sequence $\{X_n, n \geq 1\}$ of random variables converges completely to a constant C if for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n - C| > \varepsilon) < \infty.$$

By the Borel–Cantelli lemma, the inequality above implies that $X_n \to C$ almost surely (a.s., for short). Hsu and Robbins [10] proved that the sequence of arithmetic means of independent and identically distributed random variables converges completely to the expected value of the summands, provided the variance is finite. The converse was proved by Erdös [4]. This Hsu–Robbins–Erdös's result was generalized in different ways. Katz [12], Baum and Katz [1], and Chow [2] obtained a generalization of complete convergence for a sequence of

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independent and identically distributed (i.i.d., for short) random variables with normalization of Marcinkiewicz–Zygmund type (see Gut [8]). However, the independent assumption is not always reasonable in many realistic applications. A familiar extension of independence is negative association, which was introduced by Joag-Dev and Proschan [11] as follows.

DEFINITION 1.1: *A finite family of random variables* $\{X_i, 1 \le i \le n\}$ *is said to be negatively associated (NA, for short) if for every pair of disjoint subsets* A *and* B *of* $\{1, 2, \ldots, n\}$ *and any real coordinatewise non-decreasing functions* f_1 *on* $\mathbb{R}^A$ *and* f_2 *on* $\mathbb{R}^B$,

$$Cov(f_1(X_i, i \in A), f_2(X_j, j \in B)) \le 0,$$

whenever the covariance exists. An infinite family of random variables is NA if every finite subfamily is NA.

Kuczmaszewska [14] extended the Hsu-Robbins-Erdös's result from independent sequence to NA sequence and weakened the identical distribution assumption as follows.

Theorem A. *Let* $\{X_n, n \ge 1\}$ *be a sequence of NA random variables and* X *be a random variable possibly defined on a different space satisfying the condition*

$$n^{-1} \sum_{i=1}^{n} P(|X_i| > x) = cP(|X| > x)$$

for some $c > 0$, *all* $x > 0$, *and all* $n \ge 1$. *Let* $\alpha p > 1$ *and* $\alpha > 1/2$. *Moreover, additionally assume that for* $p \ge 1$, $EX_n = 0$ *for all* $n \ge 1$. *Then the following statements are equivalent:*

$$(i)\ E|X|^p < \infty;\ (ii)\ \sum_{n=1}^{\infty} n^{\alpha p - 2} P\left(\max_{1 \le k \le n} \left| \sum_{i=1}^{k} X_i \right| > \varepsilon n^{\alpha} \right) < \infty, \quad \text{for all } \varepsilon > 0.$$

The result above is meaningful but the case $\alpha p = 1$ was not considered in Kuczmaszewska [14]. Chow [3] first showed the complete moment convergence for a sequence of i.i.d. random variables by generalizing the result of Baum and Katz [1]. The concept of complete moment convergence is as follows:

Let $\{X_n, n \ge 1\}$ be a sequence of random variables and $a_n > 0, b_n > 0, q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|X_n| - \varepsilon\}_+^q < \infty, \quad \text{for all } \varepsilon > 0,$$

then X_n is said to be complete moment convergent. It is well known that complete moment convergence implies complete convergence. Thus, complete moment convergence is stronger than complete convergence. For more details about the complete moment convergence, we refer the readers to Wang and Hu [27], Liang, Li, and Rosalsky [17], Wu, Cabrea, and, Volodin [30], Guo and Zhu [7] and Shen, Xue, and Volodin [22], Shen et al. [23] among others.

Recently, Ko [13] extended Theorem A from complete convergence to complete moment convergence as follows.

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Theorem B. *Let $\alpha p > 1$, $p > 0$ and $\alpha > 1/2$. Let $\{X_n, n \geq 1\}$ be a sequence of NA random variables satisfying the condition*

$$n^{-1}\sum_{i=1}^{n} P(|X_i| > x) \leq CP(|X| > x), \quad \text{for all } x > 0 \text{ and } n \geq 1$$

with $E|X|^p < \infty$. Assume that $EX_n = 0$ for all $n \geq 1$. Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right| - \varepsilon n^{\alpha}\right)^{+} < \infty.$$

Remark 1.1: We point out that there are some mistakes in the proof of Lemma 2.4 in Ko [13]. To be specific, in the case $0 < p < 1$, the formula is wrongly calculated as $\sum_{n=1}^{m} n^{\alpha p-1-\alpha} \leq Cm^{\alpha p-\alpha}$ (see in the proof of I_1); and in the case $1 \leq p < 2$, the formula is wrongly calculated as $\sum_{n=1}^{m} n^{\alpha p-1-\alpha} \leq Cm^{\alpha p-\alpha}$ if $p = 1$ (see in the proof of I_1'). Unfortunately, these mistakes lead directly to a wrong result in the case $0 < p \leq 1$. It is desirable to correct the mistakes. On the other hand, the result when $p > 1$ can still be improved.

In order to present our work, first let us recall the concept of stochastic domination.

Definition 1.2: *A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that*

$$P(|X_n| > x) \leq CP(|X| > x)$$

for all $x \geq 0$ and $n \geq 1$.

Gut [8] introduced the concept of weak mean domination as follows.

Definition 1.3: *A sequence $\{X_n, n \geq 1\}$ of random variables is said to satisfy a weak mean dominating condition with mean dominating random variable X if for some $C > 0$,*

$$n^{-1}\sum_{i=1}^{n} P(|X_i| > x) \leq CP(|X| > x), \quad \text{for all } x > 0 \text{ and } n \geq 1.$$

To search for more general applications, we extend the concept of weak mean domination to weak weighted domination as follows.

Definition 1.4: *A sequence $\{X_n, n \geq 1\}$ of random variables is said to satisfy a weak weighted mean dominating condition with weighted mean dominating random variable X if for some $C > 0$,*

$$\sum_{i=1}^{n} W_{ni} P(|X_i| > x) \leq CP(|X| > x), \quad \text{for all } x > 0 \text{ and } n \geq 1,$$

where $W_{ni} \geq 0$ for all $1 \leq i \leq n, n \geq 1$ and $\sum_{i=1}^{n} W_{ni} = 1$, for all $n \geq 1$.

Remark 1.2: Obviously stochastic domination implies weak weighted mean domination. If we take $W_{ni} = 1/n$ for each $1 \leq i \leq n$ and $n \geq 1$, then weak weighted mean domination is equivalent to weak mean domination.

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The main purpose of this paper is to establish the complete moment convergence results for arrays of rowwise NA random variables. The results obtained in this paper correct the result obtained in Ko [13] and also improve and extend Theorems A and B for NA random variables. The meaningful case $\alpha p = 1$ that was not considered in either Kuczmaszewska [14] or Ko [13] is also considered in this paper. It is worth mentioning that the proof of our main results is different from that of Ko [13]. As an application of our main results, we present a result on complete consistency for the estimator in a non-parametric regression model based on NA errors. Finally, we present a numerical simulation to verify the validity of our result.

The layout of this paper is as follows. Some preliminary lemmas are provided in Section 2. Main results and their proofs are stated in Section 3. Some application of our main results are presented in Section 4. Throughout this paper, C represents some positive constant whose value may vary in different places. Let $\log x = \ln \max(x, e)$, and $I(A)$ be the indicator function of the set A. Denote $x^+ = xI(x \geq 0)$ and $x^- = -xI(x < 0)$. $a \ll b$ implies that there exists some positive constant c_1 such that $a \leq c_1 b$. $\lfloor x \rfloor$ stands for the integer part of x.

2. PRELIMINARY LEMMAS

To prove the main results of the paper, we need the following important lemmas. The first one comes from Sung [25].

LEMMA 2.1: *Let $\{a_i, 1 \leq i \leq n\}$ and $\{b_i, 1 \leq i \leq n\}$ be two sequences of real numbers. Then for any real number c, the following inequalities hold:*

$$(|a_i + b_i| - |c|)_+ \leq (|a_i| - |c|)_+ + |b_i|$$

and

$$\left(\max_{1\leq i\leq n} |a_i + b_i| - |c|\right)_+ \leq \left(\max_{1\leq i\leq n} |a_i| - |c|\right)_+ + \max_{1\leq i\leq n} |b_i|.$$

The following lemma is essential in proving our main results, which improve the corresponding one in Sung [25] from $r \equiv 1$ to any $r > 0$.

LEMMA 2.2: *Let $\{Y_i, 1 \leq i \leq n\}$ and $\{Z_i, 1 \leq i \leq n\}$ be sequences of random variables. Then for any $q > r > 0$, $\varepsilon > 0$, and $a > 0$, the following inequalities hold:*

$$E\left(\left|\sum_{i=1}^{n}(Y_i + Z_i)\right| - \varepsilon a\right)_+^r \leq C_r\left(\varepsilon^{-q} + \frac{r}{q-r}\right) a^{r-q} E\left|\sum_{i=1}^{n} Y_i\right|^q + C_r E\left|\sum_{i=1}^{n} Z_i\right|^r$$

and

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}(Y_i + Z_i)\right| - \varepsilon a\right)_+^r \leq C_r\left(\varepsilon^{-q} + \frac{r}{q-r}\right) a^{r-q} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} Y_i\right|^q\right) + C_r E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} Z_i\right|^r\right),$$

where $C_r = 1$ if $0 < r \leq 1$ and $C_r = 2^{r-1}$ if $r > 1$.

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PROOF: Since the proofs of the two inequalities above are completely analogous, we only present the details of the second inequality. It follows from Lemma 2.1 and C_r-inequality that

$$E\left(\max_{1\le i\le n}\left|\sum_{i=1}^{k}(Y_i+Z_i)\right|-\varepsilon a\right)_+^r \le C_rE\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}Y_i\right|-\varepsilon a\right)_+^r + C_rE\max_{1\le k\le n}\left|\sum_{i=1}^{k}Z_i\right|^r. \tag{2.1}$$

Meanwhile, we have by Markov's inequality that

$$\begin{aligned}
E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}Y_i\right|-\varepsilon a\right)_+^r &= \int_0^\infty P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}Y_i\right|-\varepsilon a>t^{\frac{1}{r}}\right)dt\\
&\le \int_0^{a^r} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}Y_i\right|>\varepsilon a\right)dt\\
&\quad+\int_{a^r}^\infty P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}Y_i\right|>t^{\frac{1}{r}}\right)dt\\
&\le \left(\varepsilon^{-q}a^{r-q}+\int_{a^r}^\infty t^{-\frac{q}{r}}dt\right)E\max_{1\le k\le n}\left|\sum_{i=1}^{k}Y_i\right|^q\\
&= \left(\varepsilon^{-q}+\frac{r}{q-r}\right)a^{r-q}E\max_{1\le k\le n}\left|\sum_{i=1}^{k}Y_i\right|^q.
\end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2), we obtain the desired result immediately. ■

The following Marcinkiewicz–Zygmund-type maximal inequality and Rosenthal-type maximal inequality for NA random variables can be found in Shao [20].

LEMMA 2.3: *Let $\{X_i, i\ge 1\}$ be a sequence of NA random variables with $EX_i=0$ and $E|X_i|^q<\infty$ for some $q>1$ and each $i\ge 1$. Then we have that*

$$E\max_{1\le j\le n}\left|\sum_{i=1}^{j}X_i\right|^q \le 2^{3-q}\sum_{i=1}^{n}E|X_i|^q, \quad \text{for } 1<q\le 2,$$

and

$$E\max_{1\le j\le n}\left|\sum_{i=1}^{j}X_i\right|^q \le 2\left(\frac{15q}{\ln q}\right)^q\left\{\sum_{i=1}^{n}E|X_i|^q+\left(\sum_{i=1}^{n}EX_i^2\right)^{q/2}\right\}, \quad \text{for } q>2.$$

LEMMA 2.4: *Let $\{X_{ni}, 1\le i\le n, n\ge 1\}$ be an array of random variables satisfying a weak weighted mean dominating condition with weighted mean dominating random variable X.*

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Let $r > 0$ and for some $A > 0$,

$$X'_{ni} = X_{ni} I(|X_{ni}| \le A), \quad X''_{ni} = X_{ni} I(|X_{ni}| > A),$$
$$X^*_{ni} = -AI(X_{ni} < -A) + X_{ni} I(|X_{ni}| \le A) + AI(X_{ni} > A)$$

and

$$X' = XI(|X| \le A), \quad X'' = XI(|X| > A),$$
$$X^* = -AI(X < -A) + XI(|X| \le A) + AI(X > A).$$

Then there exists some $C > 0$ such that

(i) if $E|X|^r < \infty$, then $\sum_{i=1}^n W_{ni} E|X_{ni}|^r \le CE|X|^r$;
(ii) $\sum_{i=1}^n W_{ni} E|X'_{ni}|^r \le C[E|X'|^r + A^r P(|X| > A)]$;
(iii) $\sum_{i=1}^n W_{ni} E|X''_{ni}|^r \le CE|X''|^r$;
(iv) $\sum_{i=1}^n W_{ni} E|X^*_{ni}|^r \le CE|X^*|^r$.

PROOF: For a random variable Y with $E|Y|^r < \infty$, we have

$$E|Y|^r = r \int_0^\infty y^{r-1} P(|Y| > y) dy,$$

then the conclusions follow. The details are omitted. ■

The following conclusion will be used frequently in the proof of the main result, so we state it here.

LEMMA 2.5: *Let $\alpha > 0$, $0 < s < p < t$ and X be a random variable with $E|X|^p < \infty$. Then we have*

$$\sum_{n=1}^\infty n^{\alpha p - \alpha s - 1} E|X|^s I(|X| > n^\alpha) \ll E|X|^p$$

and

$$\sum_{n=1}^\infty n^{\alpha p - \alpha t - 1} [E|X|^t I(|X| \le n^\alpha) + n^{\alpha t} P(|X| > n^\alpha)] \ll E|X|^p.$$

PROOF: By some standard computation, we obtain that

$$\begin{aligned}
\sum_{n=1}^\infty n^{\alpha p - \alpha s - 1} E|X|^s I(|X| > n^\alpha) &= \sum_{n=1}^\infty n^{\alpha p - \alpha s - 1} \sum_{m=n}^\infty E|X|^s I(m^\alpha < |X| \le (m+1)^\alpha) \\
&= \sum_{m=1}^\infty E|X|^s I(m^\alpha < |X| \le (m+1)^\alpha) \sum_{n=1}^m n^{\alpha p - \alpha s - 1} \\
&\ll \sum_{m=1}^\infty m^{\alpha p - \alpha s} E|X|^s I(m^\alpha < |X| \le (m+1)^\alpha) \ll E|X|^p,
\end{aligned}$$

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and

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-\alpha t-1}[E|X|^t I(|X|\le n^\alpha)+n^{\alpha t}P(|X|>n^\alpha)]\\
&\le \sum_{n=1}^{\infty} n^{\alpha p-\alpha t-1}E|X|^t I(|X|\le n^\alpha)+\sum_{n=1}^{\infty} n^{\alpha p-\alpha s-1}E|X|^s I(|X|>n^\alpha)\\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha t-1}\sum_{m=1}^{n} E|X|^t I((m-1)^\alpha<|X|\le m^\alpha)+E|X|^p\\
&= \sum_{m=1}^{\infty} E|X|^t I((m-1)^\alpha<|X|\le m^\alpha)\sum_{n=m}^{\infty} n^{\alpha p-\alpha t-1}+E|X|^p\\
&\ll \sum_{m=1}^{\infty} m^{\alpha p-\alpha t}E|X|^t I((m-1)^\alpha<|X|\le m^\alpha)+E|X|^p \ll E|X|^p.
\end{aligned}$$

The proof of the lemma is completed. ■

3. MAIN RESULTS

In this section, we will present our results, which improve and extend the corresponding ones of Kuczmaszewska [14] and Ko [13] for NA sequence and thus for independent sequence.

THEOREM 3.1: *Let $\alpha p>1$ and $\alpha>1/2$. Let $\{X_{ni},1\le i\le n,n\ge 1\}$ be an array of rowwise NA random variables satisfying a weak weighted mean dominating condition with weighted mean dominating random variable X, and $\{a_{ni},1\le i\le n,n\ge 1\}$ be an array of constants satisfying $\sum_{i=1}^{n}|a_{ni}|^q\ll n$ for some $q>\max\{p,(\alpha p-1)/(\alpha-1/2)\}$. Assume further that $EX_{ni}=0$ for $p\ge 1$. Then*

$$E|X|^p<\infty \tag{3.1}$$

implies that for any $0<r<p$ and for all $\varepsilon>0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r}E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|-\varepsilon n^\alpha\right)_+^r<\infty, \tag{3.2}$$

and thus

$$\sum_{n=1}^{\infty} n^{\alpha p-2}P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|>\varepsilon n^\alpha\right)<\infty. \tag{3.3}$$

PROOF: We first show that (3.1) implies (3.2). Note that $a_{ni}=a_{ni}^+-a_{ni}^-$. Without loss of generality, we may assume that $a_{ni}>0$. Note by Hölder's inequality that $\sum_{i=1}^{n}a_{ni}^s\ll n$ for all $0<s<q$. For fixed $n\ge 1$, denote for $1\le i\le n$ that

$$\begin{aligned}
Y_{ni}&=a_{ni}[-n^\alpha I(X_{ni}<-n^\alpha)+X_{ni}I(|X_{ni}|\le n^\alpha)+n^\alpha I(X_{ni}>n^\alpha)],\\
Z_{ni}&=a_{ni}X_{ni}-Y_{ni}=a_{ni}[(X_{ni}+n^\alpha)I(X_{ni}<-n^\alpha)+(X_{ni}-n^\alpha)I(X_{ni}>n^\alpha)].
\end{aligned}$$

Thus, $\{Y_{ni},1\le i\le n,n\ge 1\}$ and $\{Z_{ni},1\le i\le n,n\ge 1\}$ are both arrays of rowwise NA random variables. We will prove our result under the following three cases.

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Case 1. $0 < p < 1$.
Take $q^* = \min(1, q)$. It follows from Lemmas 2.2, 2.4 and 2.5 that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right| - \varepsilon n^{\alpha}\right)_+^r \\
&\quad \ll \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r}\cdot n^{\alpha(r-q^*)} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} Y_{ni}\right|^{q^*}\right) \\
&\qquad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} Z_{ni}\right|^{r}\right) \text{ (by Lemma 2.2)} \\
&\quad \ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q^*-2}\sum_{i=1}^{n} a_{ni}^{q^*}\left[E|X_{ni}|^{q^*} I(|X_{ni}|\le n^{\alpha}) + n^{\alpha q^*} P(|X_{ni}|> n^{\alpha})\right] \\
&\qquad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r}\sum_{i=1}^{n} a_{ni}^{r} E|X_{ni}|^{r} I(|X_{ni}> n^{\alpha}) \\
&\quad = \sum_{n=1}^{\infty} n^{\alpha p-\alpha q^*-2}\sum_{k=1}^{n} a_{nk}^{q^*}\sum_{i=1}^{n}\frac{a_{ni}^{q^*}}{\sum_{k=1}^{n} a_{nk}^{q^*}}\left[E|X_{ni}|^{q^*} I(|X_{ni}|\le n^{\alpha}) + n^{\alpha q^*} P(|X_{ni}|> n^{\alpha})\right] \\
&\qquad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r}\sum_{k=1}^{n} a_{nk}^{r}\sum_{i=1}^{n}\frac{a_{ni}^{r}}{\sum_{k=1}^{n} a_{nk}^{r}} E|X_{ni}|^{r} I(|X_{ni}|> n^{\alpha}) \\
&\quad \ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q^*-1}\left[E|X|^{q^*} I(|X|\le n^{\alpha}) + n^{\alpha q^*} P(|X|> n^{\alpha})\right] \\
&\qquad + \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-1} E|X|^{r} I(|X|> n^{\alpha}) < \infty. \text{ (by Lemmas 2.4 and 2.5)}
\end{aligned}$$

Case 2. $1 \le p < 2$.
Take $q^* = \min(2, q)$. For $p \ge 1$, we have by the mean zero assumption that

$$\begin{aligned}
n^{-\alpha}\max_{1\le k\le n}\left|\sum_{i=1}^{k} EY_{ni}\right| &= n^{-\alpha}\max_{1\le k\le n}\left|\sum_{i=1}^{k} EZ_{ni}\right| \\
&\le n^{-\alpha}\sum_{k=1}^{n} a_{nk}\sum_{i=1}^{n}\frac{a_{ni}}{\sum_{k=1}^{n} a_{nk}} E|X_{ni}| I(|X_{ni}|> n^{\alpha}) \\
&\ll n^{1-\alpha} E|X| I(|X|> n^{\alpha}) \le n^{1-\alpha p} E|X|^{p} I(|X|> n^{\alpha}) \\
&\to 0 \text{ as } n\to\infty. \qquad \textbf{(3.4)}
\end{aligned}$$

Therefore, when n is large enough, $\max_{1\le k\le n}|\sum_{i=1}^{k} EY_{ni}| \le \varepsilon n^{\alpha}/2$. Note that $|Z_{ni}| \le a_{ni}|X_{ni}|I(|X_{ni}| > n^{\alpha})$. Analogous to the proof of Case 1, we have by Lemma 2.3,

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C_r-inequality and Jensen's inequality that if $0 < r < 1$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right| - \varepsilon n^{\alpha}\right)_+^r \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}(Y_{ni}-EY_{ni}+Z_{ni})\right| - \varepsilon n^{\alpha}/2\right)_+^r \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q^*-2} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}(Y_{ni}-EY_{ni})\right|^{q^*}\right) + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} Z_{ni}\right|^{r}\right) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q^*-2}\sum_{i=1}^{n} a_{ni}^{q^*}\left[E|X_{ni}|^{q^*} I(|X_{ni}|\le n^{\alpha}) + n^{\alpha q^*} P(|X_{ni}|>n^{\alpha})\right] \text{ (by Lemma 2.3)} \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r}\sum_{i=1}^{n} a_{ni}^{r} E|X_{ni}|^{r} I(|X_{ni}>n^{\alpha}) < \infty,
\end{aligned}$$

and if $1 \le r < p$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right| - \varepsilon n^{\alpha}\right)_+^r \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q^*-2} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}(Y_{ni}-EY_{ni})\right|^{q^*}\right) \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}(Z_{ni}-EZ_{ni})\right|^{r}\right) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q^*-2}\sum_{i=1}^{n} a_{ni}^{q^*}\left[E|X_{ni}|^{q^*} I(|X_{ni}|\le n^{\alpha}) + n^{\alpha q^*} P(|X_{ni}|>n^{\alpha})\right] \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r}\sum_{i=1}^{n} a_{ni}^{r} E|X_{ni}|^{r} I(|X_{ni}>n^{\alpha}) < \infty. \text{ (by Lemmas 2.3–2.5)}
\end{aligned}$$

Case 3. $p \ge 2$.
Note that $\alpha p - \alpha q - 2 + q/2 < -1$ by $q > (\alpha p - 1)/(\alpha - 1/2)$ and $E|X|^p < \infty$ implies $E|X|^2 < \infty$. We have by Lemmas 2.2–2.5, C_r-inequality and Jensen's inequality that if $0 < r < 1$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right| - \varepsilon n^{\alpha}\right)_+^r \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}(Y_{ni}-EY_{ni})\right|^{q}\right) + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} Z_{ni}\right|^{r}\right) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\left\{\sum_{i=1}^{n} E|Y_{ni}|^{q} + \left(\sum_{i=1}^{n} E|Y_{ni}|^{2}\right)^{q/2}\right\}
\end{aligned}$$

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$$
\begin{aligned}
&+\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} \sum_{i=1}^{n} a_{ni}^{r} E|X_{ni}|^{r} I(|X_{ni}>n^{\alpha}) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\left\{\sum_{i=1}^{n} a_{ni}^{2}\left[E|X_{ni}|^{2} I(|X_{ni}| \leq n^{\alpha})+n^{2\alpha} P(|X_{ni}|>n^{\alpha})\right]\right\}^{q/2}+E|X|^{p} \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\left\{\sum_{k=1}^{n} a_{nk}^{2} \sum_{i=1}^{n} \frac{a_{ni}^{2}}{\sum_{k=1}^{n} a_{nk}^{2}}[E|X_{ni}|^{2} I(|X_{ni}| \leq n^{\alpha})\right. \\
&\quad \left.+n^{2\alpha} P(|X_{ni}|>n^{\alpha})]\right\}^{q/2}+E|X|^{p} \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2+q/2}\left(EX^{2}\right)^{q/2}+E|X|^{p}<\infty,
\end{aligned}
$$

and if $1 \leq r \leq 2$,

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\max_{1 \leq k \leq n}\left|\sum_{i=1}^{k} a_{ni} X_{ni}\right|-\varepsilon n^{\alpha}\right)_{+}^{r} \\
&\quad \ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} E\left(\max_{1 \leq k \leq n}\left|\sum_{i=1}^{k}\left(Y_{ni}-EY_{ni}\right)\right|^{q}\right) \\
&\qquad +\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\max_{1 \leq k \leq n}\left|\sum_{i=1}^{k}\left(Z_{ni}-EZ_{ni}\right)\right|^{r}\right) \\
&\quad \ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\left\{\sum_{i=1}^{n} E|Y_{ni}|^{q}+\left(\sum_{i=1}^{n} E|Y_{ni}|^{2}\right)^{q/2}\right\} \\
&\qquad +\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} \sum_{i=1}^{n} a_{ni}^{r} E|X_{ni}|^{r} I(|X_{ni}>n^{\alpha})<\infty,
\end{aligned}
$$

and if $2<r<p$,

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\max_{1 \leq k \leq n}\left|\sum_{i=1}^{k} a_{ni} X_{ni}\right|-\varepsilon n^{\alpha}\right)_{+}^{r} \\
&\quad \ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} E\left(\max_{1 \leq k \leq n}\left|\sum_{i=1}^{k}\left(Y_{ni}-EY_{ni}\right)\right|^{q}\right) \\
&\qquad +\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\max_{1 \leq k \leq n}\left|\sum_{i=1}^{k}\left(Z_{ni}-EZ_{ni}\right)\right|^{r}\right) \\
&\quad \ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\left\{\sum_{i=1}^{n} E|Y_{ni}|^{q}+\left(\sum_{i=1}^{n} E|Y_{ni}|^{2}\right)^{q/2}\right\}
\end{aligned}
$$

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$$
\begin{aligned}
&+\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\left\{\sum_{i=1}^{n} E|Z_{ni}|^{r}+\left(\sum_{i=1}^{n} E|Z_{ni}|^{2}\right)^{r/2}\right\}\\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2+q/2}\left(EX^{2}\right)^{q/2}+E|X|^{p}\\
&\quad+\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\left(\sum_{i=1}^{n} a_{ni}^{2} E|X_{ni}|^{2} I(|X_{ni}|>n^{\alpha})\right)^{r/2}\\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\left[nE|X|^{2} I(|X|>n^{\alpha})\right]^{r/2}\\
&\ll \sum_{n=1}^{\infty} n^{(\alpha p-1)(1-r/2)-1}\left(E|X|^{p}\right)^{r/2}<\infty.
\end{aligned}
$$

Now we will show that (3.2) implies (3.3). In fact, it can be checked that

$$
\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|-\varepsilon n^{\alpha}\right)_{+}^{r}\\
&= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r}\int_{0}^{\infty} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|-\varepsilon n^{\alpha}>t^{1/r}\right)dt\\
&\ge \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r}\int_{0}^{\varepsilon^{r} n^{\alpha r}} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|-\varepsilon n^{\alpha}>t^{1/r}\right)dt\\
&\ge \varepsilon^{r}\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|>2\varepsilon n^{\alpha}\right).
\end{aligned}
$$

Since $\varepsilon>0$ is arbitrary, therefore the desired result follows immediately. This completes the proof of the theorem. ■

Remark 3.1: As is mentioned in Remark 1.1, Theorem B is wrongly established in the case $0<p\le 1$. In Theorem 3.1, we not only present the result in the case $0<p\le 1$, but also improve the corresponding one of Ko [13] in the case $p>1$. Moreover, the proof is different from that of Ko [13]. The main improvements and difference are listed below.

(i) The array of constants $\{a_{ni}, 1\le i\le n, n\ge 1\}$ is extended from $a_{ni}\equiv 1$ to $\sum_{i=1}^{n}|a_{ni}|^{q}\ll n$;

(ii) the random variables are extended to double-indexed;

(iii) for the case $p>1$, the exponent r is improved from $r\equiv 1$ to any $0<r<p$;

(iv) Ko [13] mainly adopted the formula $E|X|=\int_{0}^{\infty} P(|X|>t)dt$ and the truncation method, by which the result in the case $0<p\le 1$ can hardly be obtained. However, in this paper, we establish a general moment inequality (see Lemma 2.2), and by utilizing the inequality, we easily obtain a much more general result only by some standard discussions.

Taking $p=2l$, $\alpha=1/l$ for $1\le l<2$ in Theorem 3.1, we have the following result.

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COROLLARY 3.1: *Let $1 \le l < 2$. Let $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of mean zero rowwise NA random variables satisfying a weak weighted mean dominating condition with weighted mean dominating random variable X and $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants satisfying $\sum_{i=1}^{n} |a_{ni}|^q \ll n$ for some $q > 2l/(2-l)$. Then $E|X|^{2l} < \infty$ implies that for any $0 < r < 2l$ and for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{-r/l} E \left(\max_{1 \le k \le n} \left| \sum_{i=1}^{k} a_{ni} X_{ni} \right| - \varepsilon n^{1/l} \right)_+^r < \infty,$$

and thus

$$\sum_{n=1}^{\infty} P \left(\max_{1 \le k \le n} \left| \sum_{i=1}^{k} a_{ni} X_{ni} \right| > \varepsilon n^{1/l} \right) < \infty.$$

The result for the case $\alpha p = 1$ is also established as follows.

THEOREM 3.2: *Let $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of rowwise NA random variables satisfying a weak weighted mean dominating condition with weighted mean dominating random variable X such that $E|X|^p < \infty$ for some $0 < p < 2$. Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants satisfying $\sum_{i=1}^{n} |a_{ni}|^q \ll n$ for some $q > p$. Assume further that $EX_{ni} = 0$ for $1 \le p < 2$. Then for any $0 < r < p$ and for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{-1-r/p} E \left(\max_{1 \le k \le n} \left| \sum_{i=1}^{k} a_{ni} X_{ni} \right| - \varepsilon n^{1/p} \right)_+^r < \infty, \tag{3.5}$$

and thus

$$\sum_{n=1}^{\infty} n^{-1} P \left(\max_{1 \le k \le n} \left| \sum_{i=1}^{k} a_{ni} X_{ni} \right| > \varepsilon n^{1/p} \right) < \infty. \tag{3.6}$$

PROOF: We state that (3.4) also holds true for $\alpha p = 1$ by the dominated convergence theorem and the rest proof is completely the same as Cases 1 and 2 in the proof of Theorem 3.1. The details are omitted. ■

For random variables $\{X_n, n \ge 1\}$ and constants $\{a_n, n \ge 1\}$, we have the following weighted version of Marcinkiewicz–Zygmund-type SLLN for NA random variables.

COROLLARY 3.2: *Let $\{X_n, n \ge 1\}$ be a sequence of NA random variables satisfying a weak weighted mean dominating condition with weighted mean dominating random variable X such that $E|X|^p < \infty$ for some $0 < p < 2$. Let $\{a_n, n \ge 1\}$ be a sequence of constants satisfying $\sum_{i=1}^{n} |a_i|^q \ll n$ for some $q > p$. Assume further that $EX_n = 0$ for $1 \le p < 2$. Then*

$$\frac{\sum_{i=1}^{n} a_i X_i}{n^{1/p}} \to 0 \ a.s., \ n \to \infty. \tag{3.7}$$

PROOF: Taking $a_{ni} = a_i$ and $X_{ni} = X_i$ for each $1 \le i \le n, n \ge 1$ in Theorem 3.2, we have that for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P \left(\max_{1 \le k \le n} \left| \sum_{i=1}^{k} a_i X_i \right| > \varepsilon n^{1/p} \right) < \infty,$$

which yields (3.7) by standard method. ■

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Taking $W_{ni} = 1/n$, $a_{ni} \equiv 1$ and $X_{ni} = X_i$ for each $1 \le i \le n, n \ge 1$ in Theorems 3.1 and 3.2, we can obtain the following result.

THEOREM 3.3: *Let $\alpha p \ge 1$ and $\alpha > 1/2$. Let $\{X_i, i \ge 1\}$ be a sequence of NA random variables satisfying a weak mean dominating condition with mean dominating random variable X. Assume further that $EX_i = 0$ for $p \ge 1$. Then*

$$E|X|^p < \infty \tag{3.8}$$

implies that for any $0 < r < p$ and for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha r} E \left(\max_{1 \le k \le n} \left| \sum_{i=1}^{k} X_i \right| - \varepsilon n^{\alpha} \right)_+^r < \infty, \tag{3.9}$$

and thus

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P \left(\max_{1 \le k \le n} \left| \sum_{i=1}^{k} X_i \right| > \varepsilon n^{\alpha} \right) < \infty. \tag{3.10}$$

Moreover, if there exists some constant $c_1 > 0$ such that $n^{-1} \sum_{i=1}^{n} P(|X_i| > x) \ge c_1 P(|X| > x)$ for any $x > 0$, then (3.8)–(3.10) are equivalent.

PROOF: The statement (3.8) $\Rightarrow$ (3.9) $\Rightarrow$ (3.10) follows immediately from Theorems 3.1 and 3.2, and similar to the proof in Kuczmaszewska [14], we have that (3.10) implies $\sum_{n=1}^{\infty} n^{\alpha p - 1} P(|X| > n^{\alpha}) < \infty$, which is equivalent to (3.8). The proof is completed. ■

Remark 3.2: Kuczmaszewska [14] only obtained that (3.8) and (3.10) are equivalent for the case $\alpha p > 1$. In Theorem 3.3, we not only consider the case $\alpha p = 1$, but also verify that (3.8) and (3.10) can be equivalent to a much stronger result (3.9). Consequently, our result markedly improve the corresponding one of Kuczmaszewska [14] for NA random variables and thus the corresponding ones for independent random variables.

4. AN APPLICATION IN NON-PARAMETRIC REGRESSION MODEL

4.1. Complete Consistency

Consider the following non-parametric regression model:

$$Y_{ni} = g(x_{ni}) + \epsilon_{ni}, \quad i = 1, 2, \ldots, n, n \ge 1, \tag{4.1}$$

where x_{ni} are known fixed design points from A where $A \subset \mathbb{R}^m$ is a given compact set for some $m \ge 1$, $g(\cdot)$ is an unknown regression function defined on A, and the ϵ_{ni} are random errors. As an estimator of $g(\cdot)$, we consider the following weighted regression estimator:

$$g_n(x) = \sum_{i=1}^{n} \omega_{ni}(x) Y_{ni}, \quad x \in A \subset \mathbb{R}^m, \tag{4.2}$$

where $\omega_{ni}(x) = \omega_{ni}(x; x_{n1}, x_{n2}, \ldots, x_{nn}), i = 1, 2, \ldots, n$ are weight functions.

The above estimator with constant weight was first proposed by Stone [24], adapted by Georgiev [6] and then constantly studied by many authors. One can refer to Roussas [18], Fan [5], Roussas, Tran, and Ioannides [19], Tran et al. [26], Hu et al. [9], Liang and Jing [16],

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Wang et al. [29], Wang and Si [28], Shen, Zhang, and Volodin [21], Liang and Baek [15], Yang et al. [31], among others.

In this section, let $c(g)$ denote the set of continuity points of the function g on A. Let $\|x\|$ denote the Euclidean norm of $x \in \mathbb{R}^m$. For any fixed design point $x \in A$, the following assumptions on weight functions $\omega_{ni}(x)$ will be used:

$$(\mathrm{A}_1) \sum_{i=1}^{n} \omega_{ni}(x) \to 1 \text{ as } n \to \infty;$$

$$(\mathrm{A}_2) \sum_{i=1}^{n} |\omega_{ni}(x)| \le C < \infty \quad \text{for all } n;$$

$$(\mathrm{A}_3) \sum_{i=1}^{n} |\omega_{ni}(x)| \cdot |g(x_{ni}) - g(x)| I(\|x_{ni} - x\| > a) \to 0 \text{ as } n \to \infty \quad \text{for all } a > 0.$$

We can see that (A_1)–(A_3) are general assumptions. Wang et al. [29] pointed out that they are satisfied for the nearest neighbor weights. Based on the assumptions above, we present the following result on complete consistency of the non-parametric regression estimator $g_n(x)$.

THEOREM 4.1: *Let $1 \le l < 2$, $\{\epsilon_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of rowwise NA random errors with mean zero which is stochastically dominated by a random variable X with*

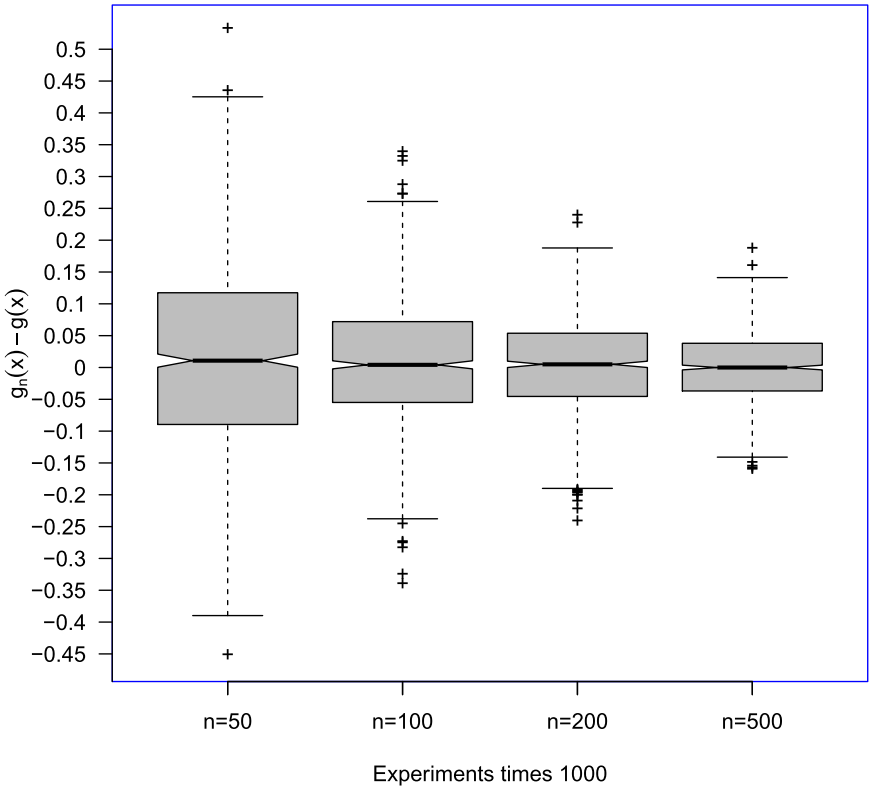


FIGURE 1. Boxplots of $g_n(x) - g(x)$ with $x = 0.25$ and $g(x) = x^2$.

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$E|X|^{2l} < \infty$. *Suppose that the conditions* (A_1)–(A_3) *hold, and*

$$\sum_{i=1}^{n} |\omega_{ni}(x)|^q \ll n^{1-q/l}, \quad \text{for some } q > 2l/(2-l). \tag{4.3}$$

Then for all $x \in c(g)$,

$$g_n(x) \to g(x) \text{ completely.} \tag{4.4}$$

PROOF: For $a > 0$ and $x \in c(g)$, we obtain from (4.1) and (4.2) that

$$\begin{aligned} |Eg_n(x) - g(x)| \le & \sum_{i=1}^{n} |\omega_{ni}(x)| \cdot |g(x_{ni}) - g(x)| I(\|x_{ni} - x\| \le a) \\ & + \sum_{i=1}^{n} |\omega_{ni}(x)| \cdot |g(x_{ni}) - g(x)| I(\|x_{ni} - x\| > a) \\ & + |g(x)| \cdot \left| \sum_{i=1}^{n} \omega_{ni}(x) - 1 \right|. \end{aligned} \tag{4.5}$$

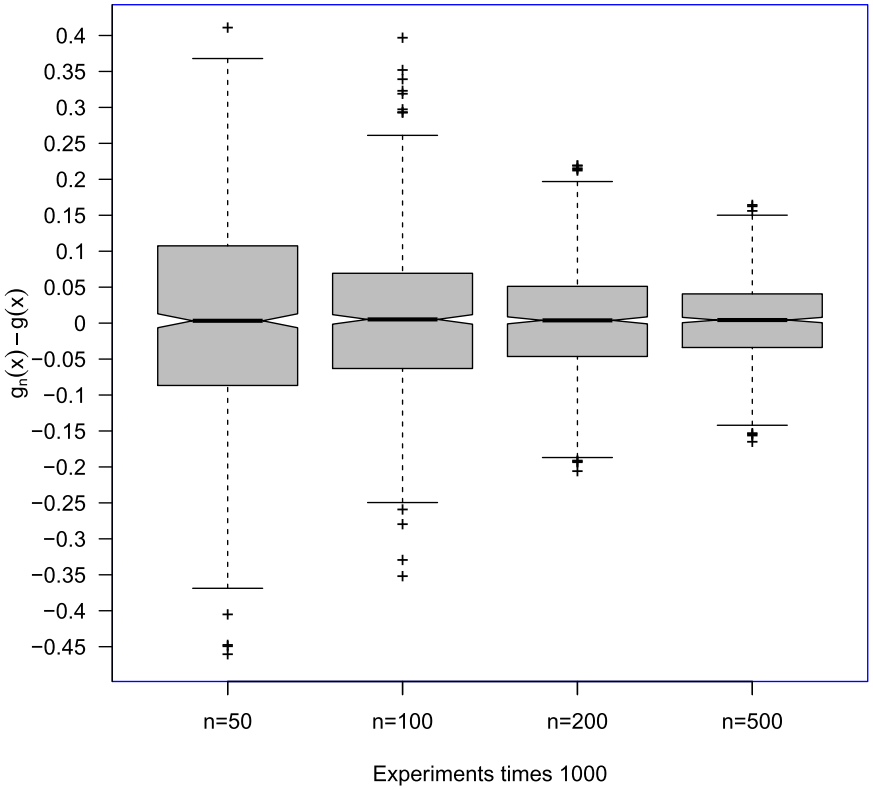


FIGURE **2.** Boxplots of $g_n(x) - g(x)$ with $x = 0.5$ and $g(x) = x^2$.

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It follows from $x \in c(g)$ that for all $\epsilon > 0$, there exists a constant $\delta > 0$ such that for all x' satisfying $\|x' - x\| < \delta$, $|g(x') - g(x)| < \epsilon$. Hence we take $0 < a < \delta$ in (4.5) to obtain

$$|Eg_n(x) - g(x)| \le \epsilon \sum_{i=1}^{n} |\omega_{ni}(x)| + \sum_{i=1}^{n} |\omega_{ni}(x)| \cdot |g(x_{ni}) - g(x)| I(\|x_{ni} - x\| > a)$$
$$+ |g(x)| \cdot \left| \sum_{i=1}^{n} \omega_{ni}(x) - 1 \right|.$$

Therefore by assumptions (A_1)–(A_3) and the arbitrariness of $\epsilon > 0$, we have that for all $x \in c(g)$,

$$\lim_{n\to\infty} Eg_n(x) = g(x). \tag{4.6}$$

In view of (4.6), to prove (4.4), it suffices to show

$$g_n(x) - Eg_n(x) = \sum_{i=1}^{n} \omega_{ni}(x)\epsilon_{ni} \to 0 \text{ completely.}$$

In other words, for all $\varepsilon > 0$, we need to verify that

$$\sum_{n=1}^{\infty} P\left(\left| \sum_{i=1}^{n} \omega_{ni}(x)\epsilon_{ni} \right| > \epsilon \right) < \infty. \tag{4.7}$$

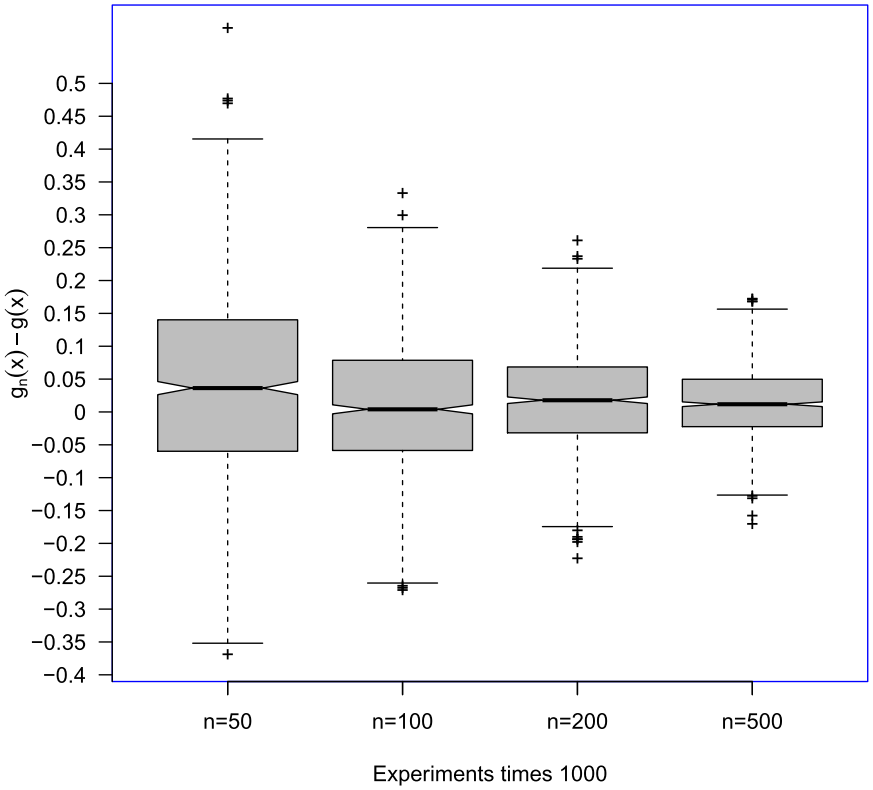


FIGURE 3. Boxplots of $g_n(x) - g(x)$ with $x = 0.75$ and $g(x) = x^2$.

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Applying Corollary 3.1 with $X_{ni} = \epsilon_{ni}$, $a_{ni} = \omega_{ni}(x)n^{1/l}$, we immediately obtain the desired result (4.7). The proof is completed. ■

4.2. Numerical Simulation

In this subsection, we will present a simulation to study the numerical performance of the consistency for the nearest neighbor weight function estimators $g_n(x)$ in non-parametric regression model. First let us recall the concept of the nearest neighbor weight function as follows.

Put $A = [0, 1]$ and let $x_{ni} = i/n$, $i = 1, 2, \ldots, n$. For any $x \in A$, we rewrite

$$|x_{n1} - x|, |x_{n2} - x|, \ldots, |x_{nn} - x|$$

as follows:

$$\left|x_{n,R_1(x)} - x\right| \leq \left|x_{n,R_2(x)} - x\right| \leq \cdots \leq \left|x_{n,R_n(x)} - x\right|,$$

if $|x_{ni} - x| = |x_{nj} - x|$, then $|x_{ni} - x|$ is located before $|x_{nj} - x|$ when $x_{ni} < x_{nj}$.

Let $1 \leq k_n \leq n$, the nearest neighbor weight function is defined as follows:

$$\omega_{ni}(x) = \begin{cases} 1/k_n, & \text{if } |x_{ni} - x| \leq \left|x_{n,R_{k_n}(x)} - x\right|, \\ 0, & \text{otherwise.} \end{cases}$$

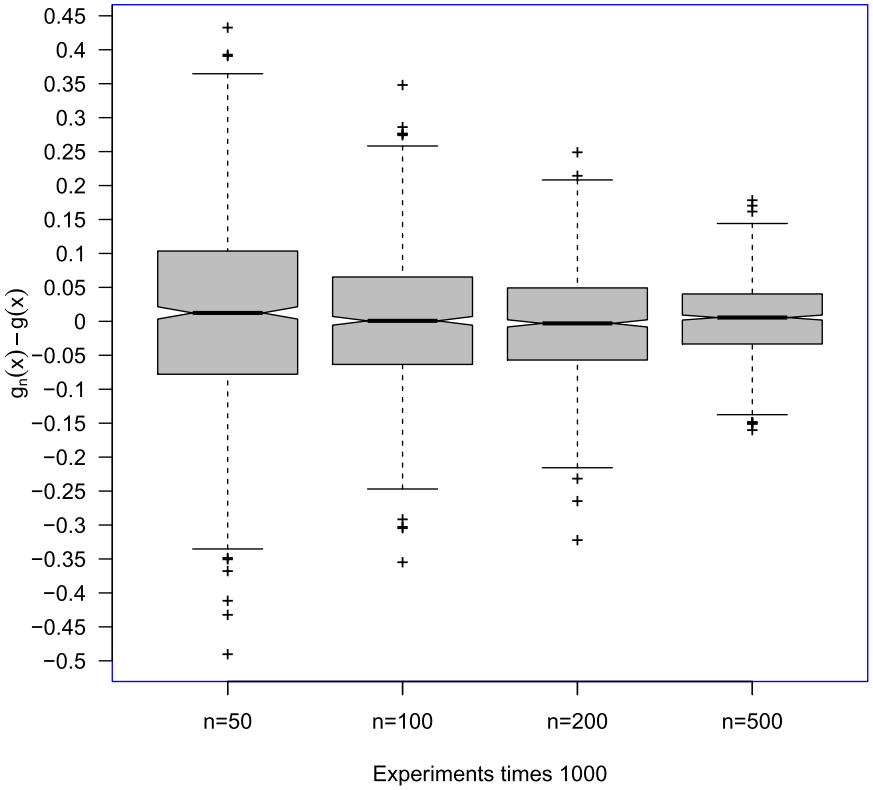


FIGURE 4. Boxplots of $g_n(x) - g(x)$ with $x = 0.25$ and $g(x) = \sin x$.

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The data are generated from model (4.1). For any fixed $n \geq 3$, let normal random vector $(\epsilon_{n1}, \epsilon_{n2}, \ldots, \epsilon_{nn}) \sim N_n(\mathbf{0}, \mathbf{\Sigma})$, where $\mathbf{0}$ represents zero vector and

$$\mathbf{\Sigma} = \begin{pmatrix} 1+\theta^2 & -\theta & 0 & \cdots & 0 & 0 & 0 \\ -\theta & 1+\theta^2 & -\theta & \cdots & 0 & 0 & 0 \\ 0 & -\theta & 1+\theta^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\theta^2 & -\theta & 0 \\ 0 & 0 & 0 & \cdots & -\theta & 1+\theta^2 & -\theta \\ 0 & 0 & 0 & \cdots & 0 & -\theta & 1+\theta^2 \end{pmatrix}_{n\times n},$$

where $0 < \theta < 1$. By Joag-Dev and Proschan [11], it can be seen that $(\epsilon_{n1}, \epsilon_{n2}, \ldots, \epsilon_{nn})$ is a NA vector for each $n \geq 3$ with finite moment of any order. We choose casually that $\theta = 0.6$, $k_n = \lfloor n^{2/3} \rfloor$, $l = 3/2$ and $q = 8$ in Theorem 4.1. As is stated in Wang et al. [29], conditions (A_1)–(A_3) hold true and it is easy to check that (4.3) is satisfied. Taking the points $x = 0.25, 0.5, 0.75$ and the sample sizes n as $n = 50, 100, 200, 500$, respectively, we use R software to compute $g_n(x) - g(x)$ with $g(x) = x^2$ and $g(x) = \sin x$ for 1000 times and obtain the Boxplots of $g_n(x) - g(x)$ in Figures 1–6 and the Mean Square Error (MSE) of $g_n(x)$ in Table 1.

Figures 1–3 are the boxplots of $g_n(x) - g(x)$ for $g(x) = x^2$ and Figures 4–6 are the boxplots of $g_n(x) - g(x)$ for $g(x) = \sin x$ with $x = 0.25, 0.5, 0.75$, respectively. We can see that no matter $g(x) = x^2$ or $g(x) = \sin x$, for $x = 0.25, 0.5, 0.75$, the differences $g_n(x) - g(x)$

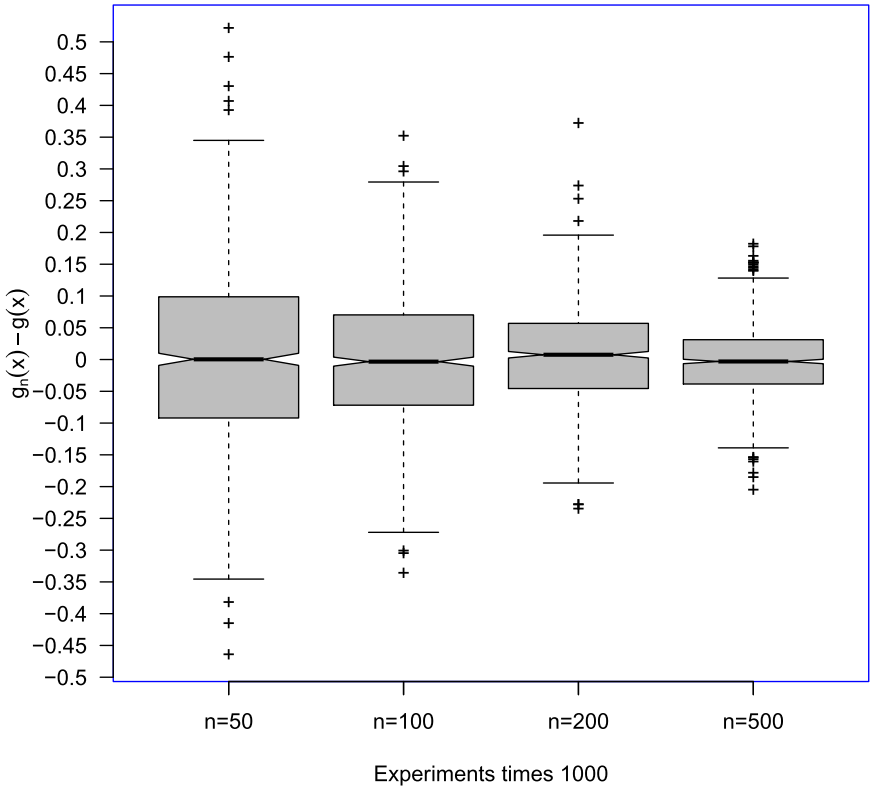


FIGURE 5. Boxplots of $g_n(x) - g(x)$ with $x = 0.5$ and $g(x) = \sin x$.

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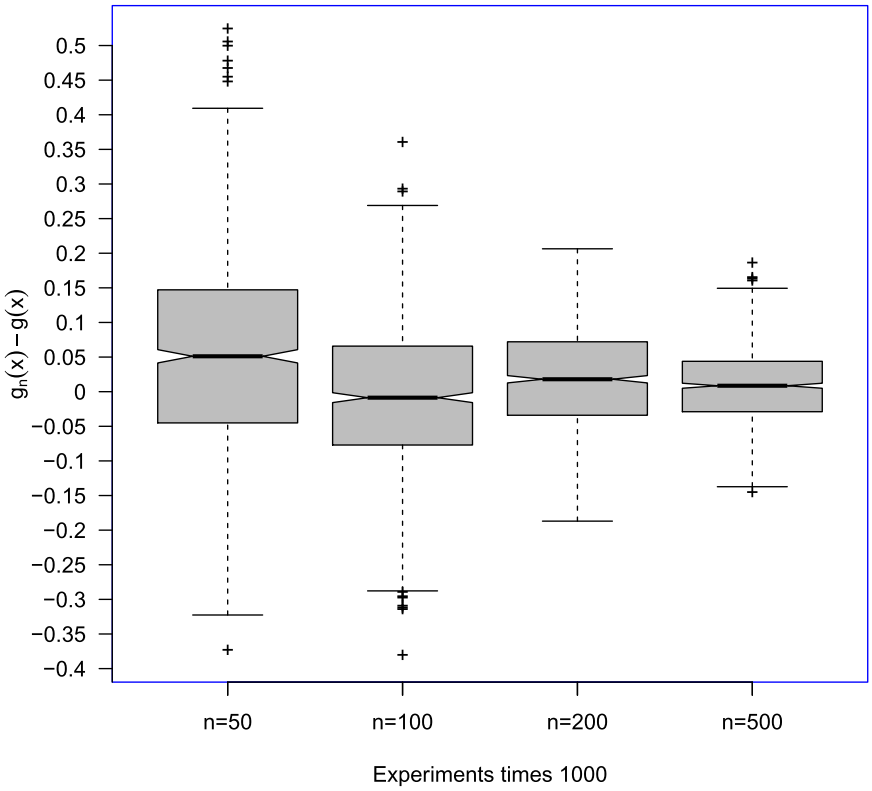


FIGURE **6.** Boxplots of $g_n(x) - g(x)$ with $x = 0.75$ and $g(x) = \sin x$.

TABLE 1. MSE of the estimator $g_n(x)$

$g(x)$	x	$n = 50$	$n = 100$	$n = 200$	$n = 500$
x^2	0.25	0.02196728	0.009856907	0.005745569	0.003043248
	0.5	0.01990434	0.01051369	0.005580355	0.002881039
	0.75	0.02290156	0.01001262	0.005982786	0.003090765
$\sin x$	0.25	0.01857444	0.009600902	0.006102704	0.002858548
	0.5	0.01886127	0.01081276	0.005891271	0.00307061
	0.75	0.0234246	0.01083584	0.005870763	0.002982542

fluctuate to zero and the variation range decreases markedly as the sample n increases. Table 1 reflects precisely that the MSE of $g_n(x)$ decreases markedly as n increases. These show a good fit of our result.

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