HAMILTONIAN STRUCTURE AND STABILITY ANALYSIS FOR A PARTIALLY FILLED CONTAINER

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ABSTRACT

Hamiltonian system is a special case of dynamical system. Mostly it is used for potential shaping of mechanical systems stabilization. In our present work, we are using Hamiltonian dynamics to study and control the fuel slosh inside spacecraft tank. Sloshing is the phenomenon which is related with the movement of fluid inside a container in micro and macro scale as well. Sloshing of fluid occurs whenever the frequency of container movement matches with the natural frequency of fluid inside the container. Such type of synchronization may cause the structural damage or could be a reason of moving object's attitude disturbance. In spacecraft technology, the equivalent mechanical model for sloshing is common to use for the representation of fuel slosh. This mechanical model may contain a model of pendulum or a mass attached with a spring. In this article, we are using mass-spring mechanical model coupled with rigid body to derive the equations for Hamiltonian system. Casimir functions are used for proposed model. Conditions for the stability and instability of moving mass are derived using Lyapunov function along with Casimir functions. Simulation work is presented to strengthen the derived results and to distribute the stable and unstable regions graphically.

Keywords: Hamiltonian system, Fluid sloshing, Equivalent mechanical model, Attitude dynamics.

1. INTRODUCTION

Control dynamics is the key interest of astronautics and many other scientists related with the filed of mechanics. Movement of fluid and its properties has not only effects in daily life, [1,2], but also having influence in the field of technology. In last few decades, it is found that fuel movement inside the tank of a moving rigid body is very important for the stability and control. There are many engineering applications for the movement of free surface inside moving containers, such as trucks carrying oil tankers, fluid movement inside storage tanks caused by earth quake, sloshing of fluid inside cargo vessels in the ships and fuel sloshing inside the tanks of a spacecraft. Frequency disturbance and shape of container play important role in sloshing [3,4]. Spacecraft motion and stability is the main concern for the researchers. The increasing amount of fuel in spacecraft, liquid management and its sloshing impact on spacecraft movement is becoming more important and become the key factor. During last several years some serious problems are noticed caused by sloshing. At the time of first lunar landing, the sloshing of the remaining propellant experienced and oscillatory motion of Apollo-11 Lunar Module, which disturbed exact control of the landing maneuver. It shows that the free surface movement of fluid not only impacts in micro-scale [5] but also could be a reason of significant change at macro-level. Rebouillat,

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S., *et al.* [6] discussed the applications of sloshing for aerospace, naval technology and in other industries. Ibrahim [7] presented detailed experimental and theoretical sloshing results. Sloshing of fluid during the landing of an airplane [8] plays an important for its control system.

Different dynamical systems, Hamiltonian [9,10] and Lagrange dynamics are used for the stability measurement of moving objects. In present article we are using Hamiltonian system for the stability of spacecraft. Hamiltonian system is a set of differential equations known as Hamilton equations. Hamilton equations give the total energy of the system and give the time evolution of the system. For a closed system, these equations are the sum of kinetic energy and potential energy of the system. Hamiltonian can be used to describe the dynamics of simple systems, such as bouncing ball, a pendulum or the oscillating mass attached with spring in which energy changes from kinetic energy to potential energy and back again over time. This system of differential equations can also be used to model the energy of other complex dynamical systems such as planetary orbits in Celestial Mechanics and also in Quantum Mechanics. A moving mass coupled with rigid body [11,12] is used for underwater actuation and for astronaut models. Leonard N. E. *et al.* [13] presented a model for under water vehicle stability by using Lyapunov for bottom heavy vehicle. Petsopoulos T. *et al.* [14] considered two-body system for aerospace control system.

Second section of this article contains the mathematical model of the problem. We propose an equivalent model for sloshing in form of a mass attached with spring. Horizontal motion of mass attached with spring representing the sloshing of fluid along 1-axis. We use the elliptical shape of spacecraft for our convenience. Force exerted by the movement of mass is taken as internal force. In section three, we discuss about the stability of spacecraft for the proposed model. Lyapunov theory along with Casimir functions is used for this reason. Conditions for stability are derived analytically and keenly reviewed. Last part of paper shows conclusions derived for stability, useful simulation results are presented in this section to show the stable and unstable region for this model.

2. MATHEMATICAL FORMULATION

Consider the model of a spacecraft having translational velocity $\vec{v}$ and angular velocity $\vec{\Omega}$. For simplicity we are assuming ellipsoid shape of spacecraft. Spring-Mass model in Fig. 1 is used by many researchers [11,13,7] to express the sloshing of fluid inside the moving object. This model is not only applicable to the spacecraft but can also be used for trucks on roads carrying liquid, air planes carrying fuel inside wings and underwater machines or rigid bodies.

Let us consider the sloshing of liquid inside the tank as an equivalent model of a mass $\bar{m}$ attached with the spring, moving linearly having position vector $\vec{r}_{\bar{m}} =< r_m,\ 0,\ \Gamma >$. O` represents the centre of body and

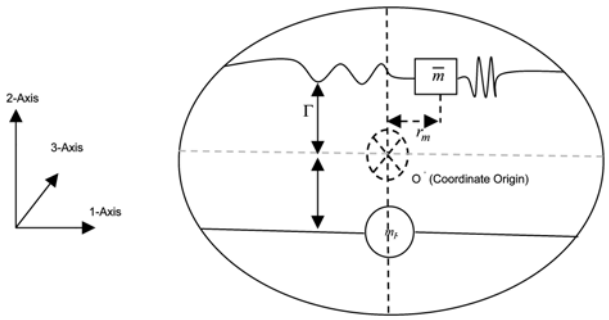


Fig. 1 Model of the Spacecraft for Sloshing

taken as an origin of the body coordinate reference frame. Rest of the mass is considered as stationary and having mass m_F with position vector $\vec{r}_F =< 0,\ 0,\ -\Gamma >$. The mass of the hull of the spacecraft is represented by m_H then the stationary mass with respect to body coordinates is $m_S = m_H + m_F$, hence complete system mass will be $m = m_S + \bar{m}$. Define the operator ^ so that the vector $\vec{a} =< a_1,\ a_2,\ a_3 >$ can be written as:

$$\hat{\vec{a}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix},$$ *i.e.* the operator ^ maps a vector $\vec{a}$ to the skew symmetric matrix. We have defined this operator notation for the cross vector product, $\hat{\vec{a}}\vec{b} = \vec{a} \times \vec{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Such type of product is used by many researchers [15,13,12,16]. This matrix expression of vector helps in the derivation of inertia matrix and makes calculations clear and easy that involve inertia matrix. Moreover vector product involving such matrix gives the same resultant vector as we can derive by using usual vector product procedure. It is novel and useful expression for vector-matrix multiplication.

The inertia matrix J_S for the stationary mass m_S with respect to body frame coordinates is given as $J_S = J_1 - m_F \hat{\vec{r}}_F \hat{\vec{r}}_F$, where J_1 denote the inertia matrix with respect to the whole body frame. Since our assumption for body frame is an elliptical shape so $J_1 =$ diag(J_{11}, J_{22}, J_{33}), hence

$$J_S = \begin{bmatrix} J_{11} & 0 & 0 \\ 0 & J_{22} & 0 \\ 0 & 0 & J_{33} \end{bmatrix} - m_F \begin{bmatrix} \Gamma^2 & 0 & 0 \\ 0 & \Gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} J_{S1} & 0 & 0 \\ 0 & J_{S2} & 0 \\ 0 & 0 & J_{S3} \end{bmatrix} \quad (1)$$

where, $J_{S1} = J_{11} - m_F\Gamma^2$, $J_{S2} = J_{22} - m_F\Gamma^2$ and $J_{S3} = J_{33}$.

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In Fig. 2 $\vec{b} =< x, y, z >$ represents the position vector from observer O to the centre of mass O' of spacecraft. Let point mass $\bar{m}$ moves along 1-Axis and having angular velocity $\vec{\Omega}$, then the kinematics is;

$$\dot{R} = R\hat{\vec{\Omega}}, \quad \dot{\vec{b}} = R\vec{v}, \quad \vec{r}_{\bar{m}} = R^T(\vec{b}_{\bar{m}} - \vec{b}) \text{ and } \vec{v}_{\bar{m}} = R^T\dot{\vec{b}}_{\bar{m}} \tag{2}$$

where R represent the rotation matrix and map vectors from body frame into inertial frame coordinates. Since m_S represents the mass which is not moving thus the translational velocity of m_S is the same as the velocity of fixed tank, *i.e.* $\vec{v}$. The inertial velocity of the moving point mass, written in body coordinates, is

$$\vec{v}_{\bar{m}} = \vec{v} + \vec{\Omega}\times\vec{r}_{\bar{m}} + \dot{\vec{r}}_{\bar{m}} \tag{3}$$

The kinetic energy of moving point mass $\bar{m}$ is

$$T_m = \frac{1}{2}\bar{m}\vec{v}_{\bar{m}}\cdot\vec{v}_{\bar{m}} = \frac{1}{2}\vec{\eta}_1\cdot \mathrm{M}_1\,\vec{\eta}_1 \tag{4}$$

where $\vec{\eta}_1 =< \vec{v}^T, \vec{\Omega}^T, \dot{\vec{r}}_{\bar{m}} >$ and $\mathrm{M}_1 = \begin{bmatrix} \bar{m}\mathrm{I} & -\bar{m}\hat{\vec{r}}_{\bar{m}} & \bar{m}\mathrm{I} \\ \bar{m}\hat{\vec{r}}_m & -\bar{m}\hat{\vec{r}}_{\bar{m}}\hat{\vec{r}}_{\bar{m}} & \bar{m}\hat{\vec{r}}_{\bar{m}} \\ \bar{m}\mathrm{I} & -\bar{m}\hat{\vec{r}}_{\bar{m}} & \bar{m}\mathrm{I} \end{bmatrix}$ is the system inertia matrix [15], where I is the identity matrix.

Similarly the kinetic energy for the stationary mass m_S is given as

$$T_S = \frac{1}{2}m_S(\vec{v}+\vec{\Omega}\times\vec{r}_S)\cdot(\vec{v}+\vec{\Omega}\times\vec{r}_S) = \frac{1}{2}\vec{\eta}_2\cdot \mathrm{M}_2\,\vec{\eta}_2 \tag{5}$$

where $\vec{r}_S$ is the position vector from O` to centre of mass m_S, $\mathrm{M}_2 = \begin{bmatrix} m_S\,\mathrm{I} & -m_S\hat{\vec{r}}_S \\ m_S\hat{\vec{r}}_S & J_S \end{bmatrix}$ and $\vec{\eta}_2 =< \vec{v}^T, \vec{\Omega}^T >$. Hence the total kinetic energy of the system will be,

$$T = T_m + T_S = \frac{1}{2}\vec{\eta}\cdot\mathrm{M}\vec{\eta} \tag{6}$$

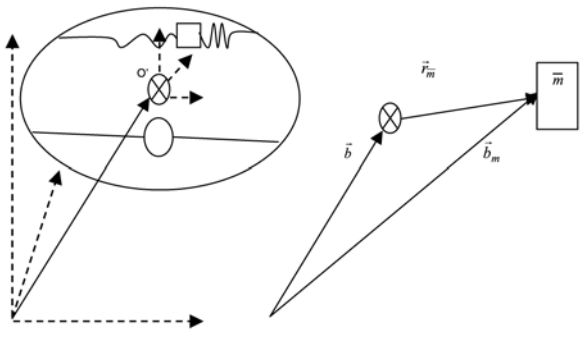


Fig. 2 Representing the frame of references and the position of moving mass

where $\mathrm{M} = \begin{bmatrix} m\mathrm{I} & -\bar{m}\hat{\vec{r}}_m - m_S\hat{\vec{r}}_S & \bar{m}\mathrm{I} \\ \bar{m}\hat{\vec{r}}_m + m_S\hat{\vec{r}}_S & J_S - \bar{m}\hat{\vec{r}}_m\hat{\vec{r}}_m & \bar{m}\hat{\vec{r}}_m \\ \bar{m}\mathrm{I} & -\bar{m}\hat{\vec{r}}_m & \bar{m}\mathrm{I} \end{bmatrix}$, $\vec{\eta} =< \vec{v}^T, \vec{\Omega}^T, \dot{\vec{r}}_{\bar{m}} >$ and $m = m_S + \bar{m}$.

Assuming that $m_H = 0$ so that $\vec{r}_S = \vec{0}$ then,

$$\mathrm{M} = \begin{bmatrix} m\mathrm{I} & -\bar{m}\hat{\vec{r}}_{\bar{m}} & \bar{m}\mathrm{I} \\ \bar{m}\hat{\vec{r}}_{\bar{m}} & J_S - \bar{m}\hat{\vec{r}}_{\bar{m}}\hat{\vec{r}}_{\bar{m}} & \bar{m}\hat{\vec{r}}_{\bar{m}} \\ \bar{m}\mathrm{I} & -\bar{m}\hat{\vec{r}}_{\bar{m}} & \bar{m}\mathrm{I} \end{bmatrix}$$

If Π is the total angular momentum of the system about the body coordinate origin written in body coordinates, P total translational momentum in body coordinates and P_m is the momentum of the moving point mass along the 1-Axis, written in body coordinates, then the momentum conjugate to $\vec{\eta}$ is given as;

$$\vec{\upsilon} = \frac{\partial T}{\partial\vec{\eta}} = \begin{bmatrix} P \\ \Pi \\ \mathrm{P}_m \end{bmatrix} = \mathrm{M}\vec{\eta} \tag{7}$$

this shows that,

$$\begin{aligned} \mathrm{P} &= \bar{m}\hat{\vec{r}}_{\bar{m}}\,\vec{v} + J_S\vec{\Omega} - \bar{m}\hat{\vec{r}}_{\bar{m}}\,\hat{\vec{r}}_{\bar{m}}\vec{\Omega} + \bar{m}\hat{\vec{r}}_{\bar{m}}\,\dot{\vec{r}}_{\bar{m}} \\ \prod &= m\vec{v} - \bar{m}\hat{\vec{r}}_{\bar{m}}\,\vec{\Omega} + \bar{m}\dot{\vec{r}}_{\bar{m}} \\ \mathrm{P}_m &= \bar{m}\vec{v} - \bar{m}\hat{\vec{r}}_{\bar{m}}\,\vec{\Omega} + \bar{m}\dot{\vec{r}}_{\bar{m}} \end{aligned} \tag{8}$$

Defining the vector $\vec{x} =< \mathrm{P}, \prod, \mathrm{P}_m, \vec{r}_{\bar{m}} >$ and knowing the fact that,

Thus the state vector gives,

$$\dot{\vec{x}} = Q(\vec{x})\nabla H(\vec{x}) + B(\vec{x})f_{\mathrm{int}} \tag{9}$$

where, $Q(\vec{x}) = \begin{bmatrix} 0 & \hat{P} & 0 & 0 \\ \hat{P} & \hat{\Pi} & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$ is the "Poisson Tensor" and $B(\vec{x}) = \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}$ is the input matrix.

$H(\vec{x})$ is Hamiltonian function defined by $H(\vec{x}) = 1/2\,\vec{\upsilon}\cdot\mathrm{M}^{-1}\vec{\upsilon}$, in the absence of external forces or moments on the system. f_{int} is the internal force exerted by the moving mass in the positive 1-Axis direction. Here the point to be noticed that for $f_{\mathrm{int}} = 0$, $H(\vec{x}) = 1/2\,\vec{\upsilon}\cdot\mathrm{M}^{-1}\vec{\upsilon}$ forms a non-canonical Hamiltonian System. In our case f_{int} correspond to the linear spring force and can be defined as inertial force of the spring, *i.e.* $f_{\mathrm{int}} = -kr_m$, where $k > 0$ is the stiffness of the spring. This control law can be written as $f_{\mathrm{int}} = -\partial\phi/\partial r_m$, where $\phi = 1/2\,kr_m^2$ denotes the artifi-

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cial potential function. The Hamiltonian System with potential function gives $H(\vec{x}) = 1/2\ \vec{\upsilon} \cdot \mathrm{M}^{-1} \vec{\upsilon} + \phi(r_m)$.

Now defining the quantities $C_1 = 1/2\ \mathrm{P} \cdot \mathrm{P}$, $C_2 = \Pi \cdot \mathrm{P}$ are known as Casimir Functions [17] and are conserved. Along with Hamiltonian system $H(\vec{x})$ these functions are important for stability analysis of a mechanical system.

3. MOTION STABILITY

We are interested to find the stability of spacecraft for Hamiltonian function $H(\vec{x}) = 1/2\ \vec{\upsilon} \cdot \mathrm{M}^{-1} \vec{\upsilon} + \phi(r_m)$. We are using Lyapunov function to find the stability for the steady rotation along 1-axis. We find the stability of rigid body in the sense to find conditions that could make rigid body stable under the impact of liquid sloshing. For such type of rotation, the equilibrium conditions are;

$$\begin{aligned} &P_e = \langle 0,\ 0,\ 0 \rangle\ ,\ \Pi_e = \langle \Pi_1^e,\ 0,\ 0 \rangle\ , \\ &P_{m_e} = \langle 0,\ 0,\ 0 \rangle \text{ and } \vec{r}_{\vec{m}_e} = \langle 0,\ 0,\ 0 \rangle \end{aligned} \tag{10}$$

Since $\| P \|$ is conserved, so if $P = \vec{0}$ initially then it will remain zero for all time intervals. For such situation we need to define another conserved sub-Casimir function *i.e.* $C_3 = 1/2\ \Pi \cdot \Pi$.

The Lyapunov function for Hamiltonian function coupled with Casimir functions [18] is defined as; $H_\Psi(\vec{x},\ \Psi) = H(\vec{x}) + \Psi(C_1,\ C_2,\ C_3)$. We define the quantities:

$$\Psi_i = \frac{\partial \Psi}{\partial C_i} \text{ and } \Psi_{ij} = \frac{\partial^2 \Psi}{\partial C_i\, \partial C_j}$$

For stability reasons, it is required that $DH_\Psi = 0$ at equilibrium Eq. (10). This condition gives, $\Psi_2^e = 0$ and $\Psi_3^e = -1/J_{11}$ (11)

Now we check the positive definiteness of H_Ψ by using second derivative and Eq. (11) at Eq. (10). Since we are taking the axis of rotation as major axis then $J_{11} > J_{22}$ and $J_{11} > J_{33}$. To make the second derivative of H_Ψ positive definite, it is required to choose Ψ such that:

$$\begin{aligned} &\Psi_{23}^e = \Psi_{33}^e = 0 \text{ and } \Psi_1^e > 0\ , \\ &\Psi_{\mathrm{p}}^{\mathrm{e}} = \Psi_1^{\mathrm{e}} + \Pi_1^{e2} \Psi_{22}^{\mathrm{e}} > 0 \text{ for } k > 0\ . \end{aligned} \tag{12}$$

3.1 Propositions

If $J_{11} > J_{22}$ and $J_{11} > J_{33}$, then the Eq. (10) is stable at one of the following proposition:

(1) $J_{11} > \dfrac{m_F\, \Gamma^2 + 1.7 m_F^2\, \Gamma^2 \Psi_p^e}{0.3 + m_F\, \Psi_p^e}$, coupled with

$1 + m_F\, \Psi_1^e > 0.7$.

Since the sloshing mass of fuel inside the container can varies from 10% to 76% of the total mass of spacecraft [19] so in our case, we assume that the sloshing mass is 70% of the total mass of spacecraft, *i.e.*, $\bar{m} = 0.7m$.

(2) $J_{11} < \dfrac{m_F\, \Gamma^2 + 1.7 m_F^2\, \Gamma^2 \Psi_p^e}{0.3 + m_F\, \Psi_p^e}$, coupled with

$1 + m_F \Psi_1^e < 0.7$.

The simulation results for these propositions are presented below;

The shaded area in this graph shows the stable region for proposition (i).

Since sloshing mass is taken as 70% of the total mass of spacecraft thus remaining fixed mass with respect to moving mass is 30% of the total mass *i.e.*, m_S = 0.3m, where $m_S = m_H + m_F$ and we have taken $m_H = 0$ in our model. Figure 3 shows that J_{11} covers the stable region from 0 to 0.3 (which is the percentage quantity of fixed mass of spacecraft). Since rotation of spacecraft is along 1-axis, thus Fig. 3 expresses the dependency of the component of moment of inertia along 1-axis *i.e.* J_{11} on the fixed mass quantity which is an important factor in spacecraft technology. Similarly, Fig. 4 shows the stable region with the variation of defined Casimir function Ψ. Above derived conditions and results (Figs. 3 and 4) not only gives the relation for moving and fixed mass but validate that the supposed model gives the potential shaping control for the considered equilibrium condition for the stability of spacecraft under the influence of fuel sloshing.

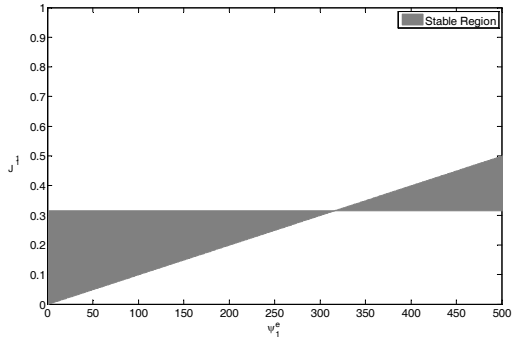


Fig. 3 Indicating the stable and unstable regions for m_S = 0.3m, for unit mass

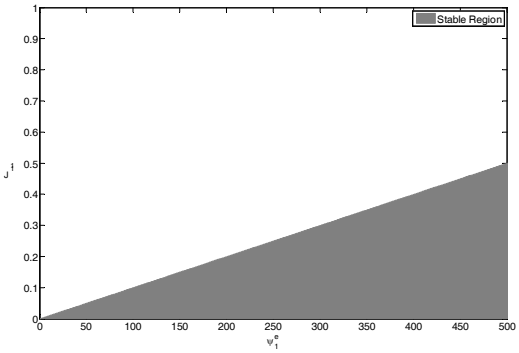


Fig. 4 Distribution of stable and unstable regions for proposition (ii)

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4. CONCLUSIONS

In this article we present the Hamiltonian model for the slosh equivalent model. Energy-Casimir functions are defined for stability analysis reasons. Lyapunov theory is used as an analytical method to find stable and unstable regions. Geometric regions for the proposition are found in Figs. 3 and 4. These results are not only providing the conditions for which the negative sloshing impact along Eq. (10) can be reduced in order to stabilize rigid body movement but also provide the proof of Lyapunov stability for our considered steady motion along the major axis. We observe that the masses and the potential function are the key roles for the stability during slosh process. This model can equivalently be considered for the vehicles on the roads and for other sloshing cases in micro and macro scales. Inclusion of flexible or moving boundaries, rotation along minor axis and non-uniform movement of container can make this analytical method more complicated, but this novel approach is reliable and effective as compare to other techniques and can be used in the field of applied mechanics.

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