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Equivariant K-theory of compact Lie groups with involution

by

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Abstract

For a compact simply connected simple Lie group G with an involution α, we compute the $G \rtimes \mathbb{Z}/2$-equivariant K-theory of G where G acts by conjugation and $\mathbb{Z}/2$ acts either by α or by $g \mapsto \alpha(g)^{-1}$. We also give a representation-theoretic interpretation of those groups, as well as of $K_G(G)$.

Key Words: equivariant K-theory, stable homotopy theory, compact Lie groups, compact symmetric spaces, representations of loop groups.

Mathematics Subject Classification 2010: 19L47, 55N15, 55P91, 53C35.

1. Introduction

Brylinski and Zhang [2] computed, for (say) a simple compact Lie group G, the G-equivariant K-theory ring $K_G(G)$ with G acting on itself by conjugation, as the ring $\Omega_{R(G)/\mathbb{Z}}$ of Kähler differentials of $R(G)$ over $\mathbb{Z}$ (see also Adem, Gomez [1] for related work). Let α be an involutive automorphism of G. Then we can consider actions of $G \rtimes \mathbb{Z}/2$ on G where G acts by conjugation and the generator of $\mathbb{Z}/2$ acts either by the automorphism α or by the map $\gamma : g \mapsto \alpha(g)^{-1}$. The main result of the present paper is a computation of $K_{G\rtimes\mathbb{Z}/2}(G)$ in both cases as a module over $K_{G\rtimes\mathbb{Z}/2}(*) = R(G \rtimes \mathbb{Z}/2)$. (Here $R(H)$ is the complex representation ring of a compact Lie group H.)

The involutive automorphism α determines a compact symmetric space G/G^α, and we were originally interested in these computations as a kind of topological invariant of symmetric pairs of compact type. (Recall, in effect, that G/G^α is a connected component of G^γ via the embedding $x \mapsto \alpha(x)x^{-1}$.) It turns out, however, that the groups $K_{G\rtimes\mathbb{Z}/2}(G)$ are a rather crude invariant of symmetric pairs, since they essentially only depend on whether α is an outer or inner automorphism of G; if α is an inner automorphism, $G \rtimes \mathbb{Z}/2$ becomes a central product, which behaves essentially the same as the direct product from our point of view.

Nevertheless, having a complete calculation is still interesting, as are some of the methods involved in it. The main ingredient of the method we present here is the

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construction of Brylinski-Zhang [2, 3] of the element $dv \in \Omega_{R(G)/\mathbb{Z}} = K_G(G)$ for a finite-dimensional complex representation v of G. That construction, unfortunately, was presented incorrectly in [2] (in fact, the elements written there are 0), and so we developed an alternate construction of those elements using induction from the normalizer of a maximal torus. However, Brylinski [3] communicated the correct construction to us. The construction [3] is completely geometric, and supersedes our previous induction method (which, for that reason, we omit from this presentation). In fact, the construction [3] turns out to be equivariant with respect to both the α and γ actions. This allows an "obvious guess" of what $K_{G\rtimes\mathbb{Z}/2}(G)$ should be. We validate that guess following the methods of Greenlees and May [9], involving Tate cohomology. (Essentially, the main point is that under suitable finiteness hypothesis, a $\mathbb{Z}/2$-equivariant map of $\mathbb{Z}/2$-CW complexes which is an equivalence non-equivariantly is an equivalence equivariantly because the Tate cohomology can be computed as an algebraic functor of the "geometric fixed points".)

We realized, however, that the construction [3] can be generalized to give a representation-theoretical interpretation of the groups $K_G(G)$, $K_{G\rtimes\mathbb{Z}/2}(G)$. Such an interpretation is strongly motivated by the work of Freed, Hopkins and Teleman [6] who showed that if τ is a regular G-equivariant twisting of K-theory on G, then the twisted equivariant K-theory $K_{G,\tau}(G)$ is isomorphic to the free abelian group on irreducible projective representations of level $\tau - h^\vee$ (where $h^\vee$ is the dual Coxeter number) of the loop group LG.

This suggests that untwisted K-theory $K_G(G)$ should correspond to representations at the critical level of the Lie algebra $L\mathfrak{g}$. We found that this is indeed true, but the representations one encounters are not lowest weight representations (which occur, for example, in the geometric Langlands program). Instead, the fixed point space of the infinite loop space $K_G(G)_0$ turns out to be the group completion of the space of *finite* representations of the loop group LG with an appropriate topology. Here by a finite representation we mean a finite-dimensional representation which factors through a projection $LG \to G^n$ given by evaluation at finitely many points (cf. [16]). (It is possible to conjecture that every finite-dimensional representation of LG is finite, although it may depend on the type of loops we consider; in this paper, we restrict our attention to continuous loops.) In fact, we also prove that this is true $\mathbb{Z}/2$-equivariantly with respect to involutions, i.e. that the fixed point space of $K_{G\rtimes\mathbb{Z}/2}(G)_0$ is the group completion of the space of representations of $LG \rtimes \mathbb{Z}/2$, where $\mathbb{Z}/2$ acts on LG via its action on G in the case of $\mathbb{Z}/2$ acting on G by α, and simultaneously on G and on the loop parameter by reversal of direction in the case when $\mathbb{Z}/2$ acts on G by γ.

The present paper is organized as follows: In Section 2, we review the construction of Brylinski-Zhang [2, 3] and study its properties with respect to the

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involution on G. In Section 3, we compute the $R(G \rtimes \mathbb{Z}/2)$-modules $K^*_{G\rtimes\mathbb{Z}/2}(G)$. In Section 4, we discuss the computation in more concrete terms, and give some examples. In Section 5, we give an interpretation of $K^*_G(G)$ in terms of representation of the loop group LG, and in Section 6, we make that interpretation $\mathbb{Z}/2$-equivariant, thus extending it to $K^*_{G\rtimes\mathbb{Z}/2}(G)$.

2. The Brylinski-Zhang construction

Let G be a simply connected compact Lie group, T a maximal torus, N its normalizer, W the Weyl group. Let $R(G)$ denote, as usual, the complex representation ring. Recall that if $u_1,\ldots,u_n$ are the fundamental weights of G ($n = rank(G)$), then the weight lattice $T^* = Hom(T,S^1)$ is freely generated by $u_1,\ldots,u_n$ and we have

$$R(T) = \mathbb{Z}[u_1,u_1^{-1},\ldots,u_n,u_n^{-1}],$$

$$R(T) \supset R(G) = \mathbb{Z}[\overline{u_1},\ldots,\overline{u_n}]$$

where $\overline{u_i}$ is the sum of elements of the W-orbit of u_i.

Let, for a map of commutative rings $S \to R$, $\Omega_{R/S}$ denote the ring of Kähler differentials of R over S. Then one easily sees by the Künneth theorem that we have an isomorphism:

$$\begin{aligned} K^*_T(T) &= \Omega_{R(T)/\mathbb{Z}} \\ &= \mathbb{Z}[u_1,u_1^{-1},\ldots,u_n,u_n^{-1}] \otimes \Lambda_{\mathbb{Z}}[du_1,\ldots,du_n]. \end{aligned} \tag{1}$$

Theorem 1 *(Brylinski-Zhang [2]) Suppose G acts on itself by conjugation. Then there is a commutative diagram of rings*

$$\begin{array}{ccc} K_G(G) & \xrightarrow{res.} & K_T(T) \\ \cong\downarrow & & \downarrow\cong \\ \Omega_{R(G)/\mathbb{Z}} & \longrightarrow & \Omega_{R(T)/\mathbb{Z}} \\ =\downarrow & & \downarrow= \\ \mathbb{Z}[\overline{u_1},\ldots,\overline{u_n}] \otimes \Lambda[d\overline{u_1},\ldots,d\overline{u_n}] & \xrightarrow{\subset} & \mathbb{Z}[u_1^{\pm 1},\ldots,u_n^{\pm 1}] \otimes \Lambda[du_1,\ldots,du_n]. \end{array} \tag{2}$$

Moreover, the isomorphisms in (2) can be chosen in such a way that the generator $dW \in K_G(G)$ for a complex (finite-dimensional) G-representation W is represented in G-equivariant K-theory by a complex of G-bundles

$$G \times \mathbb{R} \times W \xrightarrow{\phi} G \times \mathbb{R} \times W \tag{3}$$

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where ϕ is a homomorphism which is iso outside of $G \times \{0\} \times W$, and is given by

$$\begin{aligned} \phi(g,t,w) &= (g,t,tw) \qquad \text{for } t < 0 \\ \phi(g,t,w) &= (g,t,-tg(w)) \quad \text{for } t \geq 0. \end{aligned} \tag{4}$$

Proof: The proof of the diagram (2) is given in [2]. Since there is a mistake in the formula (4) in [2], (corrected in [3]), we give a proof here. In view of the commutativity of diagram (2), and injectivity of the diagonal arrows, it suffices to prove the statement for T instead of G. By Künneth's theorem, it suffices to further consider $T = S^1$.

In the case $T = S^1$, let z be the tautological 1-dimensional complex representation of S^1 (considered as the unit circle in $\mathbb{C}$). Then the element of $\widetilde{K}^{-1}_{S^1}(S^1)$ given by (4) for $W = z^n$ is equal to the element of $\widetilde{K}^0_{S^1}(S^2)$ given by $H^n - 1$ where H is the tautological bundle on $S^2 = \mathbb{C}P^1$ (with trivial action of S^1). But it is well known that

$$H = u + 1$$

where $u \in \widetilde{K}^0_{S^1}(S^2)$ is the Bott periodicity element, and thus, (recalling that $u^2 = 0$),

$$H^n - 1 = (u+1)^n - 1 = nu.$$

Thus, choosing the Bott element as u gives the required isomorphism in the right hand column of (2) for $T = S^1$, and thus the statement follows. □

Proposition 2 *Let G be as above, let α be an involutive automorphism of G, and let W be a finite-dimensional complex representation such that*

$$\alpha^*(W) \cong W \tag{5}$$

(where $\alpha^*(W)$ is the representation of G on W composed with the automorphism α). Then, given the choices described in Theorem 1, if the generator a of $\mathbb{Z}/2$ acts on G by α, then dW is in the image of the restriction (forgetful map)

$$K^1_{G \rtimes \mathbb{Z}/2}(G) \to K^1_G(G). \tag{6}$$

When the generator a of $\mathbb{Z}/2$ acts on G by γ, dW is in the image of the restriction (forgetful map)

$$K^A_{G \rtimes \mathbb{Z}/2}(G) \to K^1_G(G) \tag{7}$$

where A is the 1-dimensional real representation of $G \rtimes \mathbb{Z}/2$ given by the sign representation of the quotient $\mathbb{Z}/2$.

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Proof: Recall that when (5) holds, then a choice of the isomorphism (5) can be made to give a representation of $G \rtimes \mathbb{Z}/2$ on W. Moreover, there are precisely two such choices, differing by tensoring by the complex 1-dimensional sign representation of $\mathbb{Z}/2$.

Consider first the case when the generator a of $\mathbb{Z}/2$ acts on G by α. Then consider the $\mathbb{Z}/2$-action on the Brylinski-Zhang construction

$$\begin{array}{ccc} G \times \mathbb{R} \times W & \xrightarrow{\phi} & G \times \mathbb{R} \times W \\ \downarrow & & \downarrow \\ G \times \mathbb{R} \times W & \xrightarrow{\phi} & G \times \mathbb{R} \times W \end{array} \tag{8}$$

where the generator of $\mathbb{Z}/2$ acts by

$$\begin{array}{cc} (g,t,w) & (g,t,w) \\ \downarrow & \downarrow \\ (\alpha(g),t,\alpha(w)) & (\alpha(g),t,\alpha(w)). \end{array}$$

When the generator a of $\mathbb{Z}/2$ acts on G by γ, consider the $\mathbb{Z}/2$-action (8) where the generator of $\mathbb{Z}/2$ acts by

$$\begin{array}{cc} (g,t,w) & (g,t,w) \\ \downarrow & \downarrow \\ (\alpha(g)^{-1},-t,\alpha(w)) & (\alpha(g)^{-1},-t,\alpha(g^{-1}w)). \end{array}$$

(Note that $\alpha(\alpha(g)\alpha(g^{-1}w)) = w$, so the action of a on the right hand side is involutive. One readily sees that it also intertwines the action of G via the automorphism α.)

To verify that the homomorphism ϕ commutes with the involution in the case of a acting on G via γ, since we already know the action is involutive, it suffices to consider $t < 0$. In this case, we have

$$\begin{array}{ccc} (g,t,w) & \longmapsto & (g,t,tw) \\ \downarrow & & \downarrow \\ (\alpha(g)^{-1},-t,\alpha(w)) & \longmapsto & (\alpha(g)^{-1},-t,t\alpha(g)^{-1}\alpha(w)). \end{array}$$

□

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3. The computation of equivariant K-theory

In this section, we will compute $K_{G\rtimes\mathbb{Z}/2}(G)$ where the generator a of $\mathbb{Z}/2$ acts by α or γ. First observe that in both cases, the generator a of $\mathbb{Z}/2$ acts on $K_G^*(G) \cong \Omega_{R(G)/\mathbb{Z}}$ by automorphisms of rings. The action on $R(G)$ is given by a permutation representation given by the permutation of irreducible representations by the automorphism α. Alternately, one may think in terms of the action of α on Weyl group orbits of weights. Let $u_1,\dots,u_n$ be the fundamental weights of the simply connected group G determined by the Lie algebra $\mathfrak{g}$. Let σ be the involution on $\{1,\dots,n\}$ given by

$$\alpha^*\overline{u_i} = \overline{u_{\sigma(i)}}.$$

Consider now the short exact sequence

$$1 \to G \to G \rtimes \mathbb{Z}/2 \to \mathbb{Z}/2 \to 1. \tag{9}$$

By $\mathbb{Z}/2_+$, we shall mean the suspension spectrum of the $G \rtimes \mathbb{Z}/2$-space $\mathbb{Z}/2_+$ by the action (9). We define S^A by the cofibration sequence

$$\mathbb{Z}/2_+ \xrightarrow{\iota} S^0 \to S^A$$

where S^0 is the $G\rtimes\mathbb{Z}/2$-sphere spectrum and ι is the collapse map (for terminology, see [13]). We have, of course,

$$K^*_{G\rtimes\mathbb{Z}/2}S^0 = R(G \rtimes \mathbb{Z}/2)_{even},$$

$$K^*_{G\rtimes\mathbb{Z}/2}\mathbb{Z}/2_+ = R(G)_{even}.$$

Here, the subscript $?_{even}$ means that the given $R(G \rtimes \mathbb{Z}/2)$-module is located in the even dimension of the $\mathbb{Z}/2$-graded ring K^*. Furthermore, we have an exact sequence

$$0 \to K^0_{G\rtimes\mathbb{Z}/2}S^A \to R(G \rtimes \mathbb{Z}/2) \to R(G) \to K^1_{G\rtimes\mathbb{Z}/2}S^A \to 0$$

where the middle arrow is restriction. Therefore, $K^1_{G\rtimes\mathbb{Z}/2}S^A$ is the free abelian group on irreducible G-representations which do not extend to $G \rtimes \mathbb{Z}/2$. Recall that $\mathbb{Z}/2$ acts on the set of isomorphism classes of irreducible representations of G; $R(G \rtimes \mathbb{Z}/2)$ is the free abelian group on the regular orbits, and on two copies of each fixed orbit. Therefore, $K^0_{G\rtimes\mathbb{Z}/2}S^A$ can be thought of as the free abelian group on irreducible G-representations which do extend to $G\rtimes\mathbb{Z}/2$-representations. Equivalently,

$$K^0_{G\rtimes\mathbb{Z}/2}S^A = \mathbb{Z}\{u \in T^* \text{ dominant} \mid \alpha^*\overline{u} = \overline{u}\},$$

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$$K^1_{G\rtimes\mathbb{Z}/2}S^A = \mathbb{Z}\{\text{regular } \alpha^*\text{-orbits of dominant weights}\}.$$

Let $S^{(\epsilon)}$ for $\epsilon \in \mathbb{Z}$ denote $S^{A-1} = \Sigma^{-1}S^A$ resp. S^0 depending on whether ϵ is odd or even.

Let $\succ$ denote any chosen linear ordering of the set of subsets of $\{1,\ldots,n\}$. Let I_σ be the set of subsets

$$\{i_1 < \cdots < i_k\} \subseteq \{1,\ldots,n\} \tag{10}$$

such that

$$\{\sigma(i_1),\ldots,\sigma(i_k)\} \succ \{i_1,\ldots,i_k\}$$

and let J_σ be the set of subsets (10) such that

$$\{\sigma(i_1),\ldots,\sigma(i_k)\} = \{i_1,\ldots,i_k\}.$$

Let $orb(S)$ for a σ-invariant set S denote the number of regular (=$\mathbb{Z}/2$-free) σ-orbits of S when a acts on G by α, and of all σ-orbits of S when a acts on G by γ.

Theorem 3 *There exists an isomorphism of $R(G\rtimes\mathbb{Z}/2)$-modules*

$$\begin{aligned} &K^*_{G\rtimes\mathbb{Z}/2}(G) \cong \\ &K^*_{G\rtimes\mathbb{Z}/2}\Big(\bigvee_{\{i_1<\cdots<i_k\}\in I_\sigma} \Sigma^k\mathbb{Z}/2_+ \vee \bigvee_{\{i_1<\cdots<i_k\}\in J_\sigma} \Sigma^k S^{(orb\{i_1,\ldots,i_k\})}\Big) \end{aligned} \tag{11}$$

where $G\rtimes\mathbb{Z}/2$ acts on the wedge summands on the right hand side of (11) through the projection to $\mathbb{Z}/2$.

Proof: We first construct a $G\rtimes\mathbb{Z}/2$-equivariant stable map u_S of each wedge summand of (11) into the E_∞-algebra

$$F(\Lambda_+, K_{G\rtimes\mathbb{Z}/2}) \tag{12}$$

where Λ is G on which $g \in G \subset G\rtimes\mathbb{Z}/2$ acts by conjugation and α acts by γ such that the wedge of all the maps (12) induces an isomorphism on G-equivariant coefficients. Here $F(?,?)$, as usual, denotes the (equivariant) function spectrum, see [13].

First, recall the isomorphism [2]

$$\begin{aligned} &\pi_*(F(\Lambda_+, K_{G\rtimes\mathbb{Z}/2})^G) = K^*_G(G) \\ &\cong \Omega_{R(G)/\mathbb{Z}} = \mathbb{Z}[\overline{u_1},\ldots,\overline{u_n}]\otimes\Lambda[d\overline{u_1},\ldots,d\overline{u_n}] \end{aligned} \tag{13}$$

of Theorem 1, induced by (1).

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For regular (=σ-free) orbits, the map we need follows from G-equivariant considerations: Send, G-equivariantly,

$$S^k \to F(\Lambda_+, K_{G\rtimes\mathbb{Z}/2})$$

by the generator

$$d\overline{u_{i_1}} \wedge \cdots \wedge d\overline{u_{i_k}}$$

of (13) and then use the fact that $(G \rtimes \mathbb{Z}/2)\rtimes?$ is the left adjoint to the forgetful functor from $G \rtimes \mathbb{Z}/2$-spectra to G-spectra (cf. [13]).

Next, for σ-invariant sets $1 \leq i_1 < \cdots < i_k \leq n$ which consist of a single orbit, we have $k \leq 2$. If $k = 1$, the map follows from Proposition 2. If $k = 2$, we have a G-equivariant map

$$u : S^1 \to F(\Lambda_+, K_{G\rtimes\mathbb{Z}/2})$$

given as the generator $d\overline{u_{i_1}} \wedge d\overline{u_{i_2}}$ of $\pi_*(F(\Lambda_+, K_{G\rtimes\mathbb{Z}/2})^G) = K_G^*(G)$. The $G \rtimes \mathbb{Z}/2$-equivariant map

$$u_{\{i_1,i_2\}} : S^{1+A} \to F(\Lambda_+, K_{G\rtimes\mathbb{Z}/2})$$

we seek may then be defined as

$$N_G^{G\rtimes\mathbb{Z}/2} u$$

where N is the multiplicative norm (see [8, 11]). Finally, we may define

$$u_{S_1 \amalg \cdots \amalg S_\ell} := u_{S_1} \wedge \cdots \wedge u_{S_\ell},$$

using Bott periodicity to identify S^2 with S^{2A}. Thus, taking a wedge sum of these maps, we have a map

$$\begin{gathered} X := F(\bigvee \Sigma^k \mathbb{Z}/2_+ \vee \bigvee \Sigma^k S^{(orb\{i_1,\ldots,i_k\})}, K_{G\rtimes\mathbb{Z}/2}) \\ \downarrow f \qquad (14) \\ Y := F(\Lambda_+, K_{G\rtimes\mathbb{Z}/2}) \end{gathered}$$

of $K_{G\rtimes\mathbb{Z}/2}$-modules, inducing an isomorphism of G-equivariant coefficients (using the Wirthmüller isomorphism [13] and, again, Bott periodicity). This implies that (14) induces an equivalence on Borel cohomology:

$$F(E\mathbb{Z}/2_+, f^G) : F(E\mathbb{Z}/2_+, X^G)^{\mathbb{Z}/2} \xrightarrow{\sim} F(E\mathbb{Z}/2_+, Y^G)^{\mathbb{Z}/2}. \qquad (15)$$

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We need to conclude that (14) induces an equivalence on $G \rtimes \mathbb{Z}/2$-fixed points, i.e. that we have an equivalence

$$(f^G)^{\mathbb{Z}/2} : (X^G)^{\mathbb{Z}/2} \xrightarrow{\sim} (Y^G)^{\mathbb{Z}/2}. \tag{16}$$

To this end, consider X^G, Y^G as $\mathbb{Z}/2$-equivariant spectra.

Lemma 4 *Denote*

$$R = R_G := (R(G \rtimes \mathbb{Z}/2)/ind_G^{G\rtimes\mathbb{Z}/2} R(G)) \otimes \mathbb{Q},$$

$$\widehat{R} = \widehat{R_G} := (R(G \rtimes \mathbb{Z}/2)/ind_G^{G\rtimes\mathbb{Z}/2} R(G))_2^\wedge \otimes \mathbb{Q}.$$

Then for the $\mathbb{Z}/2$-spectra $Z = X^G, Y^G$, the spectra $\Phi^{\mathbb{Z}/2}Z$, $\widehat{Z}$ (see [13, 9]) are rational, and we have an isomorphism

$$\widehat{Z}_* \cong (\Phi^{\mathbb{Z}/2}Z)_* \otimes_R \widehat{R} \tag{17}$$

natural with respect to the map (14). (Here $ind_G^{G\rtimes\mathbb{Z}/2} : R(G) \to R(G \rtimes \mathbb{Z}/2)$ denotes the induction, and $(?)_2^\wedge$ denotes completion at 2.)

Proof of (16) using Lemma 4: Note that (14) also implies an equivalence on Borel homology:

$$(E\mathbb{Z}/2_+ \wedge f^G) : (E\mathbb{Z}/2_+ \wedge X^G)^{\mathbb{Z}/2} \xrightarrow{\sim} (E\mathbb{Z}/2_+ \wedge Y^G)^{\mathbb{Z}/2} \tag{18}$$

and Tate cohomology

$$\widehat{f^G} : \widehat{X^G} \xrightarrow{\sim} \widehat{Y^G}. \tag{19}$$

By (17), however, the map

$$\Phi^{\mathbb{Z}/2} f^G : \Phi^{\mathbb{Z}/2}(X^G) \to \Phi^{\mathbb{Z}/2}(Y^G)$$

is also an equivalence, and together with (18), this implies (16). □

Proof of Lemma 4: The spectra $\Phi^{\mathbb{Z}/2}(M)$, $\widehat{M}$ are rational for any cell module M over the E_∞-ring spectrum $K_{\mathbb{Z}/2}$ by a theorem of Greenless and May [9] which asserts this for $M = K_{\mathbb{Z}/2}$. Additionally, the methods of [9] (or a direct calculation) readily imply that

$$\begin{aligned} \Phi^{\mathbb{Z}/2}(K^G_{G\rtimes\mathbb{Z}/2})_* &= R, \\ \widehat{(K^G_{G\rtimes\mathbb{Z}/2})}_* &= \widehat{R}. \end{aligned} \tag{20}$$

Now $? \otimes_R \widehat{R}$ is clearly an exact functor on R-modules, so by using the long exact sequence in cohomology, it suffices to filter X^G, Y^G both into finite sequences of cofibrations such that the quotients Z satisfy (17).

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In the case of X^G, the quotients are either of the form $F(\mathbb{Z}/2_+, K^G_{G\rtimes\mathbb{Z}/2})$ for which the statement is trivial (both the geometric and Tate theory are 0) or $K^G_{G\rtimes\mathbb{Z}/2}$, which is covered by (20), or $\Sigma^A(K^G_{G\rtimes\mathbb{Z}/2})$, which is a cofiber of modules of the first two types.

In the case of Y^G, use the decomposition of Λ into G-orbits with respect to conjugation of skeleta of the fundamental alcove, applying also the fact that γ acts trivially on T. This is, in fact, a $G\rtimes\mathbb{Z}/2$-CW decomposition, where the cells are of type $H\rtimes\mathbb{Z}/2$ where H is a compact Lie subgroup of G associated to a sub-diagram of the affine Dynkin diagram.

Applying the computation of geometric and Tate $\mathbb{Z}/2$-fixed points of K_H, we are done if we can prove

$$\widehat{R_H} = R_H \otimes_{R_G} \widehat{R_G}. \tag{21}$$

To this end, put

$$R^0_H := R(H\rtimes\mathbb{Z}/2)/ind_H^{H\rtimes\mathbb{Z}/2}R(H).$$

Recall from [17] that $R(G\rtimes\mathbb{Z}/2)$ is a Noetherian ring and $R(H\rtimes\mathbb{Z}/2)$ is a finite module. Therefore, R^0_H is a finite R^0_G-module.

Now for any Noetherian ring P and a finite P-module M, we have

$$M^\wedge_2 = M\otimes_P P^\wedge_2. \tag{22}$$

(Consider the presentation

$$\bigoplus_n P \to \bigoplus_m P \to M \to 0$$

and right exactness of $(?)^\wedge_2$ in this case.) Rationalizing, (22) implies (21). □

4. Concrete computations and examples

Let, again, G be a simply connected simple compact Lie group and the generator a of $\mathbb{Z}/2$ act on the target G by α or by γ. To calculate $K_{G\rtimes\mathbb{Z}/2}(G)$ as an $R(G\rtimes\mathbb{Z}/2)$-module, in view of Theorem 3, it suffices to calculate the action of the automorphism α on the Weyl group orbits of the fundamental weights of the group G.

The key observation is that if α is an inner automorphism, then the action is trivial simply because an inner automorphism does not change the isomorphism class of a representation.

Outer automorphisms of simply connected simple Lie groups correspond to automorphisms of the Dynkin diagram, and therefore are necessarily trivial for all types except A, D and E_6. Furthermore, the permutation representation of $\mathbb{Z}/2$

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on orbits of fundamental weights is isomorphic to the permutation representation on the set of simple roots (using the bijection between fundamental weights and simple roots: A fundamental weight is a point of the weight lattice with which one simple root has minimal positive inner product, and the other simple roots have inner product 0).

Recall also that an automorphism α of a semisimple Lie algebra $\mathfrak{g}$ is outer if and only if $rank(\mathfrak{g}^\alpha) < rank(\mathfrak{g})$ (cf., [10]). From the point of view of symmetric pairs $(\mathfrak{g},\mathfrak{g}^\alpha)$ of compact type, we refer to the classification of such pairs (cf., [10], pp. 532-534).

For types AI and AII (corresponding to compact simply connected symmetric spaces $SU(n)/SO(n)$, $Sp(2n)/SO(n)$), the automorphism is outer, so the simple weights $v_1,\ldots,v_{n-1}$ are transformed by

$$\alpha(\overline{v_i}) = \overline{v_{n-i}}.$$

For types $AIII$ and AIV (corresponding to simply connected compact symmetric spaces $U(p+q)/U(p)\times U(q)$, the automorphism α is inner, so all the fundamental weights are fixed.

For types DI-$DIII$ (corresponding to compact simply connected symmetric spaces of type $SO(p+q)/SO(p)\times SO(q)$ with $p+q$ even), the automorphism α is outer if p (or, equivalently, q) is odd, in which case α interchanges the two fundamental weights corresponding to the spin representations. When p (or, equivalently, q) is even, the automorphism α is inner and thus, again, the action is trivial.

For E_6, the four different compact simply connected symmetric spaces are EI, EII, $EIII$, EIV. The automorphism α is outer for EI and EIV, interchanging two pairs of fundamental weights (and leaving two fundamental weights fixed). For EII, $EIII$, the automorphism is inner and all the fundamental weights are fixed.

4.1. Two explicit examples

Let us work out explicitly the cases of $G = SU(2)$, $G = SU(3)$ where the involution α is of type AI, and a acts on G by γ. In the case of $G = SU(2)$, denote by x the fundamental weight and by z the representation with character $x + x^{-1}$, as well as its chosen extension to $SU(2)\rtimes\mathbb{Z}/2$. Denote further by q the complex sign representation of $\mathbb{Z}/2$. Then

$$K^*_{SU(2)\rtimes\mathbb{Z}/2}(S^0) = R(SU(2)\rtimes\mathbb{Z}/2) = \mathbb{Z}[z,q]/(q^2-1)_{even}$$

and

$$K^*_{SU(2)\rtimes\mathbb{Z}/2}(S^A) = Ker(res : R(SU(2)\rtimes\mathbb{Z}/2) \to R(SU(2))) = \mathbb{Z}[z]_{even},$$

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generated by $q-1 \in R(SU(2) \rtimes \mathbb{Z}/2)$. The argument of K-theory on the right hand side of (11) is

$$S^0 \vee S^A.$$

Thus, we have

$$K^*_{SU(2)\rtimes\mathbb{Z}/2}(SU(2)) = (\mathbb{Z}[z,q]/(q^2-1) \oplus \mathbb{Z}[z])_{even}$$

as a $\mathbb{Z}[z,q]/(q^2-1)$-module.

For $G = SU(3)$, we have

$$K^*_{SU(3)}(*) = R(SU(3)) = \mathbb{Z}[z,t]$$

where z,t are sums of orbits of the two fundamental weights, which are formed by vertices of the two smallest equilateral triangles with center 0 in the honeycomb lattice. We have

$$K^*_{SU(3)\rtimes\mathbb{Z}/2}(*) = R(SU(3) \rtimes \mathbb{Z}/2) = \mathbb{Z}[\sigma_1,\sigma_2,q]/(q^2-1,(q-1)\sigma_1)$$

where σ_i are the elementary symmetric polynomials in z,t and q is the complex sign representation of $\mathbb{Z}/2$ (more precisely, a non-canonical choice has to be made in lifting σ_2 to a representation of $SU(3) \rtimes \mathbb{Z}/2$, but that is not important for our calculation). To compute $K^*_{SU(3)\rtimes\mathbb{Z}/2}(S^A)$, consider the exact sequence

$$\begin{array}{c} 0 \\ \downarrow \\ \mathbb{Z}[\sigma_2] \\ \downarrow q-1 \\ \mathbb{Z}[\sigma_1,\sigma_2,q]/(q^2-1,(q-1)\sigma_1) \\ \downarrow q \mapsto 1 \\ \mathbb{Z}[z,t] \\ \downarrow \begin{array}{l} z \mapsto 1 \\ t \mapsto -1 \end{array} \\ \mathbb{Z}[\sigma_1,\sigma_2]\{z\} \\ \downarrow \\ 0. \end{array}$$

The middle arrow is the restriction

$$K^0_{SU(3)\rtimes\mathbb{Z}/2}(S^0) \to K^0_{SU(3)}(S^0),$$

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so the kernel resp. cokernel is $K^0_{SU(3)\rtimes\mathbb{Z}/2}(S^A)$ resp. $K^1_{SU(3)\rtimes\mathbb{Z}/2}(S^A)$. The argument of K-theory on the right hand side of (11) is

$$S^0 \vee \Sigma\mathbb{Z}/2_+ \vee S^{1+A},$$

so we have for the maximal rank pair $(\mathfrak{su}(3),\mathfrak{h})$:

$$K^0_{SU(3)\rtimes\mathbb{Z}/2}(SU(3)) = \mathbb{Z}[\sigma_1,\sigma_2,q]/(q^2-1,(q-1)\sigma_1)\oplus\mathbb{Z}[\sigma_1,\sigma_2],$$

$$K^1_{SU(3)\rtimes\mathbb{Z}/2}(SU(3)) = \mathbb{Z}[z,t]\oplus\mathbb{Z}[\sigma_2]$$

as $R(SU(3)\rtimes\mathbb{Z}/2) = \mathbb{Z}[\sigma_1,\sigma_2,q]/(q^2-1,(q-1)\sigma_1)$-modules.

5. Representation-theoretical interpretation of $K_G(G)$

Freed, Hopkins and Teleman [6] showed that for a "regular" twisting τ, $K^*_{G,\tau}(G)$ is the free abelian group on irreducible lowest weight representations of level $\tau - h^\vee$ of the universal central extension $\widetilde{LG}$ of the loop group LG. Therefore, 0 twisting (which is not regular) corresponds to the critical level.

We found that $K^*_G(G)$ is, indeed, related to representations of LG, but not lowest weight representations (in the sense that they would be quotients of the corresponding vertex algebra - see, e.g. [7]).

Instead, we encounter finite representations. Denote by $e_x : LG \to G$ the evaluation at a point $x \in S^1$. Call a finite-dimensional complex representation of LG *finite* if it factors through a projection of the form

$$e_{x_1} \times \cdots \times e_{x_n} : LG \to G^n. \tag{23}$$

These representations are briefly mentioned in the book [16]. It is possible to conjecture that all finite-dimensional representations of LG are finite, although this may depend on what kind of loops we consider; in this paper, we restrict attention to *continuous* loops with the compact-open topology.

Let us first define the *finite representation space* $Rep(\Gamma)$ of a topological group Γ. Since we are about to do homotopy theory, let us work in the category of compactly generated spaces. A finite-dimensional representation of Γ is a finite-dimensional complex vector space V together with a continuous homomorphism

$$\Gamma \to GL(V). \tag{24}$$

Continuous homomorphisms are, in particular, continuous maps. Thus, the set of all representations (24) for a fixed V forms a topological space, denoted by $Rep(\Gamma,V)$

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with respect to the compact-open topology made compactly generated (cf.[14]). Consider the topological category $C(\Gamma)$ (both objects and morphisms are compactly generated spaces, the source S and the target T are fibrations and Id is a cofibration) with objects

$$\coprod_V Rep(\Gamma, V) \tag{25}$$

and morphisms

$$\coprod_{V,W} Rep(\Gamma, V) \times GL(V, W) \tag{26}$$

(where, say, S is the projection and $T(\phi)$ is the representation on W given by conjugating the representation on V by ϕ; $GL(V,W) = \emptyset$ when $dim(V) \neq dim(W)$). Define the *representation space* $Rep(\Gamma)$ as the bar construction on the category $C(\Gamma)$.

For $\Gamma = LG$, let $Rep_0(LG, V)$ denote the subspace of $Rep(LG, V)$ (with the induced topology made compactly generated) consisting of finite representations. Let $C_0(LG)$ be the subcategory of $C(LG)$ defined by replacing $Rep(LG, V)$ with $Rep_0(LG, V)$, and let $Rep_0(LG)$ be the bar construction on $C_0(LG)$. Our first aim is to identify a group completion of the weak homotopy type of $C_0(LG)$ with $K_G(G)$. To this end, we need a few technical tools. First of all, let IG denote the group of continuous paths $\omega : [0,1] = I \to G$ with the compact-open topology, and define $C_0(IG)$, $Rep_0(IG)$ analogously with the above, replacing LG with IG.

Lemma 5 *The inclusion $G \to IG$ via constant maps induces a homotopy equivalence*

$$\iota : Rep(G) \to Rep_0(IG).$$

Proof: Define a map

$$\kappa : Rep_0(IG) \to Rep(G)$$

by composing with e_0. Then we have $\kappa\iota = Id$. Consider now the homotopy $h_t : IG \to IG$ given on $f : I \to G$ by $h_t(f)(x) = f(tx)$. One easily checks that h_t induces a homotopy on $Rep_0(IG)$ between $\iota\kappa$ and Id. □

Next, we define a simplicial symmetric monoidal category as follows: On the simplicial n-level, we take the category $Rep_0(IG \times G^n)$ by which we mean the subcategory of $Rep(IG \times G^n)$ on representations which factor through evaluation of IG on finitely many points. We let degeneracies be given by projection

$$IG \times G^{n+1} \to IG \times G^n,$$

and faces by the maps

$$IG \xrightarrow{ev_1} G \xrightarrow{\Delta} G \times G,$$

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$$G \xrightarrow{\Delta} G \times G,$$

$$IG \xrightarrow{ev_0} G \xrightarrow{\Delta} G \times G,$$

where Δ is the diagonal. It makes sense to denote this simplicial symmetric monoidal category by $CH_{\mathbb{C}_2}(C_0(IG), C(G))$ in reference to a "Hochschild homology complex", and its realization by

$$CH_{Rep(\{e\})}(Rep_0(IG), Rep(G)). \tag{27}$$

In fact, we will also be interested in the corresponding spaces

$$CH_{\mathbb{C}_2}(C(G), C(G)),$$

and their realizations

$$CH_{Rep(\{e\})}(Rep(G), Rep(G)) \tag{28}$$

defined analogously replacing IG by the subgroup of constant paths in G. Note also that both spaces (27), (28) are E_∞ spaces, since they are classifying spaces of symmetric monoidal categories. Lemma 5 then immediately extends to the following

Lemma 6 *Inclusion of constant loops induces a homotopy equivalence of E_∞ spaces*

$$CH_{Rep(\{e\})}(Rep_0(IG), Rep(G)) \to CH_{Rep(\{e\})}(Rep(G), Rep(G)). \tag{29}$$

Next, we prove

Lemma 7 *There is an equivalence of E_∞ spaces*

$$\Omega B CH_{Rep(\{e\})}(Rep(G), Rep(G)) \sim K_G(G)_0$$

where the subscript denotes infinite loop space, and by $K_G(G)$ we denote the spectrum of G-equivariant maps $F_G(G_+, K_G)$ (where, as before, on the source G acts by conjugation). (See [13] for the standard notation.)

Proof: We need a symmetric monoidal functor from the simplicial realization of the category $CH_{C(\{e\})}(C(G), C(G))$ to vector G-bundles on G. Since the source however has topologized objects, it is convenient to consider an equivalent model of the category of bundles where objects can vary continuously parametrized by a space. In more detail, we consider the category with both objects and morphisms topologized, where the objects are formed by the disjoint union of CW-complexes $X \times \{\xi\}$ where ξ is a G-bundle on $G \times X$. Morphisms are disjoint unions of spaces

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$\Gamma_{X,\xi,Y,\eta}$ consisting of triples $x \in X$, $y \in Y$ and isomorphisms $f : \xi|G \times \{x\} \to \eta|G \times \{y\}$ topologized so that the projection $\Gamma_{X,\xi,Y\eta} \to X \times Y$ is locally a product (which is done canonically by local triviality of ξ, η). This is a symmetric monoidal category (which we will denote by $Bun_G(G)'$ by Whitney sum of pullbacks of G-bundles over $G \times X$ and $G \times Y$ to a G-bundle over $G \times X \times Y$, and moreover a groupoid whose skeleton is the ordinary symmetric monoidal category $Bun_G(G)$ of G-bundles on G.

Then, if we denote by C the simplicial realization of the category $CH_{C(\{e\})}(C(G), C(G))$, it suffices to construct a symmetric monoidal functor $C \to Bun_G(G)'$, which can be constructed from a G-bundle on $G \times Obj(C)$ which satisfies the appropriate additivity and functoriality properties. To this end, it suffices to construct a functor, symmetric monoidal over G, of the form

$$G \times CH_{C(\{e\})}(C(G), C(G)) \to C(G), \tag{30}$$

where in the source, we consider the "total" category spanned by the level morphisms as well as the simplicial structure. Construct (30) as follows: on each level, put

$$(g, V) \mapsto V.$$

All faces and degeneracies are set to the identity, except the 0'th face on each level. The 0-face from level n to level $n-1$ is sent, at $g \in G$, to multiplication by

$$(1, g, \underbrace{1, \ldots, 1}_{n-1 \text{ times}}).$$

Applying the classifying space functor, this gives an E_∞-map from $CH_{Rep(G)}(Rep(G), Rep(G))$ to the space of G-equivariant vector bundles on G. Applying an infinite loop space machine and localization at the Bott element, we obtain a map which, up to homotopy, can be expressed as

$$CH_K(K_G, K_G) \to K_G(G) \tag{31}$$

where K_G denotes the K-module of G-fixed points of G-equivariant K-theory. Now there is a spectral sequence (coming from the simplicial structure) converging to the homotopy of the left hand side of (31) whose E_2-term is

$$HH_{\mathbb{Z}}(R(G), R(G)). \tag{32}$$

Note that the source and target coefficient rings of (31), as well as (32) are rings, where in fact the ring (32) is isomorphic to the homotopy ring of the target of (31), and the generators of (32) are permanent cycles, which map to the corresponding

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generators of $K_G^*(G)$ by Theorem (1). Thus, we are done if we can prove that (31) is a map of ring spectra.

In fact, one can rigidify (32) to become a map of E_∞ ring spectra. The functor from $CH_{Rep(G)}(Rep(G), Rep(G))$ to $Bun_G(G)'$ is a weak symmetric bimonoidal functor on each simplicial level, with the simplicial structure maps weakly preserving the structure. Thus, on the "totalized" category where we combine the levels and consider simplicial structure maps as morphisms, we obtain a weak symmetric bimonoidal functor into $Bun_G(G)'$. Applying the Elmendorf-Mandell machine [5] and localization at the Bott element, an E_∞ model of (31) follows. (The Joyal-Street construction on categories with both objects and morphisms topological [12] is also relevant.) □

Finally, using the map

$$LG \to IG \times G^n \tag{33}$$

given by $f \mapsto (f \circ \pi, f(0), \ldots, f(0))$ where $\pi : I \to I/0 \sim 1 = S^1$ is the projection, we obtain a map

$$p : CH_{Rep(G)}(Rep_0(IG), Rep(G)) \to Rep_0(LG). \tag{34}$$

Lemma 8 *The map p is an equivalence.*

Proof: First, we show that p is a quasi-fibration. We use the criterion in [4], which is restated as Theorem 2.6 of [15]. We first observe that $Rep_0(G)$ is a disjoint sum of connected components indexed by dimension d of the representation. We may consider one connected component at a time. For a given d, let the i'th (increasing) filtered part be spanned by all representations which are (up to isomorphism) of the form $V \otimes W$ where W has dimension $\geq n - i$ and factors through the projection

$$e_0 : LG \to G.$$

The open neighborhoods required in Theorem 2.6 of [15] are then spanned by representations of the form $V \otimes W$ where W is of dimension $\geq n - i$, and factors through the projection of LG to $Map(U, G)$ where U is an ϵ-neighborhood of 1 in S^1. The homotopies H_t and h_t are then defined by contracting U to 1.

Once we know that p is a quasifibration, the statement follows, as it is easily checked that the inverse image of every point is contractible. □

Putting together Lemmas 5, 6, 7, 8, we now obtain the following

Theorem 9 *The group completion of the E_∞ space $Rep_0(LG)$ is weakly equivalent to the infinite loop space $K_G(G)_0$.*

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6. Representation-theoretical interpretation of $K_{G\rtimes\mathbb{Z}/2}(G)$

There is also an equivariant version of these constructions with respect to a $\mathbb{Z}/2$-action where the generator of $\mathbb{Z}/2$ acts on G either by α or by γ. In these cases, we consider the topological group $LG \rtimes \mathbb{Z}/2$. $\mathbb{Z}/2$ acts on the loop $f : S^1 \to G$ by $f \mapsto g$ where in the case of action by α on the target, $g(t) = \alpha(f(t))$, and in the case of action by γ on the target, $g(t) = \alpha(f(1-t))$ (again, we use the identification $S^1 = I/0 \sim 1$). In both cases, *finite representations* are defined as finite-dimensional representations which factor through a projection to $G^n \rtimes \mathbb{Z}/2$ by evaluation at finitely many points (where, in the case of the γ-action on the target, with each evaluation point t, we must also include $1-t$). Now restricting, again, to finite representations, we obtain from $C(G \rtimes \mathbb{Z}/2)$ the categories $C_0(LG \rtimes \mathbb{Z}/2)$ and their classifying spaces $Rep_0(LH \rtimes \mathbb{Z}/2)$.

Next, we may also similarly define

$$C_0((IG \times G^n) \rtimes \mathbb{Z}/2), \tag{35}$$

and its classifying space

$$Rep_0(IG \times G^n) \rtimes \mathbb{Z}/2). \tag{36}$$

Here the generator a of $\mathbb{Z}/2$ acts on $f : I \to G$ by $f \mapsto g$ where $g(t) = \alpha(f(t))$, in the case of α-action on the target, a acts on each of the n copies of G separately by α, in the case of γ action on the target, a acts on G^n by $(g_1,\dots,g_n) \mapsto (\alpha(g_n),\dots,\alpha(g_1))$. In the case where a acts by α on the target G, (36) forms a simplicial category, and its classifying space (36) forms a simplicial space. In the case of a acting on the target G by γ, the $\mathbb{Z}/2$-action is an automorphism over the involution of the simplicial category reversing the order of each set $\{0,\dots,n\}$, so we can still form a "simplicial realization" by letting a act on each standard simplex Δ_n by $[t_0,\dots,t_n] \mapsto [t_n,\dots,t_0]$ in barycentric coordinates. In both cases, we denote the simplicial realizations

$$CH_{\mathbb{C}_2}(C_0I(G), C(G))_{\mathbb{Z}/2}$$

and

$$CH_{Rep(G)}(Rep_0(IG), Rep(G))_{\mathbb{Z}/2}.$$

Again, the precise application of the method of proof of Lemma 5 gives

Lemma 10 *Restriction to constant loops induces a homotopy equivalence*

$$CH_{Rep(G)}(Rep_0(IG), Rep(G))_{\mathbb{Z}/2} \to CH_{Rep(G)}(Rep(G), Rep(G))_{\mathbb{Z}/2}$$

(where the target is defined the same way as the source, restricting to constant maps $I \to G$).

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Next, the map (33) defines a map

$$p_{\mathbb{Z}/2} : CH_{Rep(G)}(Rep_0(IG), Rep(G))_{\mathbb{Z}/2} \to Rep_0(LG \rtimes \mathbb{Z}/2) \tag{37}$$

in both the cases of α and γ action on the target. Again, we have an equivariant analogue of Lemma 8:

Lemma 11 *The map $p_{\mathbb{Z}/2}$ is an equivalence.*

Proof: In both cases, the filtration and the deformations used in the proof of Lemma 8 have obvious $\mathbb{Z}/2$-equivariant analogous. Therefore, the same argument applies. □

We shall now construct an E_∞ map

$$CH_{Rep(G)}(Rep(G), Rep(G))_{\mathbb{Z}/2} \to K_{G\rtimes\mathbb{Z}/2}(G) \tag{38}$$

where by the right hand side, we mean the space of $G \rtimes \mathbb{Z}/2$-equivariant maps $G \to (K_{G\rtimes\mathbb{Z}/2})_0$ where the generator a of $\mathbb{Z}/2$ acts on G by either α or γ. To this end, again, it suffices to construct a symmetric monoidal map of the realization of the category

$$CH_{\mathbb{C}_2}(C(G), C(G))_{\mathbb{Z}/2}$$

to the category of $G \rtimes \mathbb{Z}/2$-vector bundles on G. In the case of a acting on the target G by α, the construction is directly analogous to Lemma 7.

In the case of a acting by γ, we must deal with the involutive automorphism of the simplicial category. It is most convenient to give this in the form of a $G \rtimes \mathbb{Z}/2$-equivariant bundle on

$$G \times CH_{Rep(G)}(Rep(G), Rep(G))_{\mathbb{Z}/2}.$$

We let the bundle be defined by the same formula as in the non-equivariant case, with $\mathbb{Z}/2$-action

$$\alpha(g, v, [t_0, \ldots, t_n]) = (\alpha(g)^{-1}, \alpha(g^n v), [t_n, \ldots, t_0])$$

(see the proof of Lemma 7).

Theorem 12 *The induced map*

$$\Omega B(CH_{Rep(G)}(Rep(G), Rep(G))_{\mathbb{Z}/2}) \to K_{G\rtimes\mathbb{Z}/2}(G)_0 \tag{39}$$

is an equivalence.

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Proof: It is useful to note that $Rep(G \rtimes \mathbb{Z}/2)$ is a $\mathbb{Z}/2$-equivariant symmetric monoidal category in the sense that there is a transfer functor

$$\bigoplus_{\mathbb{Z}/2} : Rep(G \rtimes \mathbb{Z}/2) \to Rep(G \rtimes \mathbb{Z}/2) \tag{40}$$

extending the commutativity, associativity and unitality axioms in the usual sense. The functor (40) is given by

$$V \mapsto V \oplus V \otimes A$$

where A is the sign representation of the quotient $\mathbb{Z}/2$. We may recover $Rep(G)$ as the image of the functor (40), and then the functor (40) provides a symmetric monoidal model of the restriction $Rep(G \rtimes \mathbb{Z}/2) \to Rep(G)$. Applying this construction level-wise on the simplicial level,

$$CH_{Rep(G)}(Rep(G), Rep(G))_{\mathbb{Z}/2} \to KG \rtimes \mathbb{Z}/2(G)_0$$

becomes a map of $\mathbb{Z}/2$-equivariant E_∞-spaces, and applying $\mathbb{Z}/2$-equivariant infinite loop space theory, and localizing at the Bott element, we obtain a map of $Zfrm{-}e$-equivariant spectra (indexed over the complete universe)

$$CH_K(K_G, K_G)_{\mathbb{Z}/2} \to K_{G \rtimes \mathbb{Z}/2}(G). \tag{41}$$

In fact, it is a map of $K_{\mathbb{Z}/2}$-modules. Forgetting the $\mathbb{Z}/2$-equivariant structure, we recover the map (31), which is, of course, an equivalence. To show that the map (41), which is a non-equivariant equivalence, is, in effect, a $\mathbb{Z}/2$-equivalence, we use a variant of the finiteness argument of the section 3, this time using finiteness directly over $K_{\mathbb{Z}/2}$.

Recall again that

$$R(G) \cong \mathbb{Z}[v_1, \ldots, v_r]$$

where v_i are the fundamental irreducible representations of G. Let us call a finite-dimensional complex representation of G *of degree* n if it is a sum of subrepresentations, each of which is isomorphic to a tensor product of n elements of $\{v_1, \ldots, v_r\}$ (an element is allowed to occur more than once). We see, in fact, that we obtain degree d versions of all the symmetric monoidal categories considered, giving a stable splitting of the form

$$K_G \simeq \bigvee_{d \geq 0} K_G^d,$$

$$CH_K(K_G, K_G) \simeq \bigvee_{d \geq 0} CH_K^d(K_G, K_G).$$

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Also, these splittings are $\mathbb{Z}/2$-equivariant (since $\mathbb{Z}/2$ acts on $\{v_1,\dots,v_r\}$ by permutation), and in fact the map (41) decomposes into maps

$$CH_K(K_G,K_G)_{\mathbb{Z}/2} \to K_{G\rtimes\mathbb{Z}/2}(G).$$

On the other hand, the Brylinski-Zhang construction restricts to a map of finite $K_{\mathbb{Z}/2}$-modules

$$\begin{aligned} K^d_{G\rtimes\mathbb{Z}/2} \wedge (& \bigvee_{\substack{1\le i_1<\cdots<i_k\le n \\ \sigma\{i_1,\dots,i_k\}\succ\{i_1,\dots,i_k\}}} \Sigma^k\mathbb{Z}/2_+ \\ \vee & \bigvee_{\substack{1\le i_1<\cdots<i_k\le n \\ \sigma\{i_1,\dots,i_k\}=\{i_1,\dots,i_k\}}} \Sigma^k S^{(orb\{i_1,\dots,i_k\})}) \to CH^d_K(K_G,K_G). \end{aligned} \tag{42}$$

Since each of $K_{\mathbb{Z}/2}$-modules are finite, an equivariant map which is a non-equivariant equivalence is an equivalence by the argument of Section 3 (in fact, made simpler by the fact that we do not consider G-equivariance here). Denoting the left hand side of (42) by $\mathcal{K}^d$, we now have a diagram of $K_{\mathbb{Z}/2}$-modules of the form

$$\begin{array}{ccc} \bigvee_{d\ge 0}\mathcal{K}^d & \xrightarrow{\sim} & CH_K(K_G,K_G)_{\mathbb{Z}/2} \\ & \searrow^{\sim} & \downarrow \\ & & K_{G\rtimes\mathbb{Z}/2}(G). \end{array} \tag{43}$$

The horizontal arrow is a $\mathbb{Z}/2$-equivalence by the fact that (42) is a $\mathbb{Z}/2$-equivalence and by the stable splitting, the diagonal arrow is a $\mathbb{Z}/2$-equivalence by Theorem 3. Therefore, the vertical arrow is a $\mathbb{Z}/2$-equivalence. □

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