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Necessary and sufficient conditions for stable synchronization in random dynamical systems

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Abstract. For a composition of independent and identically distributed random maps or a memoryless stochastic flow on a compact space X, we find conditions under which the presence of locally asymptotically stable trajectories (e.g. as given by negative Lyapunov exponents) implies almost-sure mutual convergence of any given pair of trajectories ('synchronization'). Namely, we find that synchronization occurs and is 'stable' if and only if the system exhibits the following properties: (i) there is a *smallest* non-empty invariant set $K \subset X$; (ii) any two points in K are capable of being moved closer together; and (iii) K admits asymptotically stable trajectories.

1. *Introduction*

The phenomenon that different processes can be made to synchronize with each other through common exposure to the same external random influences was first described in the 1980s†. This phenomenon is called *noise-induced synchronization* or *synchronization by noise*, where the 'noise' refers to the external random influences affecting the processes. Noise-induced synchronization has, since then, been observed and studied numerous times (see e.g. [**TMHP01**] and the references therein).

In this phenomenon, one can regard the different processes as different trajectories of one *random dynamical system*, starting at different initial conditions but evolving under the same realization of the noise.

The natural question is to find conditions under which a random dynamical system (RDS) exhibits synchronization of trajectories. As we shall detail soon, there have been several results giving sufficient conditions for some synchronization phenomena

† See [**Pik84**], which considers an array of identical non-interacting self-oscillators simultaneously subjected to a series of random sharp impulses occurring at random times, and [**Ant84**], which considers synchronization of cyclic phenomena affected simultaneously by a random process. (The results in [**Ant84**] have been generalized in [**Mal14**].)

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in RDSs (mostly specific to certain settings). However, our present result is the first result that, within a very broad framework, gives *necessary and sufficient* conditions for some significant synchronization phenomenon in RDSs. Moreover, our conditions are remarkably simple, and somewhat more straightforward to verify than some of those given in previous literature.

We consider a RDS in either discrete or continuous time, under the following assumptions.

(a) The noise is strictly stationary and statistically memoryless.
(b) The phase space of the RDS is a compact metric space.
(c) For each realization of the noise, the RDS has joint continuous dependence on the initial condition and càdlàg dependence on time†.

Under these assumptions, we find necessary and sufficient conditions for the RDS to be *(globally) stably synchronizing*, by which we mean that the following two statements hold.

- Given any finite set of initial conditions in the phase space, almost every realization of the noise will cause the subsequent trajectories to synchronize as time tends to infinity.
- Given any initial condition in the phase space, under almost every realization of the noise the subsequent trajectory will be asymptotically stable‡.

Our main result (Theorem 4.2.1 of this paper) is as follows.

Under the assumptions (a)–(c) above, a RDS is stably synchronizing if and only if the following all hold.

(i) *There is a unique minimal element K of the set of non-empty invariant subsets of the phase space*§.
(ii) *Given any two distinct initial conditions in K, the set of noise realizations under which the distance between the subsequent trajectories will become (at some point in time) less than the original distance between the two initial conditions has strictly positive probability.*
(iii) *The set of noise realizations under which there exists an asymptotically stable trajectory starting in K has strictly positive probability.*

To verify condition (i), it is sufficient to show that the Markov transition probabilities admit a unique stationary distribution—or just to find a stationary distribution which is carried by every non-empty invariant set. (We then take K to be the support of this stationary distribution.) Condition (iii) is typically verified in practice by considering *maximal Lyapunov exponents* (either as a statistical average on a finite-time scale, or as an almost-sure asymptotic limit); see Remark 3.2.13 for details.

The very basic idea of the proof is as follows: due to recurrence of trajectories within K, property (iii) implies that for each $x \in K$, with *full* probability the trajectory starting at x is asymptotically stable. Property (ii) implies that any pair of trajectories in K can come arbitrarily close together; and consequently, given enough opportunity, a given pair

† See conditions (d) and (e) in §2.1 for the precise meaning of this.
‡ We say that a trajectory $(x_t)_{t\geq 0} = (\varphi_{0,t}(x_0))_{t\geq 0}$ of a dynamical system $(\varphi_{s,t})_{t\geq s\geq 0}$ is *asymptotically stable* if there exists a neighbourhood U of the initial condition x_0 such that the diameter of $\varphi_{0,t}(U)$ tends to 0 as $t \to \infty$.
§ Here, a subset G of the phase space is said to be *invariant* if G is closed and, for almost every realization of the noise, all trajectories starting in G will never escape G.

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of trajectories in K *will* come arbitrarily close together. (This is based on Lemma 2.2.6, applied to the two-point motion.) Combining this with asymptotic stability of trajectories in K yields that a given pair of trajectories in K will synchronize. The fact that K is the *only* minimal set (property (i)) enables us to extend this from pairs of trajectories in K to pairs of trajectories in X.

Let us briefly now set our present contribution within the context of previous work.

It has long been appreciated that negativity of Lyapunov exponents usually implies synchronization of *sufficiently nearby* trajectories. But concerning the ability to pass from such 'local-scale' synchronous behaviour to some kind of 'larger-scale' synchronous behaviour, the first significant result seems to be that provided in [**LeJan87**]†. There it was shown that, given an ergodic distribution ρ of the Markov transition probabilities, if ρ-almost all initial conditions have almost surely asymptotically stable trajectories, then there exist noise-dependent open subsets $V_1(\omega), \ldots, V_n(\omega)$ of the phase space (where n is *independent* of the noise) such that the following all hold.

- $V_1(\omega), \ldots, V_n(\omega)$ are mutually disjoint, with $\rho(V_i(\omega)) = 1/n$ for each $i = 1, \ldots, n$.
- With full probability, for each $i \in \{1, \ldots, n\}$, all trajectories starting in $V_i(\omega)$ synchronize with each other.
- With full probability, for each $i, j \in \{1, \ldots, n\}$ with $i \neq j$, no trajectory starting in $V_i(\omega)$ synchronizes with any trajectory starting in $V_j(\omega)$.

If $n = 1$, then we may say that the system is *almost stably synchronizing (according to ρ)*. As an example, [**LeJan87**] considers a RDS generated by random conjugations of a circle diffeomorphism f, and relates n to the periodicity of f. In particular, if f has no non-trivial periodicity, then this RDS is almost stably synchronizing; in §5 of this present paper, we will use our main result to extend this fact to the case of a *non-invertible* circle map f of degree 1 (and also to show that we have *global* stable synchronization, rather than just *almost* stable synchronization).

[**Bax91**] specifically considers RDSs generated by an autonomous Wiener-driven stochastic differential equation (SDE) on a compact manifold. It is shown that under some rather involved non-degeneracy conditions on the vector fields of the equation, if condition (ii) of our main result is satisfied by *all* pairs of distinct initial conditions on the manifold, then both the following hold.

(I) Strict negativity of the Lyapunov spectrum implies almost sure synchronization of the trajectories of any given finite set of initial conditions.

(II) The maximal Lyapunov exponent being identically 0 implies a certain kind of global-scale *synchronization in probability*.

Our main result allows the non-degeneracy condition required for (I) to be replaced with the simpler requirement that there is a unique stationary distribution. (For this, Hörmander's condition on the original SDE is sufficient, whereas Baxendale's non-degeneracy conditions involve Hörmander's condition on a more complicated 'lifted' version of the SDE.)

Sufficient conditions for a similar 'synchronization in probability' phenomenon in more general RDSs on Polish spaces can be found in [**FGS16**]. On the basis of these conditions,

† [**FGS16**, Lemma 2.19(2)] provides an analogous result in terms of convergence in probability.

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[**FGS16**] shows that large classes of ordinary differential equations in Euclidean space exhibit synchronous behaviour after the addition of Gaussian white noise to the right-hand side.

It is known that synchronization is particularly common in order-/orientation-preserving RDSs on a one-dimensional phase space (a line or a circle). See, for example, [**CF98**] (which considers monotone RDSs on $\mathbb{R}$); [**Mal14**] (which considers discrete-time orientation-preserving RDSs on the circle, as well as invertible monotone RDSs on a bounded interval); or [**Crau02**, §4] (which considers continuous-time Brownian flows on the circle). Synchronization in higher-dimensional monotone RDSs has also been considered in [**CS04**] and further developed in [**FGS16a**]; for an application, see [**CCK07**] (in particular, Proposition 5.6).

Our paper is structured as follows. In §2, we introduce all the essential notions of memoryless noise, random dynamical systems†, and invariant sets. In §3, we introduce the main ideas relating to synchronization of trajectories and asymptotic stability. In §4, we formulate and prove our main result (Theorem 4.2.1). In §5, we illustrate our result by applying it to an extended version of the example of [**LeJan87**] described above.

2. *Preliminaries*

2.1. *The basic setup.*

Let $\mathbb{T}^+$ denote either $\mathbb{N} \cup \{0\}$ or $[0, \infty)$. Let (X, d) be a compact metric space.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}^+}, \mathbb{P})$ be a filtered probability space, and write $\mathcal{F}_\infty := \sigma(\mathcal{F}_t : t \in \mathbb{T}^+)$. Let $(\theta^t)_{t\in\mathbb{T}^+}$ be a $\mathbb{P}$-preserving semigroup of $(\mathcal{F}, \mathcal{F})$-measurable functions $\theta^t : \Omega \to \Omega$ with $\theta^0 = \mathrm{id}_\Omega$, such that:

(i) θ^t is $(\mathcal{F}_{s+t}, \mathcal{F}_s)$-measurable (i.e. $\theta^{-t}\mathcal{F}_s \subset \mathcal{F}_{s+t}$) for all $s, t \in \mathbb{T}^+$;

(ii) $\mathbb{P}(E \cap \theta^{-t}(F)) = \mathbb{P}(E)\mathbb{P}(F)$ for all $E \in \mathcal{F}_t$ and $F \in \mathcal{F}_\infty$.

(Note that in (ii), we use the convention $\theta^{-t}(F) := (\theta^t)^{-1}(F)$.) It is not hard to show that in this case, for every $s \in \mathbb{T}^+ \setminus \{0\}$, θ^s is an ergodic measure-preserving transformation of the probability space $(\Omega, \mathcal{F}_\infty, \mathbb{P}|_{\mathcal{F}_\infty})$.

Remark 2.1.1. $\theta^t\omega$ represents a *time-shift* of the noise realization ω forward by time t. The fact that $\mathbb{P}$ is invariant under (θ^t) represents the fact that the noise is strictly stationary, and condition (ii) represents the fact that the noise is memoryless.

Now suppose we have a $(\mathbb{T}^+ \times \Omega)$-indexed family $\varphi = (\varphi(t, \omega))_{t\in\mathbb{T}^+,\omega\in\Omega}$ of continuous functions $\varphi(t, \omega) : X \to X$ such that the following all hold.

(a) The map $(\omega, x) \mapsto \varphi(t, \omega)x$ is $(\mathcal{F}_t \otimes \mathcal{B}(X))$-measurable for each $t \in \mathbb{T}^+$.

(b) $\varphi(0, \omega)$ is the identity function for all $\omega \in \Omega$.

(c) $\varphi(s + t, \omega) = \varphi(t, \theta^s\omega) \circ \varphi(s, \omega)$ for all $s, t \in \mathbb{T}^+$ and $\omega \in \Omega$.

(d) For any decreasing sequence (t_n) in $\mathbb{T}^+$ converging to a value t, and any sequence (x_n) in X converging to a point x, $\varphi(t_n, \omega)x_n \to \varphi(t, \omega)x$ as $n \to \infty$ for all $\omega \in \Omega$.

(e) There exists a function $\varphi_- : \mathbb{T}^+ \times \Omega \times X \to X$ such that for any strictly increasing sequence (t_n) in $\mathbb{T}^+$ converging to a value t, and any sequence (x_n) in X converging to a point x, $\varphi(t_n, \omega)x_n \to \varphi_-(t, \omega, x)$ as $n \to \infty$ for all $\omega \in \Omega$.

We refer to φ as a *random dynamical system* on X.

† We work with 'filtered RDSs', roughly as in [**Arn98**, §2.3].

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Remark 2.1.2. Conditions (d) and (e) constitute the 'càdlàg' property, with (d) being right-continuity and (e) being left limits.

Now it is easy to show that for any $x \in X$, the stochastic process $(\varphi(t, \cdot)x)_{t\in\mathbb{T}^+}$ is an $(\mathcal{F}_t)$-adapted Markov process; we denote the associated transition probabilities by

$$\varphi_x^t(A) = \mathbb{P}(\omega : \varphi(t, \omega)x \in A) = \mathbb{P}(\omega : \varphi(t, \theta^s\omega)x \in A) \quad \text{(for any } s\text{)}$$

where $x \in X$, $t \in \mathbb{T}^+$ and $A \in \mathcal{B}(X)$. By the dominated convergence theorem, the map $(t, x) \mapsto \varphi_x^t$ is jointly continuous in x and right-continuous in t (where we regard the set of Borel probability measures on X as being equipped with the topology of weak convergence).

For any $t \in \mathbb{T}^+$ we define the map $\Theta^t : \Omega \times X \to \Omega \times X$ by

$$\Theta^t(\omega, x) = (\theta^t\omega, \varphi(t, \omega)x).$$

Note that $(\Theta^t)_{t\in\mathbb{T}^+}$ forms a semigroup of measurable transformations of the measurable space $(\Omega \times X, \mathcal{F}_\infty \otimes \mathcal{B}(X))$. Moreover, for any Borel probability measure ρ on X, we have the following.

(i) ρ is a stationary distribution of the Markov transition probabilities $(\varphi_x^t)_{x\in X, t\in\mathbb{T}^+}$ if and only if $\mathbb{P}|_{\mathcal{F}_\infty} \otimes \rho$ is an invariant measure of the semigroup $(\Theta^t)_{t\in\mathbb{T}^+}$.

(ii) ρ is a ergodic stationary distribution of the Markov transition probabilities $(\varphi_x^t)_{x\in X, t\in\mathbb{T}^+}$ if and only if $\mathbb{P}|_{\mathcal{F}_\infty} \otimes \rho$ is an ergodic invariant measure of the semigroup $(\Theta^t)_{t\in\mathbb{T}^+}$.

(For a proof, see [**New15**, Theorem 143], [**Kif86**, Lemma 1.2.3, Theorem 1.2.1] or [**Oh83**, Lemma 2.1, Theorem 3.1].)

Finally, we define the *two-point motion* $\varphi \times \varphi = (\varphi \times \varphi(t, \omega))_{t\in\mathbb{T}^+, \omega\in\Omega}$ to be the $(\mathbb{T}^+ \times \Omega)$-indexed family of functions $\varphi \times \varphi(t, \omega) : X \times X \to X \times X$ given by

$$\varphi \times \varphi(t, \omega)(x, y) = (\varphi(t, \omega)x, \varphi(t, \omega)y).$$

2.2. *Invariance of sets.* We introduce the concept of invariant sets and minimal sets. This is important for the statement of our main result, and Lemma 2.2.6 will be the key ingredient in the proof.

Definition 2.2.1. Given a point $x \in X$ and an open set $U \subset X$, we will say that U is *accessible from* x (*under* φ) if

$$\mathbb{P}(\omega : \exists t \in \mathbb{T}^+ \text{ s.t. } \varphi(t, \omega)x \in U) > 0.$$

Due to right-continuity of trajectories of φ, U is accessible from x if and only if there exists $t \in \mathbb{T}^+$ such that $\varphi_x^t(U) > 0$. Now given a point $x \in X$, we have (by the second-countability of X) that an arbitrary union of open sets that are not accessible from x is itself not accessible from x. So let U_x denote the largest open set that is not accessible from x.

Definition 2.2.2. We will say that a closed set $K \subset X$ is *forward-invariant* (*under* φ) if

$$\mathbb{P}(\omega : \forall t \in \mathbb{T}^+, \varphi(t, \omega)K \subset K) = 1;$$

and we will say that an open set $U \subset X$ is *backward-invariant* if $X \setminus U$ is forward-invariant.

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Due to the continuity properties of φ, K is forward-invariant if and only if $\varphi_x^t(K) = 1$ for all $x \in K$ and $t \in \mathbb{T}^+$. In particular, it follows that K is forward-invariant if and only if for all $x \in K$, $X \setminus K$ is not accessible from x.

Note that for any stationary distribution ρ of the family of transition probabilities (φ_x^t), the support of ρ is forward-invariant. In the converse direction: by the *Krylov–Bogolyubov theorem* (see e.g. [**Kif86**, Lemma 5.2.1]), if K is a non-empty forward-invariant set, then there exists an ergodic stationary distribution ρ of (φ_x^t) such that $\rho(K) = 1$.

By the second-countability of X, an arbitrary intersection of closed forward-invariant sets is itself forward-invariant. (In fact, the set of open backward-invariant sets forms a topology on X.) So, for any $x \in X$, let G_x be the smallest closed forward-invariant set containing x. It is obvious that $X \setminus G_x$ is *contained* in U_x; but due to continuity, we actually have *equality*.

LEMMA 2.2.3. *For any $x \in X$, $X \setminus G_x = U_x$.*

Proof. We need to show that $U_x \subset X \setminus G_x$, for which it is sufficient to show that U_x is backward-invariant. So fix any $y \in X \setminus U_x$, and suppose for a contradiction that there exists $t \in \mathbb{T}^+$ such that $\varphi_y^t(U_x) > 0$. Since U_x is open, the map $\xi \mapsto \varphi_\xi^t(U_x)$ is lower semicontinuous, and so there exists a neighbourhood V of y such that $\varphi_\xi^t(U_x) > 0$ for all $\xi \in V$. Since $y \notin U_x$, V is accessible from x, so there exists $s \in \mathbb{T}^+$ such that $\varphi_x^s(V) > 0$. Hence

$$\varphi_x^{s+t}(U_x) = \int_X \varphi_\xi^t(U_x)\varphi_x^s\,(d\xi) \geq \int_V \varphi_\xi^t(U_x)\varphi_x^s\,(d\xi) > 0,$$

contradicting the fact that U_x is not accessible from x. □

Just as $G_x \subset X$ denotes the smallest forward-invariant closed set under φ containing x, so we will also write $G_{(x,y)} \subset X \times X$ to denote the smallest forward-invariant closed set under $\varphi \times \varphi$ containing (x, y).

LEMMA 2.2.4. *For any $x, y \in X$, the image of $G_{(x,y)}$ under the projection $(u, v) \mapsto u$ (respectively $(u, v) \mapsto v$) is G_x (respectively G_y).*

Proof. Let $A_{x,y} \subset X$ be the image of $G(x, y)$ under $(u, v) \mapsto u$. $A_{x,y}$ is a closed set, since $G_{(x,y)}$ is compact. Also $A_{x,y}$ is forward-invariant: for any $u \in A_{x,y}$, if we let v be such that $(u, v) \in G_{(x,y)}$, then $(X \setminus A_{x,y}) \times X$ is not accessible from (u, v), and so $X \setminus A_{x,y}$ is not accessible from u. Finally, if B is a closed proper subset of $A_{x,y}$ with $x \in B$, then B is not forward-invariant: $(X \setminus B) \times X$ is accessible from (x, y) by Lemma 2.2.3 (applied to $\varphi \times \varphi$), and so $X \setminus B$ is accessible from x. Thus we have shown that $A_{x,y}$ is the smallest closed forward-invariant set containing x, as required. □

Definition 2.2.5. We will say that a set $K \subset X$ is *minimal* (*under φ*), or that φ *has minimal dynamics on K*, if K is a non-empty closed forward-invariant set and the only closed forward-invariant subsets of K are K and $\emptyset$.

Note that for any non-empty closed $K \subset X$ the following are equivalent.

- K is minimal.
- K is a minimal element (with respect to set-inclusion) of the set of non-empty closed forward-invariant subsets of X.

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- For all $x \in K$, $G_x = K$.
- K is forward-invariant, and for any $x \in K$ and any open set $U \subset X$ with $U \cap K \neq \emptyset$, U is accessible from x.

Moreover, just as in the case of deterministic dynamical systems (see e.g. [**KH95**, Proposition 3.3.6]), every non-empty closed forward-invariant set contains a minimal set. It follows that if there is a unique minimal set $K \subset X$, then every non-empty closed forward-invariant set contains K.

LEMMA 2.2.6. *Suppose $K \subset X$ is a closed set possessing no non-empty closed forward-invariant subsets. Then for any $x \in X$, for $\mathbb{P}$-almost every $\omega \in \Omega$ there is an unbounded sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{T}^+$ such that for all $n \in \mathbb{N}$, $\varphi(t_n, \omega)x \notin K$.*

The proof of Lemma 2.2.6 is essentially the same as the proof of [**BS88**, Proposition 4.1]. We will use the following fact, which is sufficiently clear that we do not write out a proof, but is nonetheless worth stating explicitly.

LEMMA 2.2.7. *Let $(M_t)_{t\in\mathbb{T}^+}$ be an $(\mathcal{F}_t)$-adapted X-valued homogeneous Markov process with transition probabilities denoted by $P_x^t(A)$ (where $x \in X$, $t \in \mathbb{T}^+$ and $A \in \mathcal{B}(X)$). Fix $t \in \mathbb{T}^+$, let D be a countable subset of $\mathbb{T}^+$, and let $T : \Omega \to D$ be an $\mathcal{F}_t$-measurable function. Then*

$$\mathbb{E}[\mathbb{1}_A(M_{t+T})|\mathcal{F}_t] = P_{M_t}^T(A)$$

for all $A \in \mathcal{B}(X)$.

Proof of Lemma 2.2.6. Let $D := \mathbb{Q} \cap \mathbb{T}^+$. Fix $x \in X$ and let $M_t(\omega) := \varphi(t, \omega)x$ for all t and ω. For each $y \in K$, $G_y \setminus K$ is non-empty (otherwise G_y would be a non-empty closed forward-invariant subset of K) and so there exists $t \in \mathbb{T}^+$ such that $\varphi_y^t(K) < 1$†. Now the map $t \mapsto \varphi_y^t(K)$ is right upper semicontinuous for each $y \in X$; and so it follows that for each $y \in X$ there exists $t \in D$ such that $\varphi_y^t(K) < 1$. So, if we define a function $l : K \to [0, 1]$ by

$$l(y) := \inf_{t\in D} \varphi_y^t(K),$$

then l is strictly less than 1 on the whole of K. Also note that l is upper semicontinuous. Therefore l has a maximum value c', which is strictly less than 1. So, fixing an arbitrary value $c \in (c', 1)$, we have that for all $y \in K$ there exists $t \in D$ such that $\varphi_y^t(K) \leq c$; in fact, it is easy to see that one can construct a *measurable* function $\tau : K \to D$ such that $\varphi_y^{\tau(y)}(K) \leq c$ for all $y \in K$‡. We extend τ to the whole of X by setting $\tau(y) = 0$ for all $y \in X \setminus K$.

Now to obtain the desired result, it is sufficient just to show that for each $N \in \mathbb{N}$, for $\mathbb{P}$-almost every $\omega \in \Omega$ there exists $t \geq N$ such that $\varphi(t, \omega)x \notin K$. Fix any $N \in \mathbb{N}$, and define the sequence $(T_n)_{n\in\mathbb{N}\cup\{0\}}$ of functions $T_n : \Omega \to D$ by

$$T_0(\omega) = N,$$
$$T_n(\omega) = T_{n-1}(\omega) + \tau(M_{T_{n-1}(\omega)}(\omega)) \quad (n \geq 1)$$

† Since the map $y \mapsto \varphi_y^t(K)$ is upper semicontinuous and K is compact, one can show that t can be taken from a *bounded* interval $[0, T]$ (where T is independent of y). Consequently (as in [**BS88**, Proposition 4.1]) one can obtain a stronger result than stated in Lemma 2.2.6; however, we will not need this.

‡ For example, if $(s_n)_{n\in\mathbb{N}}$ is an enumeration of D, set $\tau(y) := s_{N(y)}$ where $N(y) := \min\{n \in \mathbb{N} : \varphi_y^{s_n}(K) < 1\}$.

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for all $\omega \in \Omega$. For each $n \in \mathbb{N} \cup \{0\}$, let $E_n := \{\omega \in \Omega : M_{T_r(\omega)}(\omega) \in K \text{ for all } 0 \le r \le n\}$. We will show by induction that for each $n \in \mathbb{N} \cup \{0\}$, $\mathbb{P}(E_n) \le c^n$. (Obviously, once we have shown this, we are done.) The $n = 0$ case is trivial. Now fix $m \in \mathbb{N}$ such that $\mathbb{P}(E_{m-1}) \le c^{m-1}$. It is not hard to show that $E_{m-1} \cap T_{m-1}^{-1}(\{t\}) \in \mathcal{F}_t$ for all $t \in D$; so then,

$$\begin{aligned}
\mathbb{P}(E_m) &= \int_{E_{m-1}} \mathbb{1}_K(M_{T_m(\omega)}(\omega))\mathbb{P}\,(d\omega) \\
&= \sum_{t\in D} \int_{E_{m-1}\cap T_{m-1}^{-1}(\{t\})} \mathbb{1}_K(M_{t+\tau(M_t(\omega))}(\omega))\mathbb{P}\,(d\omega) \\
&= \sum_{t\in D} \int_{E_{m-1}\cap T_{m-1}^{-1}(\{t\})} \varphi_{M_t(\omega)}^{\tau(M_t(\omega))}(K)\mathbb{P}\,(d\omega) \quad \text{(by Lemma 2.2.7)} \\
&\le \sum_{t\in D} \int_{E_{m-1}\cap T_{m-1}^{-1}(\{t\})} c\mathbb{P}\,(d\omega) \\
&= c\mathbb{P}(E_{m-1}) \\
&\le c^m.
\end{aligned}$$

So we are done. □

We have the following important corollary.

COROLLARY 2.2.8. *Let K be a minimal set, and let U be an open set with $U \cap K \neq \emptyset$. Then for each $x \in K$, for $\mathbb{P}$-almost every $\omega \in \Omega$ there exist (arbitrarily large) times $t \in \mathbb{T}^+$ such that $\varphi(t, \omega)x \in U$. Moreover, if K is the only minimal set, then for each $x \in X$, for $\mathbb{P}$-almost every $\omega \in \Omega$ there exist (arbitrarily large) times $t \in \mathbb{T}^+$ such that $\varphi(t, \omega)x \in U$.*

Proof. It is clear that $K \setminus U$ is a closed set possessing no non-empty closed forward-invariant subsets; hence Lemma 2.2.6 combined with the forward-invariance of K gives the first statement. If K is the only minimal set, then it is clear that $X \setminus U$ is a closed set possessing no non-empty closed forward-invariant subsets, so Lemma 2.2.6 gives the second statement. □

Note that the times in Lemma 2.2.6 and Corollary 2.2.8 can be selected to belong to $\mathbb{Q}$. (This follows from the fact that the trajectories of φ are right-continuous in time, but can also be seen directly within the proof of Lemma 2.2.6.)

3. *Synchronization and stability*

In this section, we introduce the basic notions of mutual convergence of trajectories that will be fundamental for stating both the hypotheses and the conclusion of our main result.

Given a sample point $\omega \in \Omega$ and a non-empty set $A \subset X$, we will say that A *contracts under* ω if $\operatorname{diam}(\varphi(t, \omega)A) \to 0$ as $t \to \infty$†. Given $\omega \in \Omega$ and points $x, y \in X$, we will say that x *and* y *mutually converge under* ω if $d(\varphi(t, \omega)x, \varphi(t, \omega)y) \to 0$ as $t \to \infty$. This is the same as saying that $\{x, y\}$ contracts under ω.

† Since X is compact, this does not depend on the metrization d of the topology of X.

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3.1. *Synchronization*

Definition 3.1.1. We say that φ is (*globally*) *synchronizing* if for every $x, y \in X$,

$$\mathbb{P}(\omega : x \text{ and } y \text{ mutually converge under } \omega) = 1.$$

The following is a generalization of [**KN04**, Corollary 2].

PROPOSITION 3.1.2. *If φ is synchronizing, then there is a unique stationary distribution of the Markov transition probabilities (φ_x^t).*

Proof. Fix a point $x \in X$, and let ρ be any stationary distribution of (φ_x^t); we will show that φ_x^t must converge (in the narrow topology) to ρ as $t \to \infty$, from which it follows that ρ is the only stationary distribution. Let $g : X \to \mathbb{R}$ be any continuous function. It is clear that for $(\mathbb{P} \otimes \rho)$-almost all $(\omega, y) \in \Omega \times X$, x and y mutually converge under ω, and so $g(\varphi(t, \omega)y) - g(\varphi(t, \omega)x) \to 0$ as $t \to \infty$. The dominated convergence theorem then gives that

$$\underbrace{\int_{\Omega \times X} g(\varphi(t, \omega)y)(\mathbb{P} \otimes \rho)\,(d(\omega, y))}_{ⓐ} - \underbrace{\int_{\Omega} g(\varphi(t, \omega)x)\mathbb{P}\,(d\omega)}_{ⓑ} \to 0 \quad \text{as } t \to \infty.$$

Observe, however, that

$$ⓐ = \int_X g(z)\rho\,(dz)$$

since $\mathbb{P}|_{\mathcal{F}_\infty} \otimes \rho$ is (Θ^t)-invariant, and that

$$ⓑ = \int_X g(z)\varphi_x^t\,(dz).$$

So we are done. □

Remark 3.1.3. If the phase space X were not compact, then global synchronization would imply that there is *at most* one stationary distribution. (In the above proof, we would take g to be a bounded uniformly continuous function.)

3.2. *Stability and stable synchronization.* We introduce a concept of local asymptotic stability, study sets admitting stable trajectories, and define stable synchronization.

Definition 3.2.1. For any non-empty open set $U \subset X$, let $E_U \subset \Omega$ be the set of sample points under which U contracts.

Now just as a trajectory of a deterministic dynamical system may be identified with its initial condition, so likewise we may identify a possible trajectory $(\varphi(t, \omega)x)_{t \in \mathbb{T}^+}$ of a *random* dynamical system φ with the pair (ω, x) consisting of the noise realization ω and the initial condition x.

Definition 3.2.2. We will say that a pair $(\omega, x) \in \Omega \times X$ is *asymptotically stable*† if there exists a neighbourhood U of x which contracts under ω.

† The more commonly given definition is that a trajectory of a dynamical system is *asymptotically stable* if it is stable in the sense of Lyapunov and its basin of attraction includes a neighbourhood of its initial condition; our definition is simpler, drastically easier to work with, and only very slightly stronger than the more commonly given definition. (This fact relies on the càdlàg properties of φ.)

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Let $O \subset \Omega \times X$ denote the set of all asymptotically stable pairs (ω, x); and for each $x \in X$, let $O_x := \{\omega : (\omega, x) \in O\}$ denote the x-section of O. It is clear that for any $x \in X$,

$$O_x = \bigcup_{n=1}^{\infty} E_{B_{1/n}(x)}$$

where $B_r(x)$ denotes the open ball of radius r about x.

LEMMA 3.2.3.

(a) *For any open $U \subset X$, E_U is $\mathcal{F}_\infty$-measurable.*

(b) *O is $(\mathcal{F}_\infty \otimes \mathcal{B}(X))$-measurable, and $\Theta^{-t}(O) \subset O$ for all $t \in \mathbb{T}^+$.*

Proof. (a) Fix an open set $U \subset X$. Let $S \subset U$ be a countable set that is dense in U, and let D be a countable dense subset of $\mathbb{T}^+$. Then we can write

$$E_U = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{t \in D \cap [j,\infty)} \bigcap_{x,y \in S} \left\{\omega \in \Omega : d(\varphi(t, \omega)x, \varphi(t, \omega)y) < \frac{1}{i}\right\}.$$

So $E_U \in \mathcal{F}_\infty$.

(b) Given a countable base $\mathcal{U}$ for the topology on X, one can easily check that $O = \bigcup_{V \in \mathcal{U}} E_V \times V$. Hence $O \in \mathcal{F}_\infty \otimes \mathcal{B}(X)$. Finally, for any $t \in \mathbb{T}^+$ and $(\omega, x) \in \Omega \times X$, if there is a neighbourhood of $\varphi(t, \omega)x$ which contracts under $\theta^t\omega$, then (due to the continuity of the map $\varphi(t, \omega)$) there is a neighbourhood of x which contracts under ω. Hence $\Theta^{-t}(O) \subset O$. □

Definition 3.2.4. For each $x \in X$, let $P_0(x) = \mathbb{P}(O_x)$ and let $P_r(x) = \mathbb{P}(E_{B_r(x)})$ for all $r > 0$.

It is clear that $P_r(x)$ is decreasing in r, with $P_0(x) = \sup_{r>0} P_r(x) = \lim_{r \to 0} P_r(x)$.

LEMMA 3.2.5. *For any $x \in X$ and $t \in \mathbb{T}^+$, $P_0(x) \geq \int_X P_0(y)\varphi_x^t(dy)$.*

Proof. Recall that $P_0(y) = \mathbb{P}(O_y) = \mathbb{P}(\theta^{-t}(O_y))$ for all y and t. Now fix any $x \in X$ and $t \in \mathbb{T}^+$.

$$\begin{aligned}
\int_X P_0(y)\varphi_x^t(dy) &= \int_\Omega P_0(\varphi(t, \omega)x)\mathbb{P}(d\omega) \\
&= \int_\Omega \mathbb{P}(\theta^{-t}(O_{\varphi(t,\omega)x}))\mathbb{P}(d\omega) \\
&= \int_\Omega \int_\Omega \mathbb{1}_O(\theta^t\tilde{\omega}, \varphi(t, \omega)x)\mathbb{P}(d\tilde{\omega})\mathbb{P}(d\omega) \\
&= \int_\Omega \mathbb{1}_O(\theta^t\omega, \varphi(t, \omega)x)\mathbb{P}(d\omega) \quad \text{(by the memoryless property)} \\
&\leq \int_\Omega \mathbb{1}_O(\omega, x)\mathbb{P}(d\omega) \quad \text{(since } O \text{ is } (\Theta^t)\text{-backward-invariant)} \\
&= P_0(x).
\end{aligned}$$

So we are done. □

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Definition 3.2.6. We will say that x *is almost surely stable* if $P_0(x) = 1$. We will say that x *is potentially stable* if $P_0(x) > 0$.

Note that x is potentially stable if and only if there is a neighbourhood U of x such that $\mathbb{P}(E_U) > 0$.

Remark 3.2.7. Given any closed $K \subset X$, if every point in K is almost surely stable, then (by the Lebesgue number lemma) $P_r(\cdot) \to 1$ uniformly on K as $r \to 0$.

LEMMA 3.2.8. *$\mathbb{P}$-almost every $\omega \in \Omega$ has the property that for any $x \in X$, if (ω, x) is asymptotically stable, then x is potentially stable.*

Proof. Let $\mathcal{U}$ be a countable base for the topology on X, and let

$$\mathcal{U}_0 := \{U \in \mathcal{U} : \mathbb{P}(E_U) = 0\}.$$

Let

$$\tilde{\Omega} := \Omega \setminus \bigcup_{U \in \mathcal{U}_0} E_U.$$

Now fix any $\omega \in \tilde{\Omega}$ and $x \in X$. If (ω, x) is asymptotically stable, then there exists $U \in \mathcal{U}$ with $x \in U$ such that $\omega \in E_U$, and hence $U \notin \mathcal{U}_0$; so $\mathbb{P}(E_U) > 0$ and therefore x is potentially stable. □

LEMMA 3.2.9. *The set $U_{ps} \subset X$ of potentially stable points is an open backward-invariant set.*

Proof. For any $x \in X$ and $r > 0$ with $P_r(x) > 0$, we clearly have that $B_r(x) \subset U_{ps}$. So U_{ps} is open. The backward-invariance of U_{ps} then follows from Lemma 3.2.5. □

We now go on to consider 'sets admitting stable trajectories'.

LEMMA 3.2.10. *For any $A \subset X$, the set*

$$O_A := \{\omega \in \Omega : \exists x \in A \text{ s.t. } (\omega, x) \text{ is asymptotically stable}\} = \bigcup_{x \in A} O_x$$

is $\mathcal{F}_\infty$-measurable.

Proof. Let $\mathcal{U}$ be a countable base for the topology on X, and let

$$\mathcal{U}_A := \{U \in \mathcal{U} : U \cap A \neq \emptyset\}.$$

It is clear that

$$O_A = \bigcup_{U \in \mathcal{U}_A} E_U,$$

giving the result. □

Definition 3.2.11. We will say that a closed forward-invariant set $K \subset X$ *admits stable trajectories* if $\mathbb{P}(O_K) > 0$.

LEMMA 3.2.12. *A closed forward-invariant set K admits stable trajectories if and only if there is a point in K that is potentially stable.*

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Proof. Follows immediately from Lemma 3.2.8. □

Remark 3.2.13. (Sufficient test for stable trajectories) Suppose either that (a) X is a Riemannian manifold and φ is a smooth RDS† on X, or that (b) X is a closed subset of a Riemannian manifold $\tilde{X}$, with φ being the X-restriction of some smooth RDS $\tilde{\varphi}$ on $\tilde{X}$ such that $\tilde{\varphi}(t, \omega)X \subset X$ for all t and ω. Let ρ be an ergodic stationary distribution of the Markov transition probabilities (φ_x^t). Provided the sizes of the spatial partial derivatives of $\varphi(t, \omega)$ at x are sufficiently well controlled in (t, ω, x) (over bounded ranges of t), we have the following test for stability: if there exists $t \in \mathbb{T}^+$ such that

$$\Lambda_t^\rho := \int_{\Omega \times X} \log \|\mathrm{d}\varphi(t, \omega)_x\| \mathbb{P} \otimes \rho\,(d(\omega, x)) < 0,$$

then ρ-almost every point in X is almost surely (exponentially) stable. (This is equivalent to saying that the 'maximal Lyapunov exponent', given by $\lim_{t\to\infty} (1/t) \log\|\mathrm{d}\varphi(t, \omega)_x\|$ for $(\mathbb{P} \otimes \rho)$-almost any (ω, x), is negative.) Precise interpretations of 'the spatial partial derivatives being sufficiently well controlled' can be found in [**LeJan87**, §3] (specifically condition (H2)‡ at the start of the section, with the proof§ given in Lemme 3) in the case of discrete time; or for SDEs, see [**Car85**] (in particular, Proposition 2.2.3). Similar results can also be found for RDSs on non-compact spaces (see e.g. [**MS99**, Corollary 3.1.1 and Remark (iii)]). Now if we did *not* assume ρ to be ergodic but only stationary, the conclusion of the test would no longer be that *ρ-almost every* point is almost surely stable; however, due to the ergodic decomposition theorem (see e.g. [**New15**, Corollary 100]), we *would still* be able to conclude that supp ρ contains at least one point that is almost surely stable.

THEOREM 3.2.14. *If K is a minimal set admitting stable trajectories, then every point in K is almost surely stable. If, in addition, K is the* only *minimal set, then every point in X is almost surely stable, and for each $x \in X$,*

$$\mathbb{P}(\omega : d(\varphi(t, \omega)x, K) \to 0 \text{ as } t \to \infty) = 1.$$

In the proof of Theorem 3.2.14 (and also in §4), we will use the following elementary lemma (which, heuristically, will play a kind of 'strong Markov property' role) in conjunction with Corollary 2.2.8.

LEMMA 3.2.15. *Let $(D, \leq)$ be a countable totally ordered set. Suppose we have, for each $s \in D$ and $n \in \mathbb{N}$, events $R_{n,s}$, $S_{n,s} \in \mathcal{F}$ with the following properties:*

- *for all n and s, $S_{n,s}$ is independent of $\sigma(R_{n,t} : t \leq s)$;*
- *for all n, $\mathbb{P}(\bigcup_{s\in D} R_{n,s}) = 1$;*
- $\inf_{s\in D} \mathbb{P}(S_{n,s}) \to 1$ *as* $n \to \infty$.

Then

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} \bigcup_{s\in D} R_{n,s} \cap S_{n,s}\right) = 1.$$

† This means that for each t and ω, $\varphi(t, \omega)$ is a smooth map.

‡ Beware that in [**LeJan87**] the characters $\|^2$ are missing from the end of the denominator in the formula for $\delta_2(T)$.

§ It is worth emphasizing that, although [**LeJan87**] works with random *diffeomorphisms*, the first of the two proofs given for Lemme 3 remains completely valid for non-invertible random smooth maps.

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Proof. First fix $n \in \mathbb{N}$. Since $\mathbb{P}(\bigcup_{s\in D} R_{n,s}) = 1$ and D is countable, we must have that for all $\varepsilon > 0$ there exist $t_1 < \cdots < t_m$ in D such that $\mathbb{P}(\bigcup_{i=1}^{m} R_{n,t_i}) > 1 - \varepsilon$, and so

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{s\in D} R_{n,s} \cap S_{n,s}\right) &\geq \mathbb{P}\left(\bigcup_{i=1}^{m} R_{n,t_i} \cap S_{n,t_i}\right) \\
&\geq \sum_{i=1}^{m} \mathbb{P}\left(R_{n,t_i} \setminus \bigcup_{j=1}^{i-1} R_{n,t_j}\right)\mathbb{P}(S_{n,t_i}) \\
&\geq \left(\sum_{i=1}^{m} \mathbb{P}\left(R_{n,t_i} \setminus \bigcup_{j=1}^{i-1} R_{n,t_j}\right)\right) \inf_{s\in D} \mathbb{P}(S_{n,s}) \\
&= \mathbb{P}\left(\bigcup_{i=1}^{m} R_{n,t_i}\right) \inf_{s\in D} \mathbb{P}(S_{n,s}) \\
&\geq (1-\varepsilon) \inf_{s\in D} \mathbb{P}(S_{n,s}).
\end{aligned}$$

This is true for all ε, and so

$$\mathbb{P}\left(\bigcup_{s\in D} R_{n,s} \cap S_{n,s}\right) \geq \inf_{s\in D} \mathbb{P}(S_{n,s}).$$

The desired result then follows from the fact that $\inf_{s\in D} \mathbb{P}(S_{n,s}) \to 1$ as $n \to \infty$. □

Proof of Theorem 3.2.14. Suppose K is a minimal set admitting stable trajectories. We first prove that there exist almost surely stable points in K. Since K is minimal, there exists an ergodic stationary distribution ρ with $\operatorname{supp} \rho = K$. Since there is at least one potentially stable point in K and the set of all potentially stable points is an open backward-invariant set (Lemma 3.2.9), it follows (by the minimality of K) that every point in K is potentially stable. Hence $\mathbb{P} \otimes \rho(O) > 0$. Since $\mathbb{P}_{|\mathcal{F}_\infty} \otimes \rho$ is ergodic with respect to (Θ^t), Lemma 3.2.3(b) then implies that $\mathbb{P} \otimes \rho(O) = 1$. So ρ-almost every point is almost surely stable.

Now let $p \in K$ be an almost surely stable point. Fix any $x \in K$, and for each $n \in \mathbb{N}$ and $s \in \mathbb{Q} \cap \mathbb{T}^+$ let

$$\begin{aligned}
R_{n,s} &= \{\omega \in \Omega : \varphi(s, \omega)x \in B_{1/n}(p)\} \\
S_{n,s} &= \theta^{-s}(E_{B_{1/n}(p)}).
\end{aligned}$$

Corollary 2.2.8 gives that $\mathbb{P}(\bigcup_s R_{n,s}) = 1$ for all n. Obviously $\mathbb{P}(S_{n,s}) = P_{1/n}(p)$ for all n (independently of s), and so $\mathbb{P}(S_{n,s}) \to 1$ as $n \to \infty$ (uniformly in s). It is clear that $R_{n,s} \cap S_{n,s} \subset O_x$ for all n and s, and so Lemma 3.2.15 yields that $\mathbb{P}(O_x) = 1$, i.e. x is almost surely stable.

Now if K is the only minimal set, then (by the second assertion in Corollary 2.2.8) all the above holds for any $x \in X$ (rather than just $x \in K$). Moreover, given any n and s, for any $\omega \in R_{n,s} \cap S_{n,s}$ we have that $d(\varphi(t, \omega)x, \varphi(t - s, \theta^s\omega)p) \to 0$ as $t \to \infty$; also, for $\mathbb{P}$-almost every $\omega \in \Omega$ we have that $\varphi(t - s, \theta^s\omega)p \in K$ for any (rational) s and all $t \geq s$. Hence Lemma 3.2.15 yields that for $\mathbb{P}$-almost all ω, $d(\varphi(t, \omega)x, K) \to 0$ as $t \to \infty$. □

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Remark 3.2.16. Note that, as a consequence of Theorem 3.2.14, if the transition probabilities (φ_x^t) admit a unique stationary distribution ρ and $\mathbb{P} \otimes \rho(O) = 1$, then every point in X is almost surely stable.

Finally, we define stable synchronization.

Definition 3.2.17. We will say that φ is (*globally*) *stably synchronizing* if φ is synchronizing and every point in X is almost surely stable.

It follows from Remark 3.2.16 that φ is stably synchronizing if and only if φ is synchronizing and (letting ρ be the unique stationary distribution of (φ_x^t)) $\mathbb{P} \otimes \rho(O) = 1$.

4. *Conditions for stable synchronization*

In this section, we give necessary and sufficient conditions for φ to be stably synchronizing. One of these conditions is a condition on the two-point motion, which we shall now introduce.

4.1. *Two-point contractibility.*

For any $A \subset X$, let $\Delta_A := \{(x, x) : x \in A\} \subset X \times X$.

Definition 4.1.1. Given points $x, y \in X$, we will say that the pair (x, y) is *contractible under* φ if every neighbourhood of Δ_X is accessible from (x, y) under $\varphi \times \varphi$. Given a closed forward-invariant set $K \subset X$, we will say that φ *is two-point contractible on* K if for all $x, y \in K$, (x, y) is contractible under φ.

Note that since Δ_X is compact, any neighbourhood of Δ_X is in fact a uniform neighbourhood, and hence we have that (x, y) is contractible under φ if and only if for all $\varepsilon > 0$,

$$\mathbb{P}(\omega : \exists t \in \mathbb{T}^+ \text{ s.t. } d(\varphi(t, \omega)x, \varphi(t, \omega)y) < \varepsilon) > 0.$$

Furthermore, it is easy to show (using Lemma 2.2.3 applied to $\varphi \times \varphi$) that for any closed forward-invariant $K \subset X$, φ is two-point contractible on K if and only if for all $x, y \in K$ with $x \neq y$,

$$\mathbb{P}(\omega : \exists t \in \mathbb{T}^+ \text{ s.t. } d(\varphi(t, \omega)x, \varphi(t, \omega)y) < d(x, y)) > 0.$$

This is, in turn, equivalent to saying that there are no non-empty forward-invariant compact subsets of $(K \times K) \setminus \Delta_K$ (which is essentially the formulation given in [**BS88**] and [**Bax91**]).

Take a sequence of values ε_n decreasing to 0; if we apply Lemma 2.2.6 to the two-point motion $\varphi \times \varphi$, replacing K with $\{(u, v) \in K \times K : d(u, v) \geq \varepsilon_n\}$, we obtain the following. If φ is two-point contractible on K, then for every $x, y \in K$,

$$\mathbb{P}(\omega : \exists \text{ unbounded } (t_n)_{n\in\mathbb{N}} \text{ in } \mathbb{T}^+ \text{ s.t. } d(\varphi(t_n, \omega)x, \varphi(t_n, \omega)y) \to 0 \text{ as } n \to \infty) = 1.$$

(See also [**BS88**, Proposition 4.1].)

It is worth observing that if φ is two-point contractible on the whole of X, then any two non-empty closed forward-invariant subsets of X must have non-trivial intersection, and so there must be a unique minimal set.

Now there may be contexts in which it is not directly clear that φ is two-point contractible on the whole of a minimal set K, and yet where it is directly clear that (for

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some given stationary probability measure supported by K) φ is two-point contractible on *almost* the whole of K (e.g. when, for some t, the support of the $C(X, X)$-valued random variable $\omega \mapsto \varphi(t, \omega)$ includes a map with a fixed point in K whose basin of attraction includes all but a zero-measure set in K; see, for example, [**Hom13**, Theorem 1.1]). For such cases, we have the following.

PROPOSITION 4.1.2. *Suppose ρ is a stationary distribution of (φ_x^t) with $K := \operatorname{supp} \rho$ being minimal. Suppose also that there is a set $A \subset K$ such that:*

- *$\rho(A) = 1$;*
- *the interior of A relative to K is non-empty;*
- *every pair of points in A is contractible under φ.*

Then φ is two-point contractible on K.

Proof. Fix $x, y \in K$; it is sufficient to show that $G_{(x,y)} \cap (A \times A) \neq \emptyset$. Let D be a countable dense subset of $\mathbb{T}^+$. Let

$$B := \{x \in K : \text{for all } t \in D,\ \varphi_x^t(A) = 1\}.$$

Since ρ is stationary and $\rho(A) = 1$, it is easy to show that $\rho(B) = 1$—and so, in particular, B is non-empty. By Lemma 2.2.4 the two projections of $G_{(x,y)}$ are both equal to K; so let (u, v) be a point in $G_{(x,y)}$ such that $u \in B$. Let $U \subset X$ be an open set such that $U \cap K$ is a non-empty subset of A; so U is accessible from v. Since the map $t \mapsto \varphi_v^t(U)$ is right lower semicontinuous, there must exist $t \in D$ such that $\varphi_v^t(U) > 0$. Since K is forward-invariant, it follows that $\varphi_v^t(A) > 0$. So then, there exists a $\mathbb{P}$-positive measure set of sample points ω such that $\varphi(t, \omega)u$ and $\varphi(t, \omega)v$ are both in A. So $G_{(u,v)}$ has non-trivial intersection with $A \times A$, and hence $G_{(x,y)}$ has non-trivial intersection with $A \times A$. □

4.2. *Main result*

THEOREM 4.2.1. *φ is stably synchronizing if and only if the following conditions hold.*

(i) *There is a unique minimal set $K \subset X$.*
(ii) *φ is two-point-contractible on the minimal set K.*
(iii) *The minimal set K admits stable trajectories.*

Remark 4.2.2. If we are given a stationary distribution ρ, then we can replace condition (i) with the condition that supp ρ is the unique minimal set; for conditions (ii) and (iii), we then set $K := \operatorname{supp} \rho$.

Proof of Theorem 4.2.1. If φ is synchronizing, then it is obviously two-point contractible on X, so (i) and (ii) hold. If φ is stably synchronizing, then (iii) also holds. Now suppose that (i), (ii) and (iii) hold. By Theorem 3.2.14, (i) and (iii) together imply that every point in X is almost surely stable, and so we just need to establish that φ is synchronizing. Using the second assertion in Theorem 3.2.14, it is clear that for any closed non-empty forward-invariant $C \subset X \times X$, C has non-trivial intersection with $K \times K$; consequently, by (iii), C has non-trivial intersection with Δ_K—and therefore, since Δ_K is clearly minimal under $\varphi \times \varphi$, C contains the whole of Δ_K. Thus we see that Δ_K is the unique minimal set under

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$\varphi \times \varphi$. Now fix any $x, y \in X$. Fix a point $p \in K$, and for each $n \in \mathbb{N}$ and $s \in \mathbb{Q} \cap \mathbb{T}^+$, let

$$R_{n,s} = \{\omega \in \Omega : (\varphi(s, \omega)x, \varphi(s, \omega)y) \in B_{1/n}(p) \times B_{1/n}(p)\},$$
$$S_{n,s} = \theta^{-s}(E_{B_{1/n}(p)}).$$

Corollary 2.2.8 (applied to $\varphi \times \varphi$) gives that $\mathbb{P}(\bigcup_s R_{n,s}) = 1$ for all n. Since p is almost surely stable, we have once again that $\mathbb{P}(S_{n,s}) \to 1$ as $n \to \infty$ (uniformly in s). Obviously, given any n and s, for any $\omega \in R_{n,s} \cap S_{n,s}$ we have that $d(\varphi(t, \omega)x, \varphi(t, \omega)y) \to 0$ as $t \to \infty$. So Lemma 3.2.15 yields that for $\mathbb{P}$-almost all ω, $d(\varphi(t, \omega)x, \varphi(t, \omega)y) \to 0$ as $t \to \infty$. So φ is synchronizing. □

5. *Example*

The following is based on the example described in [**LeJan87**]†. Let $(\Omega, \mathcal{F}, \mathbb{P}) = ([0, 1)^{\mathbb{N}}, \mathcal{B}([0, 1))^{\otimes\mathbb{N}}, \nu^{\otimes\mathbb{N}})$, where ν is the Lebesgue measure on $[0, 1)$. For each $n \in \mathbb{N} \cup \{0\}$, define the projection $c_n : \Omega \to [0, 1)^n$ by $c_n((\alpha_r)_{r\in\mathbb{N}}) = (\alpha_r)_{r\in\{1,\dots,n\}}$, and let $\mathcal{F}_n := \sigma(c_n)$. Writing $\theta^n : (\alpha_r)_{r\in\mathbb{N}} \mapsto (\alpha_{n+r})_{r\in\mathbb{N}}$ for the canonical shift dynamical system, it is clear that $(\Omega, \mathcal{F}, (\mathcal{F}_n), \mathbb{P}, (\theta^n))$ is a memoryless noise space.

We identify the circle $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z}$ in the obvious manner (where $\mathbb{R}/\mathbb{Z}$ is equipped with the obvious topology and Riemannian structure), and we write $[x] := \{x + n : n \in \mathbb{Z}\} \in \mathbb{S}^1$ for all $x \in \mathbb{R}$. Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a smooth map, and $F : \mathbb{R} \to \mathbb{R}$ a lift of f, such that $F(x) - x$ is periodic in x (i.e. $\deg f = 1$). For any $\alpha \in [0, 1)$, we define the map $f_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ by

$$f_\alpha([x]) = [F(x + \alpha) - \alpha].$$

We will say that a value $\alpha \in (0, 1)$ is a *subperiod of* f if $f_\alpha = f$. (This is equivalent to saying that $F(x) - x$ is α-periodic in x.) It is clear that for any $\alpha_1, \alpha_2 \in (0, 1)$, α_1 is a subperiod of f if and only if it is a subperiod of f_{α_2}.

We define the RDS φ^f on $\mathbb{S}^1$ by

$$\varphi^f(n, (\alpha_r)) = f_{\alpha_n} \circ \cdots \circ f_{\alpha_1}.$$

It is clear that the Lebesgue measure l on $\mathbb{S}^1$ is a stationary distribution for the Markov transition probabilities associated to φ^f.

PROPOSITION 5.1. *$\mathbb{S}^1$ is minimal under φ^f if and only if f is not a rational rotation.*

Proof. If f is a rotation, then $\varphi^f(n, \omega) = f^n$ for all n and ω; hence it is clear that if f is a rational rotation, then φ^f is not minimal on $\mathbb{S}^1$. Now suppose that f is not a rational rotation; so there must exist $p \in \mathbb{R}$ such that $w := F(p) - p$ is irrational. For all $y \in \mathbb{S}^1$ let $\alpha(y) \in [0, 1)$ be such that $[p - \alpha(y)] = y$. Fix $x \in \mathbb{S}^1$, with $x' \in \mathbb{R}$ being a lift of x, and any non-empty open $U \subset \mathbb{S}^1$. Let $n \in \mathbb{N}$ be such that $[x' + nw] \in U$. Let $\alpha_1 := \alpha(x)$, and recursively define $\alpha_m := \alpha(f_{\alpha_{m-1}} \circ \cdots \circ f_{\alpha_1}(x))$ for $2 \le m \le n$. Then by construction, $f_{\alpha_n} \circ \cdots \circ f_{\alpha_1}(x) \in U$. It is then clear that for a sufficiently small neighbourhood V of $(\alpha_1, \dots, \alpha_n)$ in $[0, 1)^n$, $\varphi^f(n, \omega)x \in U$ for all $\omega \in c_n^{-1}(V)$. Since ν has full support on $[0, 1)$, it follows that $\mathbb{P}(c_n^{-1}(V)) > 0$. Thus U is accessible from x. This is true for all x and U, so φ^f is minimal. □

† A related study (a preliminary version of which is cited in [**LeJan87**]) can also be found in [**Kai93**]; it is from this paper that we have taken our term 'subperiod'.

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PROPOSITION 5.2. *φ^f is two-point contractible on $\mathbb{S}^1$ if and only if f has no subperiods.*

Proof. We work with the metric $d(x, y) = \min\{|y' - x'| : [x'] = x, [y'] = y\}$ (under which the diameter of $\mathbb{S}^1$ is $\frac{1}{2}$). First suppose f has a subperiod; without loss of generality, let $\alpha \in (0, \frac{1}{2}]$ be a subperiod. Since α is a subperiod of $f_{\alpha'}$ for all $\alpha' \in [0, 1)$, it follows that for any $x, y \in \mathbb{S}^1$ with $d(x, y) = \alpha$, $d(\varphi^f(n, \omega)x, \varphi^f(n, \omega)y) = \alpha$ for all n and ω. So φ^f is not two-point contractible on $\mathbb{S}^1$.

Now suppose f has no subperiods. Fix any $\alpha \in (0, \frac{1}{2}]$, and define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(z) = F(z + \alpha) - F(z) - \alpha$. Since α is not a subperiod, there exists $q \in \mathbb{R}$ such that $g(q) \neq 0$. We now consider separately the possibilities that $g(q) > 0$ and $g(q) < 0$. First suppose $g(q) > 0$. Given the continuity of F, let $\varepsilon \in (0, 1)$ be such that for all $x \in [q, q + \varepsilon)$, $F(x) - F(q) - (x - q) < g(q)$. Let $r \geq 2$ be an integer such that $\langle r\alpha \rangle \in [0, \varepsilon)$ (where $\langle a \rangle := a - \lfloor a \rfloor$ denotes the fractional part of a). We have

$$\begin{aligned}\sum_{i=0}^{r-1} g(q + i\alpha) &= F(q + r\alpha) - F(q) - r\alpha \\ &= F(q + \langle r\alpha \rangle) - F(q) - \langle r\alpha \rangle \quad (\text{since } x \mapsto F(x) - x \text{ is 1-periodic}) \\ &< g(q).\end{aligned}$$

Hence there must exist $i \in \{1, \ldots, r - 1\}$ such that $g(q + i\alpha) < 0$. Now in the case that $g(q) < 0$, by the same argument we can find $i \in \mathbb{N}$ such that $g(q + i\alpha) > 0$. In either case, g attains both positive and negative values, and so by the intermediate value theorem, the range of g includes a neighbourhood of 0. So let $p \in \mathbb{R}$ be such that $g(p) \in [-\alpha, 0)$. Now fix any points $x, y \in \mathbb{S}^1$ with $d(x, y) = \alpha$; without loss of generality, assume there exist lifts $x', y' \in \mathbb{R}$ of x and y respectively such that $y' - x' = \alpha$. Let $\beta := \langle p - x' \rangle$. Then

$$\begin{aligned}d(f_\beta(x), f_\beta(y)) &= d([F(x' + \beta)], [F(y' + \beta)]) \\ &= d([F(p)], [F(p + \alpha)]) \\ &< \alpha.\end{aligned}$$

Hence (by continuity) there must exist a non-empty open $U \subset [0, 1)$ such that

$$d(f_{\tilde{\beta}}(x), f_{\tilde{\beta}}(y)) < \alpha$$

for all $\tilde{\beta} \in U$. Note that $\mathbb{P}(c_1^{-1}(U)) = \nu(U) > 0$. Hence φ^f is two-point contractible on $\mathbb{S}^1$. □

Now define the quantity $\lambda_f \in \mathbb{R} \cup \{-\infty\}$ by

$$\lambda_f := \int_0^1 \log |F'(y)| \, dy.$$

It is easy to show that for any $n \in \mathbb{N} \cup \{0\}$,

$$\int_{\Omega \times \mathbb{S}^1} \log \|\mathrm{d}\varphi^f(n, \omega)_x\| \mathbb{P} \otimes l\,(d(\omega, x)) = n\lambda_f.$$

THEOREM 5.3. *If f has no subperiods and $\lambda_f < 0$, then φ^f is stably synchronizing.*

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Note that a partial converse also holds: if f has a subperiod, then clearly φ^f cannot be synchronizing.

Proof. Obviously if f has no subperiods, then, in particular, f is not a rotation. So Propositions 5.1 and 5.2 give that φ^f satisfies conditions (i) and (ii) of Theorem 4.2.1, with K being the whole of $\mathbb{S}^1$. Now it is clear that for any $n \in \mathbb{N} \cup \{0\}$, every order derivative of $\varphi^f(n, \omega)$ is uniformly bounded in ω; this is easily sufficient to be able to apply the test described in Remark 3.2.13, so if $\lambda_f < 0$, then $\mathbb{S}^1$ admits stable trajectories. Theorem 4.2.1 then gives the result. □

Now [**LeJan87**] specifically considers the case that f is an orientation-preserving diffeomorphism—in which case (as is mentioned in [**LeJan87**]), if f is not a rotation, then $\lambda_f < 0$ (by the strict Jensen inequality). It is stated that if $1/n$ is the least period of $x \mapsto F(x) - x$, then the number of the open regions of synchronization described in the Introduction is equal to n. The justification (as is very briefly outlined in [**LeJan87**]) is essentially as follows: due to f being a homeomorphism, each region of synchronization must be a *connected* set of length $1/m$ (where m is the number of such regions); and moreover (due to the shift map θ being $\mathbb{P}$-preserving) the images of each of these sets under the RDS φ^f must *remain* of length $1/m$. Since at each step the random parameter α can be selected anywhere from $[0, 1)$, m must itself be a subperiod of f, and therefore n is a multiple of m; but it is also clear that m is a multiple of n.

Note in particular the '$n = 1$ case' of Le Jan's statement: if f is a diffeomorphism with no subperiods, then φ^f is almost stably synchronizing according to *l*. Theorem 5.3 strengthens this to *global* stable synchronization, and moreover allows the statement to be extended *beyond* the case of invertible f.

As an example, consider the map f whose lift is given by

$$F(x) = x + a \sin(2\pi x), \quad a > 0.$$

If $a \in (0, (1/2\pi))$, then f is an orientation-preserving diffeomorphism with no subperiods, and so φ^f is stably synchronizing. If we wish to increase a beyond $1/2\pi$, such that f will no longer be a diffeomorphism or even an injective map, there will still exist a range of a-values close enough to $1/2\pi$ from above such that $\lambda_f < 0$ and so φ^f remains stably synchronizing.

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