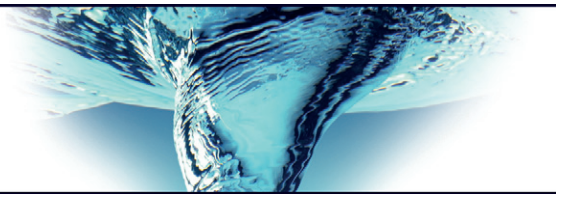
J. Fluid Mech. (2014), *vol.* 743, R2, doi:10.1017/jfm.2014.81



JFM RAPIDS
journals.cambridge.org/rapids

Estimating wave heights from pressure data at the bed

A. Constantin†

Department of Mathematics, King's College London, Strand, London WC2R 2LS, UK
Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria

(Received 17 December 2013; revised 3 February 2014; accepted 6 February 2014)

We provide some estimates for the wave height of a two-dimensional travelling gravity water wave from pressure measurements at the flat bed. The approach is applicable without limitations on the wave amplitude. It improves the classical estimates available if one relies on the hydrostatic approximation or on the linear theory of waves of small amplitude.

Key words: surface gravity waves, waves/free-surface flows

1. Introduction

The techniques used to observe waves at sea are of two types: remote sensing and *in situ*. Remote-sensing techniques, by means of electromagnetic radiation emitted from aircraft or satellites, provide a wide synoptic view of the sea surface and represent the only practical way to obtain data from secluded regions. However, there are some basic limitations, due to the fact that remote sensing is an indirect measurement and the desired data is retrieved by solving an inverse problem. Because of data sensitivity and resulting instability issues in the retrieval process, remote-sensing outcomes must be calibrated against *in situ* data, provided by instruments deployed in the water at an accessible location. In this context, the widespread use of pressure sensors is due to their low cost, ease of use and low liability to damage (cf. Jones & Monismith 2007). Pressure sensors are often placed at fixed positions on the rigid bed, and they measure and record pressure fluctuations at these locations. The primary information from any wave observation technique consists of an estimate of wave height, wave speed and wave period. While pressure sensors provide very accurate pressure measurements, the conversion of this data into wave height information is quite delicate. The most basic approach, often employed for tsunami detection (cf. Oliveras *et al.* 2011), uses the hydrostatic formula

$$\eta(x) + d = \frac{P(x, -d) - P_{atm}}{\rho\, g}, \quad x \in \mathbb{R}, \tag{1.1}$$

† Email address for correspondence: adrian.constantin@kcl.ac.uk

© Cambridge University Press 2014

https://doi.org/10.1017/jfm.2014.81 Published online by Cambridge University Press

to approximate the profile $s \mapsto \eta(s)$ of a periodic unidirectional wave travelling at constant speed $c > 0$, without change of shape, at the surface of water over the flat bed $y = -d$. Here $d > 0$ is the average depth, g is the (constant) acceleration due to gravity, P_{atm} is the (constant) atmospheric pressure and $P(x, -d)$ is the pressure evaluated at the flat bed, as a function of the spatial variable x. Since the theory underlying (1.1) completely ignores wave effects, it is not surprising that (1.1) is quite inaccurate (cf. the discussion in Bishop & Donelan (1987)). Linear theory accounts for the presence of waves of small amplitude and within this framework one can easily check by inspection the validity of the formula

$$\eta(x) = \cosh(kd)\left(\frac{P(x, -d) - P_{atm}}{\rho g} - d\right), \quad x \in \mathbb{R}, \tag{1.2}$$

where $k = 2\pi/L$ is the wavenumber determined by the wavelength $L > 0$; see, e.g., Constantin (2014) and Escher & Schlurmann (2008) for the explicit expressions for the pressure and the free surface wave that are available in this setting. In this context we may draw attention to the fact that, often, the superposition of the multiple frequency components of the pressure on the flat bed, $P(x, -d)$, is used when reconstructing the free surface profile $\eta(x)$ from (1.2). However, this procedure is not consistent with the asymptotic expansions in the small-amplitude parameter used in linear wave theory (cf. the discussion in Constantin (2014)). This is a caveat missed by many authors. In the limit $kd \to 0$ of (1.2) we recover (1.1). While (1.2) represents an improvement with respect to (1.1), it turns out that it overestimates the wave height by more than 10 % even for waves of moderate amplitude (cf. the data provided in Tsai *et al.* (2005)). Recent investigations of the governing equations, without approximations, were initiated in Oliveras *et al.* (2011) and Deconinck *et al.* (2012), and were further pursued in Constantin (2012), Clamond & Constantin (2013) and Clamond (2013), the final outcome being that an exact recovery of the surface wave profile from pressure measurements at the flat bed is possible. However, this recovery is explicit only in the case of solitary water waves (cf. Constantin 2012), while for periodic travelling waves it involves either solving a nonlinear differential equation (cf. Clamond & Constantin 2013), or solving an implicit functional equation (cf. Clamond 2013). Even if the numerical solution by means of an iterative process is easy, the obtained results are not that transparent. From a practical point of view, it is reassuring to know that the knowledge of the pressure at the bed completely determines a periodic travelling surface wave. Nevertheless, some simple explicit estimates of the type provided by (1.1) and (1.2), with the added benefit of accuracy, would be of considerable practical relevance. In addressing this issue, we will develop an approach that differs from those pursued by Oliveras *et al.* (2011), Clamond & Constantin (2013) and Clamond (2013).

2. Preliminaries

We investigate the most regular water wave patterns: periodic travelling waves, propagating on the surface of water with a flat, horizontal bed, in a given direction and at constant speed. Compelling practical grounds for the study of this type of water flows are the waves propagating in a canal, at the surface of water of relatively small average depth, as well as surface waves in sea regions with a flat bed: note that abyssal plains, the flattest areas on Earth, with an almost total absence of geographic features, are found in all major sea and ocean basins, usually adjacent to land and

https://doi.org/10.1017/jfm.2014.81 Published online by Cambridge University Press

covering overall almost a third of the Earth's surface. We will also make some other simplifying but physically reasonable assumptions: (i) the density ρ of the water is constant; (ii) the flow is inviscid; (iii) the influence exerted by the air above the water is in the form of atmospheric pressure acting on the surface, taken to be constant; (iv) the evolution of the waves is governed by the balance between the restoring force of gravity and the inertia of the system, so that gravity is the only external force of significance.

To describe a two-dimensional water wave it suffices to consider a cross-section of the flow in the direction of wave propagation. We choose Cartesian coordinates (X, Y) with the X-axis pointing in the direction of wave propagation and the Y-axis pointing vertically upwards, while the origin is located on the mean water level $Y = 0$; see figure 1. Let $(U(X, Y, T),\ V(X, Y, T))$ be the velocity field of the two-dimensional flow propagating in the X direction over the flat bed $Y = -d$, where $d > 0$ is the average depth, and let $Y = \eta(X, T)$ be the water's free surface with mean water level $Y = 0$. For water of constant density the equation of mass conservation is

$$U_X + V_Y = 0 \tag{2.1}$$

throughout the fluid. Since the assumption of inviscid flow is physically realistic for water waves, the equation of motion is Euler's equation

$$\left.\begin{aligned} U_T + UU_X + VU_Y &= -\frac{1}{\rho} P_X, \\ V_T + UV_X + VV_Y &= -\frac{1}{\rho} P_Y - g, \end{aligned}\right\} \tag{2.2}$$

where $P(X, Y, T)$ is the pressure, g is the (constant) acceleration of gravity and ρ is the constant density. The boundary conditions associated with (2.1) and (2.2) are

$$V = 0 \quad \text{on } Y = -d, \tag{2.3}$$

$$V = \eta_T + U\eta_X \quad \text{on } Y = \eta(X, T), \tag{2.4}$$

$$P = P_{atm} \quad \text{on } Y = \eta(X, T). \tag{2.5}$$

The kinematic boundary conditions (2.3) and (2.4) reflect the fact that both boundaries are interfaces, while the dynamic boundary condition (2.5), in which P_{atm} represents the (constant) atmospheric pressure at the surface, decouples the motion of the water from that of the air above. The absence of non-uniform underlying currents is captured by imposing an irrotational flow, that is,

$$U_Y - V_X = 0 \quad \text{throughout the fluid.} \tag{2.6}$$

Field data shows that for waves entering a region of still water the assumption of irrotational flow is realistic. We consider waves with no underlying current.

Throughout this paper, a *Stokes wave* is an irrotational smooth solution to the governing equations for which there exists a period $L > 0$ and a wave speed $c > 0$ such that the free surface profile η, the fluid velocity (U, V) and the pressure P have period L in the X variable, η depends only on $(X - cT)$, while U, V and P depend only on $(X - cT)$ and Y. Moreover, there is a single crest and trough per period, the wave profile is strictly monotone between successive crests and troughs, and η, U and P are symmetric while V is antisymmetric about the crest line. By a crest line

https://doi.org/10.1017/jfm.2014.81 Published online by Cambridge University Press

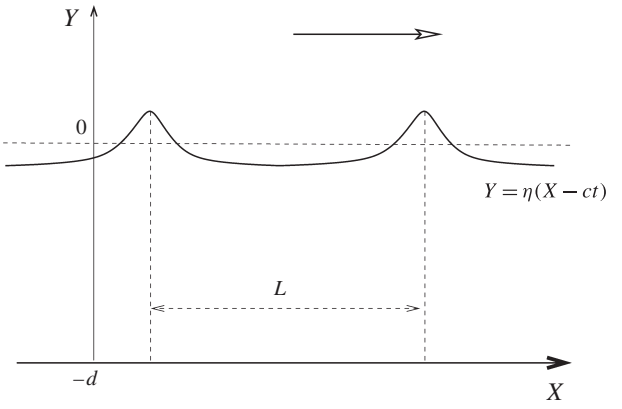


FIGURE 1. The choice of Cartesian coordinates (X, Y): X points in the direction of wave propagation and Y points vertically upwards, with $Y = 0$ being the mean water level.

(trough line), we mean the vertical line directly below a crest (trough). This definition excludes flows with a flat free surface, as well as the *Stokes wave of greatest height*, for which the symmetric free surface is not continuously differentiable, the profile having at the crest a corner with an angle of 120°. For a prescribed wavelength, this is the highest two-dimensional periodic travelling wave. However, the highest wave is not the fastest nor the most energetic (see the discussion in Constantin (2011)). We refrain from discussing this flow pattern, as it presents peculiar features and it requires many of specific highly technical considerations that are beyond our scope. Note also that some of the features required in the definition of a Stokes wave are not restrictive, e.g. any periodic, smooth, irrotational travelling wave with a profile that is monotone between successive crests and troughs must be symmetric, with $\eta_x = 0$ only at crests and troughs (see the discussion in Constantin (2011)).

Passing to the moving frame

$$x = X - cT, \quad y = Y, \tag{2.7}$$

we can take advantage of the (X, T)-dependence of the form $(X - cT)$ to write the governing equations for Stokes waves as

$$\left.\begin{aligned} &uu_x + vu_y = -\frac{1}{\rho} P_x \quad \text{for} \quad -d \leqslant y \leqslant \eta(x), \\ &uv_x + vv_y = -\frac{1}{\rho} P_y - g \quad \text{for} \quad -d \leqslant y \leqslant \eta(x), \\ &u_x + v_y = 0 \quad \text{for} \quad -d \leqslant y \leqslant \eta(x), \\ &u_y = v_x \quad \text{for} \quad -d \leqslant y \leqslant \eta(x), \\ &v = 0 \quad \text{on} \quad y = -d, \\ &v = u\eta_x \quad \text{on} \quad y = \eta(x), \\ &P = P_{atm} \quad \text{on} \quad y = \eta(x), \end{aligned}\right\} \tag{2.8}$$

where we denoted

$$u = U - c, \quad v = V. \tag{2.9}$$

https://doi.org/10.1017/jfm.2014.81 Published online by Cambridge University Press

An important feature of Stokes waves is that throughout the flow the horizontal fluid velocity can never attain the value c, that is, $U < c$ throughout the fluid, or

$$u < 0 \quad \text{throughout the fluid;} \tag{2.10}$$

see the discussion in Constantin (2006). For convenience, we assume that in the moving frame the wave crests/troughs are located at $x = nL$ and $x = (2n+1)L/2$ with n an integer, respectively. Note that the fourth relation in (2.8), in combination with the fact that, due to symmetry, v vanishes on the trough lines, yields

$$\partial_y \int_{-L/2}^{L/2} u(x, y)\, \mathrm{d}x = 0 \quad \text{for} \quad -d < y < \eta(L/2). \tag{2.11}$$

Consequently, the average of the horizontal velocity, at any depth beneath the trough level $y = \eta(L/2)$, is constant. In the physical frame, the expression $\int_{-L/2}^{L/2} U(x, y)\, \mathrm{d}x$ represents the uniform current underlying the surface waves (see also Constantin & Strauss 2010). The absence of an underlying current is therefore characterized by

$$c + \frac{1}{L} \int_{-L/2}^{L/2} u(x, y)\, \mathrm{d}x = 0, \quad -d \leqslant y \leqslant \eta(L/2). \tag{2.12}$$

In view of the third and fourth relations in (2.8), one can see that the Euler equations can be reformulated as Bernoulli's law, stating that

$$\frac{u^2 + v^2}{2} + g(y + d) + \frac{1}{\rho} P = Q \tag{2.13}$$

with Q constant throughout the fluid. Furthermore, the third relation in (2.8) ensures the existence of a stream function $\psi(x, y)$, determined up to an additive constant by requiring

$$\psi_x = -v, \quad \psi_y = u. \tag{2.14}$$

The fifth and sixth relations in (2.8) merely express the fact that ψ is constant on the upper and on the lower boundary of the fluid domain $\mathcal{D} = \{(x, y) : \ -d < y < \eta(x)\}$. For convenience, we set

$$\psi(x, -d) = m \quad \text{and} \quad \psi(x, \eta(x)) = 0 \quad \text{for all } x \in \mathbb{R}, \tag{2.15}$$

where

$$m = -\int_{-d}^{\eta(x)} u(x, y)\, \mathrm{d}y > 0 \quad \text{for all } x \in \mathbb{R}, \tag{2.16}$$

is the constant relative mass flux, relative to the uniform flow at speed c (recall that $u = U - c$). These considerations are useful in expressing the governing equations in the equivalent form

$$\left.\begin{aligned} &\Delta_{(x,y)} \psi = 0 \quad \text{for} \quad -d \leqslant y \leqslant \eta(x), \\ &\psi = 0 \quad \text{on } y = \eta(x), \\ &\psi = m \quad \text{on } y = -d, \\ &\frac{\psi_x^2 + \psi_y^2}{2} + g(y + d) + \frac{1}{\rho} P_{atm} = Q \quad \text{on } y = \eta(x). \end{aligned}\right\} \tag{2.17}$$

The harmonic conjugate function of ψ is the velocity potential ϕ, determined uniquely up to a constant by requiring that $\phi_x = u$ and $\phi_y = v$ throughout the fluid. The function ϕ is uniquely determined if we specify that $\phi = 0$ on the crest line $x = 0$. While ψ is L-periodic and even in the x variable, ϕ is odd in the x variable and $[\phi(x, y) + cx]$ has period L in the x variable (see Constantin 2006).

https://doi.org/10.1017/jfm.2014.81 Published online by Cambridge University Press

3. Lower bounds for the wave height

Several features of the fluid velocity field in a Stokes wave were established theoretically and confirmed experimentally, cf. Constantin (2006), Constantin & Strauss (2010) and Umeyama (2012), while others are at the stage of conjectures supported by numerical evidence, cf. Clamond (2012). To make the present discussion self-contained, we will first present proofs of some facts that are useful for the considerations made in this section.

By the third and fourth relations in (2.8), v is a harmonic function. Moreover, due to the fifth relation in (2.8) and to symmetry, $v = 0$ on the lower boundary and on the lateral boundaries of the domain $\Omega = \{(x, y) : \ 0 < x < L/2, \ \ -d < y < \eta(x)\}$. On the other hand, the strict monotonicity of the wave profile between the crest and a successive trough, together with (2.10) and the sixth relation in (2.8), yield that $v > 0$ on the upper boundary of Ω. By the strong maximum principle, $v > 0$ throughout Ω, while Hopf's maximum principle yields $v_x(0, y) > 0$ for $-d < y < \eta(0)$, $v_x(L/2, y) < 0$ for $-d < y < \eta(L/2)$, and $v_y(x, -d) > 0$ for $0 < x < L/2$. Using the third and fourth relations in (2.8), we get

$$\left.\begin{array}{l} u_y(0, y) > 0 \quad \text{for} \ \ -d < y < \eta(0), \\ u_y(L/2, y) < 0 \quad \text{for} \ \ -d < y < \eta(L/2), \\ u_x(x, -d) < 0 \quad \text{for} \ 0 < x < L/2. \end{array}\right\} \tag{3.1}$$

Evaluating Bernoulli's law (2.13) at the wave crest/trough and on the flat bed at $x = 0$ and $x = L/2$, respectively, we obtain

$$\eta(0) + d > \frac{P(0, -d) - P_{atm}}{\rho g}, \tag{3.2}$$

and

$$\eta(L/2) + d < \frac{P(L/2, -d) - P_{atm}}{\rho g}, \tag{3.3}$$

due to (2.10) and (3.1). This yields at once the estimate

$$H > \frac{P(0, -d) - P(L/2, -d)}{\rho g} \tag{3.4}$$

for the wave height

$$H = \eta(0) - \eta(L/2). \tag{3.5}$$

Since throughout the fluid the pressure is maximal at $(0, -d)$, while $P(L/2, -d)$ is the minimum value of the pressure along the flat bed (cf. Constantin & Strauss 2010), we see that (3.4) expresses the fact that the variation of the pressure on the bed, normalized by means of ρg, provides a lower bound for the wave height. In this context, it is worth pointing out that (cf. Constantin & Strauss 2010) the averaged pressure on the flat bed equals $P_{atm} + \rho g d$. Therefore, the right-hand side of (3.2) exceeds d, while the right-hand side of (3.3) is smaller than d. Note that the absence of waves corresponds to equality in (3.2) and in (3.3).

The estimates (3.2) and (3.3) show that the hydrostatic formula (1.1) is always underestimating the wave elevation and overestimating the wave trough level. On the other hand, the so-called pressure transfer formula (1.2) of linear theory approximates H by the right-hand side of (3.4) times the multiplicative factor $\cosh(kd) > 1$. Field data shows that this overestimates the wave height.

https://doi.org/10.1017/jfm.2014.81 Published online by Cambridge University Press

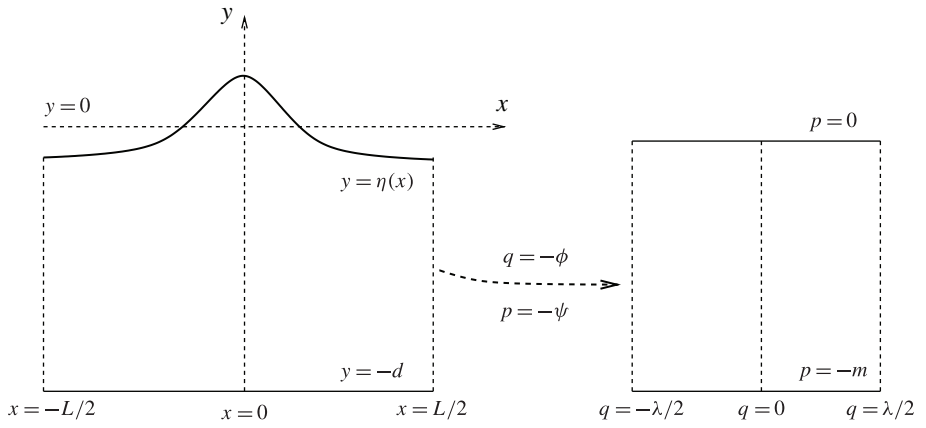


FIGURE 2. The hodograph transform mapping the free-boundary problem in the moving frame to a boundary-value problem in a known strip. The horizontal period L in the moving plane becomes the horizontal period $\lambda = cL$ in the conformal plane.

4. Upper bounds for the wave height

To provide upper estimates for the wave height in terms of pressure data on the bed, let us first discuss some straightforward recovery of certain wave characteristics. Since the pressure along the flat bed in maximal precisely under the wave crest and minimal exactly under the wave trough (cf. Constantin & Strauss 2010) from pressure transducers covering an area on the bed, the direction of wave propagation, the wave length L and the time L/c needed for the wave crests to advance a wave length can be determined. This yields at once the wave speed c.

To derive upper bounds for the wave height, it is convenient to transform by means of the conformal change of variables

$$q = -\phi(x, y), \quad p = -\psi(x, y), \tag{4.1}$$

the free-boundary problem (2.17) into the equivalent boundary value problem

$$\left.\begin{array}{l} \Delta_{(q,p)} h = 0 \quad \text{for} \quad -m < p < 0, \\ h = 0 \quad \text{on} \ p = -m, \\ 2\left(Q - \dfrac{1}{\rho} P_{atm} - gh\right)(h_q^2 + h_p^2) = 1 \quad \text{on} \ p = 0, \end{array}\right\} \tag{4.2}$$

for the function

$$h(q, p) = y + d, \tag{4.3}$$

(cf. Constantin & Strauss 2010); see figure 2. Note that

$$\left.\begin{array}{l} \partial_q = h_p\, \partial_x + h_q\, \partial_y, \\ \partial_p = -h_q\, \partial_x + h_p\, \partial_y, \end{array}\right\} \qquad \left.\begin{array}{l} \partial_x = -u\, \partial_q + v\, \partial_p, \\ \partial_y = -v\, \partial_q - u\, \partial_p, \end{array}\right\} \tag{4.4}$$

where

$$h_q = -\frac{v}{u^2 + v^2}, \quad h_p = -\frac{u}{u^2 + v^2}. \tag{4.5}$$

https://doi.org/10.1017/jfm.2014.81 Published online by Cambridge University Press

Bernoulli's law (2.13) yields

$$\frac{1}{\rho}P_* = Q - gh - \frac{1}{2(h_q^2+h_p^2)}, \quad -m \leqslant p \leqslant 0, \tag{4.6}$$

where we use the symbols P_*, u_* and v_*, to indicate the dependence of the pressure P and the velocity field $(u,\, v)$ on the conformal variables (q, p); e.g. $P(x, y) = P_*(q, p)$. Using (4.4), from the first two relations in (2.8) we deduce that

$$\frac{1}{\rho}\Delta_{(q,p)}P_* = -2\,(u_{*q}^2 + u_{*p}^2), \quad -m < p < 0. \tag{4.7}$$

Note that $q = cL/2$ corresponds to $x = L/2$, so that if we set

$$\alpha = \frac{P_*(cL/2, -m) - P_{atm}}{m} = \frac{P(L/2, -d) - P_{atm}}{m} > 0, \tag{4.8}$$

then (2.10), (3.1), (4.5) and (4.6) yield that

$$\begin{aligned} P_*(q, -m) &= \rho\, Q - \frac{\rho\, u_*^2(q, -m)}{2} \geqslant \rho\, Q - \frac{\rho\, u_*^2(cL/2, -m)}{2} \\ &= P_*(cL/2, -m) = P_{atm} + \alpha\, m \end{aligned} \tag{4.9}$$

for all $q \in [0, cL/2]$. By (4.7), the function

$$f(q, p) = \frac{P_*(q, p) + \alpha p}{\rho} \tag{4.10}$$

is superharmonic in the strip $-m < p < 0$. Note that f equals P_{atm}/ρ on the upper boundary $p = 0$, and by (4.9) it does not attain lower values on the lower boundary $p = -m$. Consequently, the strong minimum principle yields that the minimum of f throughout the strip is attained on the upper boundary $p = 0$, and Hopf's principle subsequently ensures $f_p(q, 0) < 0$ for all $q \in \mathbb{R}$. Using (4.4), (4.5) and the sixth and the first four relations in (2.8), we express this inequality as

$$\frac{\alpha}{\rho} < \frac{2\,u^2u_x\eta_x + u^2u_y(\eta_x^2 - 1) - gu}{u^2(1+\eta_x^2)} \quad \text{on } y = \eta(x). \tag{4.11}$$

Indeed,

$$\begin{aligned} f_p(q, 0) &= \frac{1}{\rho}P_{*p}(q, 0) - \frac{\alpha}{\rho} = \left\{\frac{v}{\rho(u^2+v^2)}P_x\right\}_{y=\eta(x)} - \left\{\frac{u}{\rho(u^2+v^2)}P_y\right\}_{y=\eta(x)} - \frac{\alpha}{\rho} \\ &= \left\{\frac{-v(uu_x + vu_y) + u(uv_x + vv_y + g)}{u^2+v^2}\right\}_{y=\eta(x)} - \frac{\alpha}{\rho} \\ &= \left\{\frac{-2uvu_x + (u^2 - v^2)u_y + gu}{u^2+v^2}\right\}_{y=\eta(x)} - \frac{\alpha}{\rho} \\ &= \left\{\frac{-2u^2u_x\eta_x + u^2(1-\eta_x^2)u_y + gu}{u^2(1+\eta_x^2)}\right\}_{y=\eta(x)} - \frac{\alpha}{\rho}. \end{aligned} \tag{4.12}$$

https://doi.org/10.1017/jfm.2014.81 Published online by Cambridge University Press

On the other hand, differentiating the fourth relation of (2.17) along the free surface $y=\eta(x)$ with respect to the x-variable yields

$$0=uu_x+uu_y\eta_x+vv_x+vv_y\eta_x+g\eta_x \quad \text{on } y=\eta(x). \tag{4.13}$$

Using the third, fourth and sixth relation in (2.8) in the above expression, we obtain

$$uu_x(1-\eta_x^2)+2uu_y\eta_x+g\eta_x=0 \quad \text{on } y=\eta(x). \tag{4.14}$$

Due to (2.10), and the fact that $\eta_x(x)<0$ for $x\in(0,L/2)$, by multiplying (4.14) by u and (4.11) by η_x, we get

$$\frac{\alpha}{\rho}\,\eta_x(x)>u_x(x,\eta(x))+u_y(x,\eta(x))\,\eta_x(x), \quad 0<x<L/2. \tag{4.15}$$

Integrating the above inequality on $[0,L/2]$, we obtain

$$\frac{\alpha}{\rho}\,[\eta(0)-\eta(L/2)]<u(0,\eta(0))-u(L/2,\eta(L/2)). \tag{4.16}$$

Evaluating (2.13) at the wave crest and at the wave trough, due to (2.5), we can write the above as

$$\frac{\alpha}{2\rho g}\,[u^2(L/2,\eta(L/2))-u^2(0,\eta(0))]<u(0,\eta(0))-u(L/2,\eta(L/2)). \tag{4.17}$$

Taking (2.10) into account, we get

$$|u(0,\eta(0))|+|u(L/2,\eta(L/2))|<\frac{2\rho g}{\alpha}, \tag{4.18}$$

since by (3.1), $u(0,\eta(0))>u(L/2,\eta(L/2))$. On the other hand, $\int_{-L/2}^{L/2}\eta(x)\,\mathrm{d}x=0$ in combination with (2.10), (2.14) and (2.15), yields

$$\begin{aligned} L^2m^2 &= \left(\int_{-L/2}^{L/2}\int_{-d}^{\eta(x)}|u(x,y)|\,\mathrm{d}y\,\mathrm{d}x\right)^2\leqslant Ld\int_{-L/2}^{L/2}\int_{-d}^{\eta(x)}u^2(x,y)\,\mathrm{d}y\,\mathrm{d}x\\ &= Ld\int_{-cL/2}^{cL/2}\int_{-m}^{0}\frac{u_*^2}{u_*^2+v_*^2}\,\mathrm{d}p\,\mathrm{d}q\leqslant Ld\int_{-cL/2}^{cL/2}\int_{-m}^{0}1\,\mathrm{d}p\,\mathrm{d}q=mcL^2d, \end{aligned} \tag{4.19}$$

by performing the change of variables (4.1) in the middle term. Thus

$$m\leqslant cd. \tag{4.20}$$

Combining (4.8), (4.18) and (4.20), we obtain

$$|u(0,\eta(0))|+|u(L/2,\eta(L/2))|<\frac{2\rho gcd}{P(L/2,-d)-P_{atm}}. \tag{4.21}$$

On the other hand, evaluating (2.13) at the wave crest/trough and recalling (2.5) and (2.10), we get

$$H=\eta(0)-\eta(L/2)=\frac{u^2(L/2,\eta(L/2))-u^2(0,\eta(0))}{2g}<\frac{[u(0,\eta(0))+u(L/2,\eta(L/2))]^2}{2g}. \tag{4.22}$$

https://doi.org/10.1017/jfm.2014.81 Published online by Cambridge University Press

The above inequality and (4.21) yield the upper bound

$$H < \frac{2\rho^2 g c^2 d^2}{[P(L/2, -d) - P_{atm}]^2}, \tag{4.23}$$

on the wave height. The bound (4.23) holds for all Stokes waves. Its accuracy improves with an increase of amplitude, since for waves close to the wave of greatest height we have $u(0, \eta(0)) \to 0$, and therefore the intermediate estimate between (4.21) and (4.23) is sharper. Note also that in the shallow-water limit the pressure is approximately hydrostatic (cf. Johnson 1997), so that the upper bound in (4.21) is approximated by $2c$, while in this regime $|u(0, \eta(0))| \approx c$ and $|u(L/2, \eta(L/2))| \approx c$, with $c \approx \sqrt{gd}$. In this limiting case, the upper bound in (4.23) is approximated by $2d$.

Acknowledgement

The support of the ERC Advanced Grant "Nonlinear studies of water flows with vorticity" is acknowledged. The author is grateful to all three referees for useful suggestions.

References

BISHOP, C. T. & DONELAN, M. A. 1987 Measuring waves with pressure transducers. *Coast. Engng* **11**, 309–328.

CLAMOND, D. 2012 Note on the velocity and related fields of steady irrotational two-dimensional surface gravity waves. *Phil. Trans. R. Soc. Lond.* A **370**, 1572–1586.

CLAMOND, D. 2013 New exact relations for easy recovery of steady wave profiles from bottom pressure measurements. *J. Fluid Mech.* **726**, 547–558.

CLAMOND, D. & CONSTANTIN, A. 2013 Recovery of steady periodic wave profiles from pressure measurements at the bed. *J. Fluid Mech.* **714**, 463–475.

CONSTANTIN, A. 2006 The trajectories of particles in Stokes waves. *Invent. Math.* **166**, 523–535.

CONSTANTIN, A. 2011 In *Nonlinear Water Waves with Applications to Wave-Current Interactions and Tsunamis* CBMS-NSF Reg. Conf. Ser. Appl. Math., vol. 81, SIAM.

CONSTANTIN, A. 2012 On the recovery of solitary wave profiles from pressure measurements. *J. Fluid Mech.* **699**, 373–384.

CONSTANTIN, A. 2013 Mean velocities in a Stokes wave. *Arch. Rat. Mech. Anal.* **207**, 907–917.

CONSTANTIN, A. 2014 Stokes waves in water with a non-flat bed. *J. Fluid. Mech.* **740**, 17–27.

CONSTANTIN, A. & STRAUSS, W. 2010 Pressure beneath a Stokes wave. *Commun. Pure Appl. Maths* **63**, 533–557.

DECONINCK, B., OLIVERAS, K. L. & VASAN, V. 2012 Relating the bottom pressure and the surface elevation in the water wave problem. *J. Nonlinear Math. Phys.* **19**, 1240014.

ESCHER, J. & SCHLURMANN, T. 2008 On the recovery of the free surface from the pressure within periodic travelling water waves. *J. Nonlinear Math. Phys.* **15**, 50–57.

JOHNSON, R. S. 1997 *A Modern Introduction to the Mathematical Theory of Water Waves*. Cambridge University Press.

JONES, N. L. & MONISMITH, S. G. 2007 Measuring short-period wind waves in a tidally forced environment with a subsurface pressure gauge. *Limnol. Oceaogr.: Methods* **5**, 317–327.

OLIVERAS, K. L., VASAN, V., DECONINCK, B. & HENDERSON, D. 2012 Recovering the water-wave profile from pressure measurements. *SIAM J. Appl. Maths* **72**, 897–918.

TSAI, C. -H., HUNAG, M. -C., YOUNG, F. -J., LIN, Y. -C. & LI, H. W. 2005 On the recovery of surface wave by pressure transfer function. *Ocean Engng* **32**, 1247–1259.

UMEYAMA, M. 2012 Eulerian–Lagrangian analysis for particle velocities and trajectories in a pure wave motion using particle image velocimetry. *Phil. Trans. R. Soc. Lond.* A **370**, 1687–1702.

https://doi.org/10.1017/jfm.2014.81 Published online by Cambridge University Press