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The KAM theorem and renormalization group

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Abstract. We give an elementary proof of the analytic KAM theorem by reducing it to a Picard iteration of a certain PDE with quadratic nonlinearity, the so-called Polchinski renormalization group equation studied in quantum field theory.

1. *Introduction*

An integrable Hamiltonian system is given in action–angle coordinates $(I, \theta) \in \mathbb{T}^d \times \mathbb{R}^d$ by a Hamiltonian $H_0(I)$ which depends only on the actions. It generates the flow

$$\dot{I} = 0, \quad \dot{\theta} = \omega(I) \tag{1.1}$$

where $\omega = \partial_I H_0$. The orbit $\theta(t) = \omega(I(0))t$ is quasiperiodic and spans the torus $I = I(0)$ for almost all $I(0)$ in a neighbourhood of a point I where the Hessian $D^2 H_0$ is non-degenerate. The KAM theorem constructs invariant tori in such a neighbourhood when H_0 is perturbed by a Hamiltonian $\lambda H_1(I, \theta)$ that is suitably small in the analytic [**1, 2**] or C^k [**3**] category. These solutions are found by a Newton iteration that constructs coordinates in which the flow takes the simple form (1.1).

In the analytic case, an alternative approach is to use a perturbation expansion of the solutions in powers of λ. The resulting Lindstedt series converges as a consequence of the KAM theorem [**1**]; a proof that relies on directly bounding the coefficients was obtained much later by Eliasson [**4**], who showed the presence of huge cancellations in the terms of the series. Subsequently, Eliasson's work was simplified and extended by Gallavotti [**5–7**], Chierchia and Falcolini [**8**], and Bonetto *et al* [**10**]. In particular, these later works revealed the analogy of Lindstedt series to perturbative expansions in quantum field theory (QFT), and applied ideas that had been developed for QFT, such as the renormalization group (RG), to Lindstedt series.

The papers [**5–8, 10**] in turn inspired the study [**12**], which developed a new iterative scheme based on the RG. With this approach, the small denominators that plague the Lindstedt series are treated scale by scale, through the introduction of renormalized problems for the unresolved scales. In the iterative scheme, these renormalized problems

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become simpler as the scale of the denominators is reduced, a fact that was shown to be a consequence of symmetries in the problem and which provides an explanation for the cancellations that Eliasson [**4**] uncovered in his proof. The renormalization group iteration has subsequently been extended to PDEs [**13**], to the homoclinic splitting problem [**17**], and also to the case of a C^k perturbation [**18**] where the Lindstedt series is not defined.

In this paper, we return to the analytic KAM theorem and present a proof of it which uses a RG where the scale is changed continuously. The renormalized problems are given as solutions of a simple evolution PDE with quadratic nonlinearity. This equation is well known in QFT where it is called the Polchinski equation [**14**]; see equation (2.11). Although, conceptually, this proof is just an infinitesimal version of the one given in [**12**], we feel it is worth writing down for two reasons.

First, the KAM theorem is reduced to a very simple Picard iteration with elementary estimates.

Second, we wish to bring into focus the Polchinski equation in this context. It is well known that as the size λ of the perturbation is increased, the torus with a given frequency ω disappears. For certain 'scale invariant' ω (e.g. in the case of $d = 2$, $\omega = (1, \gamma)$ with γ a 'noble' irrational), the solution at the critical λ is believed to have 'universal' scaling properties [**15, 16**]. Such universality is presumably related to fixed points of the RG equation (composed with a scaling). It would be very interesting to see if the RG equation possesses scaling solutions of this kind. The simplicity of the equation means that this hope might not be entirely unrealistic.

2. *The RG equation*

We will prove the KAM theorem in the special case where the perturbation H_1 is a function of the angles θ only, i.e. the so-called *Thirring model* (see [**5, 8**]). In this model one encounters the same difficulties as in the general case (for the RG formalism in the general case, see [**12**]). Thus, given a real analytic function $v : \mathbb{T}^d \to \mathbb{R}$ where $\mathbb{T}$ is the circle $\mathbb{R}/2\pi\mathbb{Z}$, we look for quasiperiodic solutions to the equation

$$\ddot{\theta} = -\lambda \partial v(\theta), \tag{2.1}$$

where λ is a perturbation parameter. The solutions are constructed by first fixing a vector $\omega \in \mathbb{R}^d$ whose components are independent over the integers and then looking for a function $X : \mathbb{T}^d \to \mathbb{R}^d$ such that

$$\theta(t) = \omega t + X(\omega t)$$

solves (2.1). This would follow if X solves the equation

$$-(\omega\partial)^2 X = U(X), \tag{2.2}$$

where $\omega\partial := \sum_i \omega_i \partial_i$ (throughout the paper we shall use this notation for the inner product), and

$$U(\theta, X) = \lambda \partial v(\theta + X(\theta)). \tag{2.3}$$

Note that if X solves (2.3), then so does

$$X_\beta(\theta) = X(\theta + \beta) + \beta \tag{2.4}$$

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for all $\beta \in \mathbb{R}^d$. We shall work mostly in Fourier space, using lower-case letters to denote the Fourier transform

$$x(q) = \int_{\mathbb{T}^d} e^{-iq\theta} X(\theta)$$

for $q \in \mathbb{Z}^d$. Equation (2.2) can be written as the fixed-point equation

$$x(q) = (\omega q)^{-2} u(q, x), \quad q \neq 0, \ u(0, x) = 0, \tag{2.5}$$

which is non-trivial because the multiplier $(\omega q)^{-2}$ is unbounded (the denominator can get arbitrarily small).

For the RG, we will cut out the small denominators ωq in (2.5) and study how the problem changes as the cutoff is changed. Let $\chi : \mathbb{R}^+ \to [0, 1]$ be a smooth monotone function with $\chi = 0$ on $[0, 1]$ and $\chi = 1$ on $[2, \infty)$, and let

$$\eta(t) = te^{\alpha t}$$

where $\alpha > 0$ will be chosen later. Note that t here is not the dynamical time. Define

$$\gamma_t(\omega q) = \chi(\eta(t)|\omega q|)(\omega q)^{-2}. \tag{2.6}$$

Thus, γ_t has had the small denominators $|\omega q| \leq \eta(t)^{-1}$ cut off. We chose $\eta(t)$ in such a way that it interpolates between the original problem at $t = \infty$ and the zero problem at $t = 0$.

We shall look for a solution to the cutoff problem

$$x(t) = \gamma_t u(x(t)) \tag{2.7}$$

by studying a slightly more general problem. Suppose we can find a functional $f(t, y)$, defined in a suitable space of maps $Y : \mathbb{T}^d \to \mathbb{R}^d$, which satisfies

$$f(t, y) = y + \gamma_t u(f(t, y)). \tag{2.8}$$

Then $x(t) = f(t, 0)$. Define also

$$w(t, y) = u(f(t, y)). \tag{2.9}$$

Then, upon differentiating (2.8) in t and y, we have $(1 - \gamma Du(f))\dot{f} = \dot{\gamma} w$ and $(1 - \gamma Du(f))Df = 1$ (where D stands for the Fréchet derivative); in other words, we should look for f as a solution to the differential equation

$$\dot{f}(t, y) = Df(t, y)\dot{\gamma}_t w(t, y), \quad f(0, y) = y, \tag{2.10}$$

where the initial condition is due to $\gamma_0 = 0$. For w, we get from (2.9) that $\dot{w} = Du(f)\dot{f} = Du(f)Df\dot{\gamma}\, w$, i.e. we should look for the solution to

$$\dot{w}(t, y) = Dw(t, y)\dot{\gamma}_t w(t, y), \quad w(0, y) = u(y). \tag{2.11}$$

Our strategy is therefore to solve the RG equations (2.10) and (2.11) in a suitable space and then construct the solution of (2.5) as the limit

$$x = \lim_{t \to \infty} f(t, 0). \tag{2.12}$$

We need the usual Diophantine property: for some $a, \nu > 0$ and all $q \neq 0$,

$$|\omega q| \geq a|q|^{-\nu}. \tag{2.13}$$

Then we shall prove the following result.

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THEOREM 1. *Let v be real analytic. Then the equation (2.2) has a real analytic solution which is unique up to translations (2.4) and analytic in λ in some neighbourhood of the origin.*

First let us make a remark about notation: our functions will depend on various arguments, but we shall not always write out all of the dependencies explicitly. For instance, the map w in (2.11) may be written as $w(t, y)$, or as $w(t, q, y)$ when all its arguments are displayed.

3. *Identities*

The flow possesses two symmetries, exhibited in [**12**], that are crucial for the analysis. Before defining these symmetries, let us write (2.11) explicitly as

$$\dot{w}(t, q, y) = \sum_{r\in\mathbb{Z}^d} Dw(t, q, r, y)\dot{\gamma}_t(\omega r)w(t, r, y), \tag{3.1}$$

where $Dw(t, q, r, y)$ is a $d \times d$ matrix and, setting $\kappa = \omega r$,

$$\dot{\gamma}_t = \kappa^{-2}\frac{d}{dt}\chi(\eta(t)|\kappa|). \tag{3.2}$$

The functions w and f will be given by their Taylor series; for instance,

$$w(t, q, y) = \sum_{n=0}^{\infty}\sum_{\boldsymbol{p}} w_n(t, q, p_1, \ldots, p_n)(y(p_1), \ldots, y(p_n)), \tag{3.3}$$

where $\boldsymbol{p} = (p_1, \ldots, p_n)$ runs through $(\mathbb{Z}^d)^n$ and the kernels w_n are linear maps $w_n \in \mathcal{L}((\mathbb{R}^d)^n, \mathbb{R}^d)$ which are symmetric in $p_1, \ldots, p_n$. For f we have a similar expansion. The equation (3.1) then becomes a set of equations for the kernels:

$$\dot{w}_n(t, q, \boldsymbol{p}) = \sum_{k=1}^{n+1}\sum_{r\in\mathbb{Z}^d} kw_k(t, q, r, \boldsymbol{p}')w_{n+1-k}(t, r, \boldsymbol{p}'')\dot{\gamma}_t(\omega r) \equiv v_n(t, q, \boldsymbol{p}), \tag{3.4}$$

with $\boldsymbol{p} = (\boldsymbol{p}', \boldsymbol{p}'')$. Equation (3.4) should be symmetrized in the $\boldsymbol{p}$, but we shall make this implicit in the notation.

3.1. $\mathbb{Z}^d$ *action.*

To define this, first observe that at $t = 0$ we get, from (2.3),

$$u_n(q, \boldsymbol{p}) = \frac{\lambda v(r)}{n!}(ir)^{\otimes^{n+1}}, \quad r = q - \sum p_i \tag{3.5}$$

after some calculation, where from now on we identify $\mathcal{L}((\mathbb{R}^d)^n, \mathbb{R}^d)$ with $(\mathbb{R}^d)^{n+1}$. Hence, trivially,

$$u_n(q + p, p_1 + p, \boldsymbol{p}') = u_n(q, p_1, \boldsymbol{p}') \tag{3.6}$$

for $p \in \mathbb{Z}^d$. For $t > 0$ this symmetry will take a peculiar form. Let

$$\pi(t, q, q', y) = Dw(t, q, q', y).$$

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Then (3.6) can be paraphrased as $\tau_p\pi(0, q, q', y) \equiv \pi(0, q+p, q'+p, y) = \pi(0, q, q', y)$. On the other hand, (3.1) yields

$$\dot\pi(t, q, q', y) = \sum_r (D\pi(t, q, q', r, y)w(t, r, y) + \pi(t, q, r, y)\pi(t, r, q', y))\dot\gamma_t(\omega r).$$

Upon shifting r by p in the second term and denoting ωp by κ, the translated function satisfies

$$\begin{aligned}\tau_p\dot\pi(t, q, q', y) = &\sum_r D\tau_p\pi(t, q, q', r, y)w(t, r, y)\dot\gamma_t(\omega r)\\ &+ \sum_r \tau_p\pi(t, q, r, y)\tau_p\pi(t, r, q', y)\dot\gamma_t(\omega r + \kappa).\end{aligned} \tag{3.7}$$

Now suppose that we can find a one-parameter family of functions $\pi(t, q, q', y, \kappa)$, $\kappa \in \mathbb{R}$, which solve equation (3.7) with $\tau_p\pi$ replaced by $\pi(\cdot, \kappa)$ and satisfy the initial condition $\pi|_{t=0} = Du$ independently of κ. Uniqueness of solutions would then yield

$$Dw(t, q+p, q'+p, y) = \pi(t, q, q', y, \omega p), \tag{3.8}$$

a relation which will be used in §3.2. Equation (3.7) for $\pi(\cdot, \kappa)$ can be written in terms of the kernels $w_n(q, q', \boldsymbol{p}, \kappa)$, which are symmetric in $\boldsymbol{p} \in (\mathbb{R}^d)^{(n-1)}$ and coincide with $w_n(q, q', \boldsymbol{p})$ at $n = 0$ or $\kappa = 0$. For $n > 0$, equation (3.4) is modified to

$$\dot w_n(t, q, q', \boldsymbol{p}, \kappa) = v_n(t, q, q', \boldsymbol{p}, \kappa), \tag{3.9}$$

with

$$\begin{aligned}v_n(t, q, q', \boldsymbol{p}, \kappa) = \sum_{k+l=n+1} \frac{k}{n} \sum_r [&(k-1)w_k(t, q, q', r, \boldsymbol{p}', \kappa)w_l(t, r, \boldsymbol{p}'')\dot\gamma_t(\omega r)\\ &+ l w_k(t, q, r, \boldsymbol{p}', \kappa)w_l(t, r, q', \boldsymbol{p}'', \kappa)\dot\gamma_t(\omega r + \kappa)]\end{aligned} \tag{3.10}$$

where, again, symmetrization with respect to $\boldsymbol{p}$ is implicit. Equation (3.9) obviously coincides with equation (3.4) at $\kappa = 0$, the two positions of q' giving rise to symmetrization in $(q', \boldsymbol{p})$.

The $\mathbb{Z}^d$ action is then given by

$$w_n(t, q+r, q'+r, \boldsymbol{p}, \kappa) = w_n(t, q, q', \boldsymbol{p}, \kappa + \omega r) \tag{3.11}$$

and, given a w satisfying (3.11), the vector field v in (3.10) will satisfy it too. Equation (3.11) finds its use in the relation

$$v_1(t, q, q, 0) = v_1(t, 0, 0, \omega q). \tag{3.12}$$

3.2. *T^d action.* Consider $\beta \in \mathbb{T}^d$ and define the action $W \mapsto \tau_\beta W$ by

$$\tau_\beta W(\theta, Y) = W(\theta - \beta, Y_\beta) \quad \text{with } Y_\beta(\theta) = Y(\theta + \beta) + \beta. \tag{3.13}$$

From (2.5) we have, for $t = 0$, $\tau_\beta U = U$. Moreover, the vector field on the right-hand side of (2.11) is invariant under τ_β. Indeed, we have

$$(D\tau_\beta W)(\theta, \theta', Y) = DW(\theta - \beta, \theta' - \beta, Y_\beta),$$

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so the invariance follows from the fact that the kernel of the operator $\dot\gamma$ in θ-space, $\dot\Gamma(\theta-\theta')$, commutes with translations. Thus we will look for solutions satisfying $\tau_\beta w = w$. Infinitesimally, $\partial_\beta \tau_\beta w = 0$ gives the so-called *Ward identity*,

$$a\partial_\theta W(\theta, Y) = \int_{\mathbb{T}^d} DW(\theta, \theta', Y)(a + a\partial_\theta Y(\theta'))\, d\theta' \tag{3.14}$$

for any $a \in \mathbb{R}^d$. In Fourier space, this is

$$iaqw(q, y) = \sum_{q'} Dw(q, q', y)(a\delta_{q'0} + iaq'y(q')) \tag{3.15}$$

or, in terms of the kernels,

$$iaqw_n(q, \boldsymbol{p}) = (n+1)w_{n+1}(q, 0, \boldsymbol{p})a + i\sum_i ap_i w_n(q, \boldsymbol{p}). \tag{3.16}$$

The $n = 0$ relation

$$iw_0(q) \otimes q = w_1(q, 0) \tag{3.17}$$

will play a crucial role later. Furthermore, by using equation (2.11) together with (3.17), we get

$$\dot w_0(0) = \sum_r w_1(0, r)\dot\gamma(r)w_0(r) = -i\sum_r r \otimes w_0(-r)\dot\gamma(r)w_0(r) = 0, \tag{3.18}$$

since the summand is antisymmetric in r. Thus, because $w_0(0)|_{t=0} = u_0(0) = \int \lambda\partial_\theta v(\theta)\, d\theta = 0$, equation (3.18) implies that

$$w_0(0) = 0. \tag{3.19}$$

To see the consequences of equations (3.19) and (3.17), note that equation (3.5) implies that

$$u_1(q, q') = u_1(-q', -q)^T. \tag{3.20}$$

It is readily checked that $\tilde w_m(q, q', \boldsymbol{p}, \kappa) \equiv w_m(-q', -q, \boldsymbol{p}, -\kappa)^T$ satisfies equation (3.9) as well. Therefore, given the uniqueness of solutions to (3.9), we conclude that

$$w_n(q, q', \boldsymbol{p}, \kappa) = w_n(-q', -q, \boldsymbol{p}, -\kappa)^T. \tag{3.21}$$

Consider, in particular, equation (3.9) for $n = 1$ and $q = q' = 0$. By (3.21), $w_2(t, 0, 0, \boldsymbol{p}, \kappa)$ is even in κ. Hence

$$\dot w_1(0, 0, \kappa) = \sum_r w_1(0, r, \kappa)w_1(0, -r, -\kappa)^T\dot\gamma_t(\omega r + \kappa) + \text{even}, \tag{3.22}$$

where (3.21) has been used. Since $\dot\gamma_t$ is even, we deduce that the right-hand side is even in κ and therefore $w_1(0, 0, \kappa)$ is also even. We will show below that the w_n are C^2 in κ. Hence

$$\partial_\kappa w_1(0, 0, 0) = 0. \tag{3.23}$$

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4. *Estimates*

We now describe the functional space in which (3.9) is solved. Consider u given by (3.5). By assumption, v is analytic in $|\Im\theta_i| \le 2b$, for some $b > 0$. Then equation (3.5) implies that

$$|u_n| \le C|\lambda| R^n e^{-b|r|} \tag{4.1}$$

for some R depending on b. Given $\boldsymbol{q} \in (\mathbb{Z}^d)^n$ and $\kappa \in \mathbb{R}$, let $w_n(\boldsymbol{q}, \kappa)$ be an element of the tensor product $(\mathbb{R}^d)^{\otimes^{n+1}}$ which has the following properties. It is symmetric under simultaneous exchanges of q_i and q_j and their corresponding $\mathbb{R}^d$'s for $i, j \ge 3$ and, for $\kappa = 0$, this property holds for $i, j \ge 2$; furthermore, $w_n(\boldsymbol{q}, \kappa)$ satisfies (3.16), (3.19) and (3.21). Let

$$\Lambda_t = \{q \in \mathbb{Z}^d : \eta(t)|\omega q| \le 3\} \tag{4.2}$$

and, for $0 < \beta < b$, let

$$\beta_t = \frac{1}{2}\left(1 + \frac{1}{1+t}\right)\beta. \tag{4.3}$$

Define

$$\|w_n\|_t = \sup_{\boldsymbol{p} \in \Lambda_t^n} \sum_{q \in \Lambda_t} e^{\beta_t |q - \sum p_i|} |w_n(q, \boldsymbol{p}, \cdot)|_t, \tag{4.4}$$

where $|\cdot|_t$ is the $\mathcal{C}^2$ norm given by

$$|f|_t = \sup_{\eta(t)|\kappa| \le 1} \sum_{i=0}^{2} \eta(t)^{-i} |\partial_i f(\kappa)|, \tag{4.5}$$

and $|\cdot|$ is the standard norm in $(\mathbb{R}^d)^{\otimes^{n+1}}$.

Let $w(t) = \{w_n(t)\}_{n \in \mathbb{N}}$, and define

$$\|w\| = \sup_{t \ge 0}\left(e^{2t}\|w_0(t)\|_t + \eta(t)^2\|w_1(t)\|_t + \sup_{n>1} n^2 \rho^n e^{(\frac{3}{2}-n)t}\|w_n(t)\|_t\right). \tag{4.6}$$

We then have the following.

PROPOSITION 1. *The equation (2.11) has a unique solution which is analytic in λ in some ball around zero and satisfies*

$$\|w\| \le C(v)|\lambda|, \tag{4.7}$$

where the constant C depends on v.

Proof. We write (3.9) as a fixed-point equation

$$w(t) = u + \int_0^t v(s)\, ds \equiv u + \Phi(t, w), \tag{4.8}$$

where v_n is given by (3.4) for $n = 0$ and by (3.10) for $n > 0$.

We start by showing that

$$\|u\| \le C\lambda. \tag{4.9}$$

We use the bound (4.1) and take $\beta < b$ to get

$$\|u_n\|_0 \le C R^n |\lambda|. \tag{4.10}$$

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For $n = 0$, we have

$$\sum_{q\in\Lambda_t} |u_0(q)|e^{\beta_t|q|} \le \sup_{0\neq q\in\Lambda_t} e^{-(\beta-\beta_t)|q|}\|u_0\|_0 \tag{4.11}$$

where $q \neq 0$ since $u_0(0) = 0$ by (3.5). By the Diophantine condition equation (2.13), $|\omega q|\eta(t) < 3$ implies $|q| \ge ((a/3)\eta(t))^{1/\nu}$. Since $\beta - \beta_t = (1/2)\beta t/(1+t)$, the supremum is super-exponential in t, and hence

$$\|u_0\|_t \le C|\lambda|e^{-2t}. \tag{4.12}$$

For $n = 1$, we obtain from (3.17) that $u_1(q, q) = 0$, and so

$$\sum_{q\in\Lambda_t} |u_1(q, p)|e^{\beta_t|q-p|} \le \sup_{\substack{q,p\in\Lambda_t\\ q\neq p}} e^{-(\beta-\beta_t)|q-p|}\|u_1\|_0. \tag{4.13}$$

Moreover, since $\eta(t)|\omega(q-p)| \le 6$, we get as in (4.12) that

$$\|u_1\|_t \le CR|\lambda|\eta(t)^{-2}. \tag{4.14}$$

Finally, for u_n with $n > 1$, we just use $\beta_t \le \beta_0$ to conclude that

$$\sum_{q\in\mathbb{Z}^d} |u_n(q, \boldsymbol{p})|e^{\beta_t|q-\boldsymbol{p}|} \le \|u_n\|_0 \le CR^n e^{(n-\frac{3}{2})t}|\lambda| \le Cn^{-2}\rho^{-n}e^{(n-\frac{3}{2})t}|\lambda|, \tag{4.15}$$

provided we take $\rho R < 1$. Equation (4.9) then follows from (4.12), (4.14), (4.14) and (4.6).

Thus it suffices to prove that Φ is a contraction from the ball $B_u(R|\lambda|)$ to the ball $B_0(R|\lambda|)$, for a suitable R. We shall treat Φ_n for $n = 0$, $n = 1$ and $n \ge 2$ as separate cases.

Case $n = 0$. From (3.4), we have

$$v_0(s, q) = \sum_{r\in\Lambda_s} w_1(s, q, r)w_0(s, r)\dot\gamma_s(\omega r), \tag{4.16}$$

where we may impose the restriction $r \in \Lambda_s$ since, by (3.2), $\dot\gamma_s(x)$ is supported in $\eta(s)|x| \in [1, 2]$. Using $e^{\beta_t|q|} \le e^{\beta_t|q-r|+\beta_t|r|}$ and $\beta_t \le \beta_s$ together with the obvious facts that

$$|\cdot|_t \le |\cdot|_s, \quad |AB|_s \le c|A|_s|B|_s, \tag{4.17}$$

we obtain

$$\begin{aligned}&\sum_{q\in\Lambda_t} e^{\beta_t|q|}|\Phi_0(t, q)|_t\\ &\quad\le C\sum_{q\in\Lambda_t}\int_0^t \sum_{r\in\Lambda_s} e^{\beta_s|q-r|}|w_1(s, q, r)|_s e^{\beta_s|r|}|w_0(s, r)|_s|\dot\gamma_s|_s e^{-(\beta_s-\beta_t)|q|}\,ds.\end{aligned} \tag{4.18}$$

From (3.2) we infer that

$$|\dot\gamma_s|_s \le C\eta(s)^2, \tag{4.19}$$

so

$$\|\Phi_0(t)\|_t \le C\|w\|^2\int_0^t e^{-2s}\inf_{0\neq q\in\Lambda_t} e^{-(\beta_s-\beta_t)|q|}\,ds. \tag{4.20}$$

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Since $\beta_s - \beta_t \geq \beta(1+t)^{-2}(t-s)/2$, the Diophantine property implies that

$$e^{-(\beta_s-\beta_t)|q|} \leq Ce^{-(t-s)a(t)}, \tag{4.21}$$

where $a(t) \to \infty$ as $t \to \infty$. Hence

$$\|\Phi_0(t)\|_t \leq Ce^{-2t}\|w\|^2. \tag{4.22}$$

Case $n = 1$. Write $v_1 = v_a + v_b$, with

$$v_a(s, q, q', \kappa) = \sum_{r\in\Lambda_s} w_1(s, q, r)w_1(s, r, q')\dot{\gamma}_s(\omega r + \kappa), \tag{4.23}$$

$$v_b(s, q, q', \kappa) = \sum_{r\in\Lambda_s} w_2(s, q, r, q', \kappa)w_0(s, r)\dot{\gamma}_s(\omega r), \tag{4.24}$$

and do similarly for Φ_1. In (4.23) we can also impose the restriction $r \in \Lambda_s$, because $\eta(t)|\kappa| \leq 1$ and so $\eta(s)|\omega r| \leq 2 + \eta(s)|\kappa| \leq 3$ since $\eta(t)$ is increasing in t.

We first consider $q \neq q'$ and proceed as in the $n = 0$ case. For v_a, use

$$e^{\beta_t|q-q'|} \leq e^{\beta_s[|q-r|+|r-q'|]}e^{-(\beta_s-\beta_t)|q-q'|}$$

along with (4.21), (4.6), (4.17) and (4.19) to get

$$\sum_{q\neq q'} |e^{\beta_t|q-q'|}\Phi_a(t, q, q')|_t \leq C\|w\|^2 \int_0^t \eta(s)^{-2}e^{-(t-s)a(t)}\,ds \leq C\eta(t)^{-2}\|w\|^2. \tag{4.25}$$

Similarily, for Φ_b we use

$$e^{\beta_t|q-q'|} \leq Ce^{\beta_s(|q-r-q'|+|r|)}e^{-(\beta_s-\beta_t)|q-q'|}$$

to get

$$\sum_{q\neq q'} e^{\beta_t|q-q'|}|\Phi_b(t, q, q')|_t \leq C\|w\|^2 \int_0^t e^{s/2-2s}\eta(s)^2e^{-(t-s)a(t)} \leq C\eta(t)^{-2}\|w\|^2, \tag{4.26}$$

provided we take $2\alpha \leq 3/2$ in the definition of $\eta(t)$.

Now let $q = q'$ and call the summands in (4.23) and (4.24) v_{ar} and v_{br}, respectively. First consider $s \geq s_0$ in the integral of (4.8), with $\eta(s_0)^{-1} = 4\eta(t)^{-1}$. Using

$$|v_{ar}(s, q, q)|_t \leq Ce^{-2\beta_s|q-r|}\eta(s)^{-2}\|w\|^2, \tag{4.27}$$

we obtain

$$\int_{s_0}^t |v_a(s, q, q)|_t\,ds \leq C\eta(t)^{-2}\|w\|^2. \tag{4.28}$$

In the same way, we find that

$$\int_{s_0}^t |v_b(s, q, q)|_t \leq C\|w\|^2 \int_{s_0}^t e^{s/2-2s}\eta(s)^2\,ds \leq C\eta(t)^{-2}\|w\|^2. \tag{4.29}$$

Finally, consider $v_a(s, q, q)$ for $s \leq s_0$. For such s, the $r = q$ term is zero. Indeed, Supp $\dot{\gamma}_s \subset [\eta(s)^{-1}, 2\eta(s)^{-1}]$ but $|\omega q + \kappa| \leq 4\eta(t)^{-1} \leq \eta(s)^{-1}$. For $r \neq q$, we have

$$|\partial_\kappa^i v_{ar}| \leq Ce^{-2\beta_s|q-r|}\eta(s)^{-2+i}\|w\|^2 \tag{4.30}$$

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for $i \leq 2$ and $|\kappa| \leq \eta(s)^{-1}$. From $|q-r| > C\eta(s)^{1/\nu}$, we conclude that

$$\left|\partial_\kappa^i \int_0^{s_0} v_a(s,q,q)\,ds\right| \leq C\|w\|^2, \tag{4.31}$$

which holds for $|\kappa| \leq 4\eta(t)^{-1}$ since $\eta(s_0)^{-1} \geq 4\eta(t)^{-1}$. In a similar manner, for $|\kappa| \leq 4\eta(t)^{-1}$ we obtain the bound

$$\left|\partial_\kappa^i \int_0^{s_0} v_b(s,q,q)\,ds\right| \leq C\|w\|^2. \tag{4.32}$$

By the symmetry (3.12),

$$v_1(s,q,q,\kappa) = v_1(s,0,0;\omega q+\kappa). \tag{4.33}$$

Let $\Psi(q,\kappa) = \int_0^{s_0} v_1(s,q,q,\kappa)$. Since $\partial_\kappa^i \Psi(0,0) = 0$ for $i < 2$, we have

$$\Psi(q,\kappa) = \int_0^{\omega q+\kappa} \partial_\kappa^2 \Psi(0,\kappa')(\omega q+\kappa-\kappa')d\kappa'. \tag{4.34}$$

As $\Psi(q,\kappa)$ satisfies (4.31) for $|\kappa| \leq 4\eta(t)^{-1}$ and $|\omega q+\kappa| \leq 3\eta(t)^{-1} + \eta(t)^{-1}$, (4.34) is defined and we infer from (4.31) that

$$|\Psi|_t \leq C\|w\|^2\eta(t)^{-2}. \tag{4.35}$$

Thus (4.25), (4.26) and (4.35) together give

$$\|\Phi_1\|_t \leq C\eta(t)^{-2}\|w\|^2. \tag{4.36}$$

The reader might notice how the argument leading to control of the diagonal part of Φ_1 is much simpler than the one that was used to bound the corresponding quantity in [**12**, Section 6, p. 22]. This is due to the simplification achieved through reducing an originally discrete problem to a PDE.

To finish the proof, we still need to deal with the case $n > 1$ in (4.8). Very crude bounds suffice. Using

$$e^{\beta_t(q-q'\sum p_i)} \leq e^{\beta_s(|q-r-q'-\sum p_i'|+|r-\sum p_i''|)} \tag{4.37}$$

for the first term and $|q-r-\sum p_i'| + |r-q'-\sum p_i''|$ in the exponent of the second term, we get

$$\begin{aligned}
\sum_q |\Phi_n|_t e^{\beta_t|q-q'-\sum p_i|} &\leq \sum_{k=1}^{n+1} k\int_0^t \|w_k\|_s\|w_{n+1-k}\|_s\eta(s)^2\,ds \\
&\leq C\|w\|^2\rho^{-n}\int_0^t \Big\{\eta(s)^{-2}n^{-2}e^{(n-3/2)s} \\
&\quad + (n+1)^{-1}\rho^{-1}e^{(n-1/2)s}e^{-2s} \\
&\quad + \rho^{-1}\sum_{k=2}^{n}\frac{1}{k}\frac{1}{(n+1-k)^2}e^{(n-2)s}\Big\}\eta(s)^2\,ds \\
&\leq C\|w\|^2\rho^{-n-1}e^{(n-3/2)t}n^{-2}\left[\frac{\rho}{n}+1+n\sum_{k=2}^{n}\frac{1}{k}\frac{1}{(n+1-k)^2}\right],
\end{aligned} \tag{4.38}$$

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provided that $\eta(s)^2 e^{-s/2} < C$, which holds if $\alpha < 1/4$. The expression within the square brackets is bounded in n, and hence

$$\|\Phi_n\|_t \le Cn^{-2}\rho^{-n-1}e^{(n-3/2)t}\|w\|^2. \tag{4.39}$$

Equations (4.22), (4.36) and (4.39) imply that Φ maps the ball $B_u(r)$ to the ball $B_0(r)$ with $r = C\rho^{-1}\lambda^2$ provided $\rho^{-1}\lambda$ is small enough. Contraction in this ball is similar. □

5. *The solution*

In this final section, let us solve the f-equation (2.10), written as

$$\begin{aligned} f_n(t, q, \boldsymbol{p}) &= a\delta_{n,1}\delta_{q,p}\mathbb{1} + \sum_{k=1}^{n+1}\int_0^t \sum_{q'} k f_k(s, q, q', \boldsymbol{p}')\dot\gamma_s(\omega q')w_{n+1-k}(s, q', \boldsymbol{p}'')\,ds \\ &\equiv g_n + \mathcal{F}_n, \end{aligned} \tag{5.1}$$

where $a = 1$ for (2.10) but we keep $a \ge 0$ for later purposes. Let

$$\|f_n\|_t = \sup_{\boldsymbol{p}\in\Lambda_t^n}\sum_{q\in\mathbb{Z}^d} e^{\beta_t|q-\sum p_i|}|f_n(q, \boldsymbol{p})| \tag{5.2}$$

and

$$\|f\| = \sup_{t\ge 0}\Big(\|f_0\|_t + \|f_1\|_t + \sup_{n\ge 2} n^2\rho^n e^{(1-n)t}\|f_n\|_t\Big). \tag{5.3}$$

Note that (5.2) differs from (4.4) in the constraint on q and the absence of κ; however, with an abuse of notation we write the norms in the same way.

PROPOSITION 2. *Equation (5.1) has a unique solution in the ball* $\|f - g\| \le Ca\rho^{-1}\|w\|$.

Proof. We proceed as in the proof of Proposition 1 to get

$$\|\mathcal{F}_n\|_t \le \sum_{k=1}^{n+1}\int_0^t k\|f_k(s)\|_s\|\dot\gamma_s\|\|w_{n+1-k}\|_s\,ds. \tag{5.4}$$

For $n = 0$, this gives

$$\|\mathcal{F}_0\| \le C\|w\|\|f\|\int_0^t \eta(s)^2 e^{-2s}\,ds \le C\|w\|\|f\| \tag{5.5}$$

and, for $n > 1$,

$$\begin{aligned} \|\mathcal{F}_n\|_t \le C\rho^{-n-1}\|w\|\|f\|\int_0^t \Big[&\rho n^{-2}e^{(n-3/2)s} + (n+1)^{-1}e^{ns}e^{-2s} + \rho n^{-1}e^{(n-1)s}\eta(s)^{-2} \\ &+ \sum_{k=2}^{n-1} k^{-1}e^{(k-1)s}e^{(n+1-k-3/2)s}\Big]\eta(s)^2\,ds \le C\|w\|\|f\|\rho^{-n-1}n^{-2}e^{(n-1)t}. \end{aligned} \tag{5.6}$$

For $n = 1$, the $k = 2$ term in (5.4) is bounded by

$$C\rho^{-1}\|w\|\|f\|\int_0^t e^s e^{-2s}\eta(s)^2\,ds \le C\rho^{-1}\|w\|\|f\|. \tag{5.7}$$

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For the $k = 1$ term, we need to improve (5.4) a little and extract, as before, the factor $e^{(\beta_t - \beta_s)|q-q'|}$, which is bounded by $e^{-(t-s)\alpha(s)}$, to get the bound

$$C\|w\|\|f\| \int_0^t \eta(s)^{-2}\eta(s)^2 e^{-(t-s)\alpha(s)}\, ds \leq C\|w\|\|f\| \tag{5.8}$$

for the $q = p \neq q'$ terms in (5.1). For $q = p = q'$, observe that $\dot{\gamma}_s(\omega q)$ vanishes for $3\eta(s) \leq \eta(t)$ since $q \in \Lambda_t$. Thus, the integral in (5.1) is restricted to $s > t - \mathcal{O}(1)$ and therefore bounded by (5.8) again.

So, altogether, we have

$$\|\mathcal{F}_1\|_t \leq C\rho^{-1}\|w\|\|f\|. \tag{5.9}$$

We can now prove Theorem 1. First, note that the limit (2.12) exists and is real analytic. Indeed, let $x(t, q) = f_0(t, q)$; from Proposition 2 we deduce that

$$\|f_0(t) - f_0(s)\|_t = \mathcal{O}(e^{-2s}\eta(s)^2), \tag{5.10}$$

i.e. $x(t, q)$ converges uniformly to an exponentially decaying sequence $x(q)$. We will shortly prove that $x(t)$ solves the equations

$$x(t) = \gamma_t u(x(t)), \quad u(x(t)) = w(t, 0). \tag{5.11}$$

The first equation implies that $x(t, 0) = 0$ and that for $|\omega q| \geq 2\eta(t)^{-1}$, $(\omega q)^2 x(t, q) = u(q, x(t))$. Hence, for $q \neq 0$,

$$(\omega q)^2 x(q) = u(q, x) \tag{5.12}$$

and $x(0) = 0$. The second equation of (5.11), in combination with (3.19), yields $u(0, x(t)) = 0$; so, by taking limits, the second equation of (2.5) follows. Our solution is clearly analytic in λ and is thus unique (up to the translations) since (2.2) determines uniquely the Taylor coefficients of its analytical solution.

We still need to prove (5.11). Consider the functionals

$$\phi(t, y) = y + \gamma_t u(f(t, y)), \quad \psi(t, y) = u(f(t, y)), \tag{5.13}$$

which are analytic in y in a neighbourhood of 0 in $\ell^1(\Lambda_t)$. It suffices to show that $\phi = f$ and $\psi = w$ because, at $y = 0$, (5.11) would follow. Upon differentiating and using (2.10), we get

$$\dot{\phi} = \dot{\gamma}\psi + \gamma Du\dot{f} = \dot{\gamma}\psi + \gamma Du Df\dot{\gamma}w = D\phi\dot{\gamma}w + \dot{\gamma}\rho, \tag{5.14}$$

where $\rho = \psi - w$. A similar calculation gives $\dot{\psi} = D\psi\dot{\gamma}w$ and, by combining this with (2.11), we get

$$\dot{\rho} = D\rho\dot{\gamma}w. \tag{5.15}$$

The initial conditions are $\phi(0, y) = f(0, y)$ and $\rho(0, y) = 0$. By applying Proposition 2 with $a = 0$, we get that $\rho = 0$; then, applying the proposition with $a = 1$, we obtain $\phi = 0$. Hence (5.11) follows.

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