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THE BAIRE METHOD FOR THE PRESCRIBED SINGULAR VALUES PROBLEM

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1. *Introduction*

The Baire category method was developed by De Blasi and Pianigiani [**10**–**13**] in order to establish the existence of solutions to the Cauchy problem for certain classes of differential inclusions in Banach spaces in the absence of convexity or compactness assumptions on the multivalued vector field. For results in this line of research and further references, see Pianigiani [**21**] and the monograph by Hu and Papageorgiou [**17**].

More recently, the Baire method was adapted by Dacorogna and Marcellini [**5**–**9**] to prove the existence of solutions to the Dirichlet problem for some partial differential inclusions of scalar or vectorial type. A somehow parallel approach, based on Gromov convex integration theory, was developed by Müller and Sverák [**19**] and Müller and Sychev [**20**].

In their fundamental contribution [**7**], Dacorogna and Marcellini investigated, among other vectorial problems, the classical prescribed singular value problem with Dirichlet boundary condition

$$\begin{cases} \sigma_j(\nabla u(x)) = 1, & x \in \Omega \text{ a.e., } j = 1, \dots, n, \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{D}$$

Here $\Omega \subset \mathbb{R}^n$ is a nonempty open bounded set with boundary $\partial\Omega$, and $\nabla u(x)$ is the gradient of $u : \overline{\Omega} \longrightarrow \mathbb{R}^n$ evaluated at $x \in \Omega$.

Throughout this paper $\sigma_j(A)$, $j = 1, \dots, n$, denote the singular values of an $n \times n$ real matrix A, repeated according to multiplicity and numbered in decreasing order, that is, $\sigma_1(A) \geqslant \dots \geqslant \sigma_n(A)$.

If $\varphi : \overline{\Omega} \longrightarrow \mathbb{R}^n$ is continuous and satisfies either of the following assumptions:

(d_1) $\varphi \in W^{1,\infty}(\Omega, \mathbb{R}^n)$, $\sigma_1(\nabla\varphi(x)) \leqslant 1 - \delta$, $x \in \Omega$ almost everywhere, $0 < \delta < 1$;

(d_2) $\varphi \in C^1(\Omega, \mathbb{R}^n)$, $\sigma_1(\nabla\varphi(x)) < 1$, $x \in \Omega$;

then Dacorogna and Marcellini show that problem (D) has a solution, that is, there exists a continuous $u : \overline{\Omega} \longrightarrow \mathbb{R}^n$ with $u \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ which satisfies (D). For the case $\mathrm{n} = 3$ and $\varphi = 0$, a similar existence result was previously obtained by Cellina and Perrotta [**3**].

A function $\varphi : D \longrightarrow \mathbb{R}^m$, $D \subset \mathbb{R}^n$ is called *nonexpansive*, *contractive*, *strictly contractive* if φ satisfies conditions (l_1), (l_2), (l_3), respectively.

(l_1) $|\varphi(x) - \varphi(y)| \leqslant |x - y|$ for every $x, y \in D$.

(l_2) $|\varphi(x) - \varphi(y)| < |x - y|$ for every $x, y \in D$, $x \neq y$.

(l_3) $|\varphi(x) - \varphi(y)| \leqslant \lambda|x - y|$ for every $x, y \in D$, where $0 \leqslant \lambda < 1$.

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Moreover, $\varphi : D \longrightarrow \mathbb{R}^m$ is called *locally nonexpansive, locally contractive, locally strictly contractive* if, for every $z \in D$, there exists an open neighborhood $U_z \subset D$ of z such that φ restricted to U_z is nonexpansive, contractive, strictly contractive, respectively, with constant λ depending on U_z.

Let Ω be a nonempty open bounded subset of $\mathbb{R}^n$. As usual, $C(\overline{\Omega}, \mathbb{R}^n)$ denotes the Banach space of all continuous functions $\varphi : \overline{\Omega} \longrightarrow \mathbb{R}^n$ endowed with the norm of uniform convergence. Define

$$\begin{aligned}
\mathcal{N}(\overline{\Omega}) &= \{\varphi \in C(\overline{\Omega}, \mathbb{R}^n) \,|\, \varphi \text{ is locally nonexpansive on } \Omega\} \\
\mathcal{K}(\overline{\Omega}) &= \{\varphi \in C(\overline{\Omega}, \mathbb{R}^n) \,|\, \varphi \text{ is locally contractive on } \Omega\} \\
\mathcal{S}(\overline{\Omega}) &= \{\varphi \in C(\overline{\Omega}, \mathbb{R}^n) \,|\, \varphi \text{ is locally strictly contractive on } \Omega\}.
\end{aligned}$$

Throughout, the space $\mathcal{N}(\overline{\Omega})$ is equipped with the induced metric of $C(\overline{\Omega}, \mathbb{R}^n)$. It is evident that $\mathcal{S}(\overline{\Omega}) \subset \mathcal{C}(\overline{\Omega}) \subset \mathcal{N}(\overline{\Omega})$.

The aim of this paper is to establish the following two existence results for the vectorial Dirichlet problem (D).

(I) If $\varphi : \overline{\Omega} \longrightarrow \mathbb{R}^n$ is continuous on $\overline{\Omega}$ and locally contractive on Ω, then problem (D) has solutions.

(II) If $\varphi : \partial\Omega \longrightarrow \mathbb{R}^n$ is contractive on $\partial\Omega$, then problem (D) has solutions.

Actually the above statements are special cases of two slightly more general results, namely Theorem 5.1 and Theorem 5.2.

The existence result (I) is in a certain sense optimal, for it is no longer true under the weaker assumption that $\varphi : \overline{\Omega} \longrightarrow \mathbb{R}^n$ is continuous on $\overline{\Omega}$ and locally nonexpansive on Ω. It is worth pointing out that the set $\mathcal{K}(\overline{\Omega})$ is residual in $\mathcal{N}(\overline{\Omega})$, that is, $\mathcal{N}(\overline{\Omega}) \setminus \mathcal{K}(\overline{\Omega})$ is of the Baire first category in $\mathcal{N}(\overline{\Omega})$. Moreover, the set of all continuous $\varphi : \overline{\Omega} \longrightarrow \mathbb{R}^n$ satisfying condition (d_1) or (d_2) is contained in $\mathcal{K}(\overline{\Omega})$ and is of the Baire first category in $\mathcal{N}(\overline{\Omega})$.

Similarly, the existence result (II) can fail under the weaker assumption that $\varphi : \partial\Omega \longrightarrow \mathbb{R}^n$ is nonexpansive on $\partial\Omega$.

In order to prove (I), we adapt the procedure developed in [**10**–**13**]. We associate with (D) the 'convexified' problem

$$\begin{cases} \sigma_j(\nabla u(x)) \leqslant 1, & x \in \Omega \text{ a.e., } j = 1, \dots, n, \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{C}$$

To solve problem (D), we first show that the solution set $\mathcal{M}_{\mathrm{C}}$ of (C) is a nonempty complete metric space under the metric of uniform convergence. We next prove that each set $\mathcal{M}_{\alpha_k}$, $\alpha_k = 1/2^k$, consisting of all solutions $u \in \mathcal{M}_{\mathrm{C}}$ such that the singular values $\sigma_j(\nabla u(x))$, $j = 1, \dots, n$, are α_k-close (in a sense to be made precise) to 1, is an open and dense subset of $\mathcal{M}_{\mathrm{C}}$. Since the solution set $\mathcal{M}_{\mathrm{D}}$ contains the intersection of all $\mathcal{M}_{\alpha_k}$, it follows that $\mathcal{M}_{\mathrm{D}}$ is residual in $\mathcal{M}_{\mathrm{C}}$ and hence nonempty.

To understand better the relation between problems (D) and (C), we recall (see Horn and Johnson [**16**, p. 158]) that if $\mathcal{K}$ is the set of all $n \times n$ real matrices A with singular values $\sigma_j(A) = 1$, $j = 1, \dots, n$, then the closed convex envelope $\overline{\mathrm{co}}\,\mathcal{K}$ is the set of all $n \times n$ real matrices A with singular values $\sigma_j(A) \leqslant 1$, $j = 1, \dots, n$. In view of this, problem (C) can be considered as the convexified version of problem (D). It is worth observing that the rank-1 convexity plays a crucial role in vectorial problems, yet for the specific problem (D) that we consider, the usual convexity turns out to be appropriate, as $\overline{\mathrm{co}}\,\mathcal{K}$ coincides with the rank-1 closed convex envelope of $\mathcal{K}$.

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In (I), as in Dacorogna and Marcellini [**7**], the datum $\varphi : \Omega \longrightarrow \mathbb{R}^n$ is actually a solution of problem (C). Thus (I) can be rephrased by stating that problem (D) has solutions provided that the convexified problem (C) admits a solution which is continuous on $\overline{\Omega}$ and locally contractive in Ω. In (II), the datum $\varphi : \partial\Omega \longrightarrow \mathbb{R}^n$ is supposed to be defined and contractive on $\partial\Omega$. In this case we construct a contractive extension $\overline{\varphi} : \overline{\Omega} \longrightarrow \mathbb{R}^n$ of φ by establishing a Kirszbraun–Valentine type extension theorem for contractive mappings. Then, by virtue of (I), it follows that the Dirichlet problem (D) has solutions.

The Baire method has also been used by Dacorogna and Marcellini [**9**] to study more general problems, including the prescribed singular values problem of the form

$$\begin{cases} \sigma_j(\nabla u(x)) = a_j, & x \in \Omega \text{ a.e., } j = 1, \dots, n, \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{E}$$

where $a_1 \geqslant a_2 \geqslant \dots \geqslant a_n$. In this case the existence of solutions is established under a C^1 assumption on the datum $\varphi : \overline{\Omega} \longrightarrow \mathbb{R}^n$.

It is not clear if the procedure used here to solve problem (D) can be adapted for problem (E) in order to obtain existence results without assuming that the datum φ is C^1.

The paper consists of five sections, the first of which is the introduction. Section 2 contains notation and preliminary results. In Section 3 the problem of extending contractive mappings is considered. Section 4 contains two lemmas which play a crucial role in our existence results. The existence theorems for the vectorial problem (D) are established in Section 5.

2. *Notations and preliminaries*

In this section we introduce notation and terminology and state some preliminary results. Throughout, $\mathbb{R}^n$ is the n-dimensional Euclidean space with scalar product $\langle ., . \rangle$ and induced norm $|\cdot|$, and Ω is a nonempty open bounded subset of $\mathbb{R}^n$.

If $A \subset \mathbb{R}^n$, the boundary, interior, closure, diameter of A are denoted by ∂A, $\operatorname{int} A$, $\overline{A}$, $\operatorname{diam} A$, respectively. Furthermore, $m(A)$ stands for the Lebesgue measure of A. $B(x, r)$ (respectively $B[x, r]$) denotes the open (respectively closed) ball in $\mathbb{R}^n$ with center x and radius r. By an ϵ-net of A, $\epsilon > 0$, we mean a finite set of points $a_i \in A$, $i = 1, \dots, p$, such that $\bigcup_{i=1}^{p} B(a_i, \epsilon) \supset A$.

If $x \in \mathbb{R}^n$ and A, B are nonempty subsets of $\mathbb{R}^n$, we put $d(x, A) = \inf\{|x - a| \,|\, a \in A\}$, $d(A, B) = \inf\{|a - b| \,|\, a \in A,\ b \in B\}$; moreover, if A, B are also bounded, then we set

$$h(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.$$

In the space of the nonempty compact subsets of $\mathbb{R}^n$, h is the usual Pompeiu–Hausdorff metric.

If $a, b \in \mathbb{R}^n$, then we denote by $[a, b]$ (respectively (a, b)) the closed (respectively open) linear interval with endpoints a, b.

In a metric space M we denote by $B_M(x, r)$ (respectively $B_M[x, r]$) the open (respectively closed) ball in M with center x and radius r; by $\operatorname{cl}_M(A)$ we mean the closure in M of the set $A \subset M$. If $\varphi : M \longrightarrow \mathbb{R}^n$ is bounded, then we put $\|\varphi\|_M = \sup\{|\varphi(x)| \,|\, x \in M\}$.

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For other notations and notions that we use without explanation, we refer to Ekeland and Temam [**14**] and Horn and Johnson [**16**].

PROPOSITION 2.1. *Let $\varphi : \overline{\Omega} \longrightarrow \mathbb{R}^n$ be continuous. Then the following hold.*
(i) $\varphi \in \mathcal{K}(\overline{\Omega}) \Leftrightarrow |\varphi(x) - \varphi(y)| < |x - y|$ *for all* $x, y \in \overline{\Omega}$, $x \neq y$, *with* $(x, y) \subset \Omega$.
(ii) $\varphi \in \mathcal{N}(\overline{\Omega}) \Leftrightarrow |\varphi(x) - \varphi(y)| \leqslant |x - y|$ *for all* $x, y \in \overline{\Omega}$, *with* $(x, y) \subset \Omega$.

Moreover, the following also holds.
(iii) *If* $\varphi \in \mathcal{K}(\overline{\Omega})$ (*respectively* $\mathcal{N}(\overline{\Omega})$, $\mathcal{S}(\overline{\Omega})$) *and* $B = B[a, r] \subset \Omega$, *where* $0 < r < d(a, \partial\Omega)$, *then the restriction of* φ *to* B *is contractive* (*respectively nonexpansive, strictly contractive with constant* $\lambda(B) < 1$).

Proof. Let $\varphi \in \mathcal{K}(\overline{\Omega})$ and take $x, y \in \overline{\Omega}$, $x \neq y$, with $(x, y) \subset \Omega$. Suppose that $x, y \in \Omega$. Put $T = \{t \in (0, 1] \,|\, |\varphi(z_t) - \varphi(x)| < |z_t - x|\}$, where $z_t = (1 - t)x + ty$. A simple continuation argument gives $\sup T = 1$, which implies that $|\varphi(y) - \varphi(x)| \leqslant |y - x|$. Then, by taking any $t \in T$, one has $|\varphi(y) - \varphi(x)| \leqslant |\varphi(y) - \varphi(z_t)| + |\varphi(z_t) - \varphi(x)| < |y - x|$. Whenever $x \in \Omega$, $y \in \partial\Omega$, the previous argument gives $|\varphi(z_t) - \varphi(x)| < |z_t - x|$ for every $t \in (0, 1)$ and, as before, $|\varphi(y) - \varphi(x)| < |y - x|$. The latter inequality remains valid if $x, y \in \partial\Omega$, because $|\varphi(y) - \varphi(z)| < |y - z|$, $|\varphi(z) - \varphi(x)| < |z - x|$, where $z = (x + y)/2$. Therefore the right implication in (i) holds. The opposite one being obvious, (i) holds. The proof of (ii) is analogous. Finally, (iii) is immediate, as $B \subset \Omega$ is compact convex. This completes the proof. □

PROPOSITION 2.2. *The space $\mathcal{N}(\overline{\Omega})$ endowed with the metric of uniform convergence is complete. Moreover,*

$$\mathrm{cl}_{\mathcal{N}(\overline{\Omega})}\mathcal{S}(\overline{\Omega}) = \mathrm{cl}_{\mathcal{N}(\overline{\Omega})}\mathcal{K}(\overline{\Omega}) = \mathcal{N}(\overline{\Omega}). \tag{2.1}$$

Proof. Let $\{\varphi_k\} \subset \mathcal{N}(\overline{\Omega})$ converge to $\varphi \in C(\overline{\Omega}, \mathbb{R}^n)$, and let $z \in \Omega$ and $0 < r < d(z, \partial\Omega)$ be arbitrary. By Proposition 2.1(iii), each φ_k is nonexpansive on $B(z, r)$, whence so is φ and $\varphi \in \mathcal{K}(\overline{\Omega})$. Therefore $\mathcal{N}(\overline{\Omega})$ is closed in $C(\overline{\Omega}, \mathbb{R}^n)$ and thus complete. Furthermore, $\mathcal{S}(\overline{\Omega})$ is dense in $\mathcal{N}(\overline{\Omega})$, because for each $\varphi \in \mathcal{N}(\overline{\Omega})$ the sequence $\{(1 - 1/k)\varphi\} \subset \mathcal{S}(\overline{\Omega})$ converges to φ. As $\mathcal{S}(\overline{\Omega}) \subset \mathcal{K}(\overline{\Omega}) \subset \mathcal{N}(\overline{\Omega})$, (2.1) also holds, completing the proof. □

PROPOSITION 2.3. *Let Ω be a nonempty open bounded subset of $\mathbb{R}^p$. Then there exists a sequence $\{\Omega_n\}$ of nonempty open sets $\Omega_n \subset \Omega$ such that the following hold.*
(i) $\overline{\Omega}_n \subset \Omega_{n+1}$, $n \in \mathbb{N}$.
(ii) $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$.
(iii) $\lim_{n\to\infty} h(\partial\Omega_n, \partial\Omega) = 0$.
(iv) $\lim_{n\to\infty} \rho_n = 0$, *where* $\rho_n = d(\partial\Omega_n, \partial\Omega_{n+1}) > 0$, $n \in \mathbb{N}$.

Proof. Set $\Omega_n = \{x \in \Omega \,|\, d(x, \partial\Omega) > \varepsilon_n\}$, where $\varepsilon_n > 0$, $\varepsilon_{n+1} < \varepsilon_n$, $\lim_{n\to\infty} \varepsilon_n = 0$. Each Ω_n is open nonempty (if ε_1 is small enough), and $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$. Moreover $x \in \overline{\Omega}_n$ implies that $d(x, \partial\Omega) \geqslant \varepsilon_n$; hence $x \in \Omega_{n+1}$, as $\varepsilon_{n+1} < \varepsilon_n$, and so (i) and (ii) are proved.

Clearly $\partial\Omega_n \subset \{x \in \Omega \,|\, d(x, \partial\Omega) = \varepsilon_n\}$. For each $x \in \partial\Omega_n$, we have $d(x, \partial\Omega) = \varepsilon_n$ and thus $\max_{x\in\partial\Omega_n} d(x, \partial\Omega) = \varepsilon_n$. On the other hand, for $x \in \partial\Omega$, we have $d(x, \partial\Omega_n) = d(x, \Omega_n)$. Further, $d(x, \Omega_n)$ decreases monotonically to 0, and so, by Dini's theorem, $\max_{x\in\partial\Omega} d(x, \partial\Omega_n) \to 0$, proving (iii). Finally, (iv) holds because

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$\overline{\Omega}_n \subset \Omega_{n+1}$ implies that $\rho_n > 0$ and, moreover, $\rho_n \leqslant h(\partial\Omega_n, \partial\Omega) + h(\partial\Omega_{n+1}, \partial\Omega)$. This completes the proof. □

REMARK 2.1. From the construction of Ω_n, it follows that for every closed ball $B \subset \Omega$, there exists $n_0 \in \mathbb{N}$ such that $B \subset \Omega_n$ for all $n \geqslant n_0$.

For the following proposition see Horn and Johnson [**16**, p. 148, p. 197].

PROPOSITION 2.4. *Let $M^{n\times n}$ be the space of all $n \times n$ real matrices. Then the following hold.*

(i) $\sigma_1(A) = \|A\|$ *for each* $A \in M^{n\times n}$.

(ii) *The function* $A \longrightarrow \sum_{j=1}^{k} \sigma_j(A)$ *is convex on* $M^{n\times n}$ *for* $k = 1, \ldots, n$.

3. *Extension and category properties of contractive maps*

In this section we prove an extension result of Kirszbraun–Valentine type [**18**, **25**] for contractive maps. We also show that most (in the Baire category sense) nonexpansive maps are actually contractive.

THEOREM 3.1. *Let $\mathbb{H}, \mathbb{K}$ be real Hilbert spaces and let D be a compact subset of $\mathbb{H}$ with* $\operatorname{diam} D > 0$. *Then each contractive map $\varphi : D \longrightarrow \mathbb{K}$ has a contractive extension $\psi : \mathbb{H} \longrightarrow \mathbb{K}$ to all of $\mathbb{H}$.*

Proof. Let $\mu : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be the modulus of continuity of φ, that is,

$$\mu(t) = \max\{\|\varphi(x) - \varphi(y)\| \,|\, x, y \in D,\ \|x - y\| \leqslant t\}.$$

Clearly μ is nondecreasing, upper semicontinuous, bounded above with $\mu(0) = 0$. Furthermore, $\mu(t) < t$ for all $t > 0$. To show this, let $t > 0$, and suppose that $\mu(t) > 0$ (the case $\mu(t) = 0$ is trivial). Then, for some $x, y \in D$, with $0 < \|x - y\| \leqslant t$, we have $\mu(t) = \|\varphi(x) - \varphi(y)\|$, which implies that $\mu(t) < t$ as φ is contractive.

Let $\omega : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be the lower envelope of all affine majorants of μ

$$\omega(t) = \inf\{\alpha(t) \,|\, \alpha \text{ is affine, } \alpha(s) \geqslant \mu(s) \text{ for all } s \in \mathbb{R}^+\}.$$

The function ω is concave and satisfies $\mu(t) \leqslant \omega(t) \leqslant t$ for $t \geqslant 0$. Moreover, we claim that

$$\omega(t) < t \qquad \text{for every } t > 0. \tag{3.1}$$

Suppose, by contradiction, that there exists $\theta > 0$ such that $\omega(\theta) = \theta$. For $t \geqslant \theta$, the real function $t - \mu(t)$ is strictly positive and lower semicontinuous, and tends to $+\infty$ as t $\to$ $+\infty$, and thus it has a strictly positive minimum, say ε. Hence $\mu(t) \leqslant t - \varepsilon$ for all $t \geqslant \theta$. As μ is bounded, there exists $\tau > \theta$ such that

$$\mu(t) < \tau - \varepsilon \quad \text{for all } t > \tau. \tag{3.2}$$

Let α be the affine mapping given by

$$\alpha(t) = \theta - \varepsilon + \lambda(t - \theta + \varepsilon), \quad \text{where } \lambda = \frac{\tau - \theta}{\tau - \theta + \varepsilon} < 1.$$

It is easily seen that α is an affine majorant of μ. In fact, if $t \in [0, \theta - \varepsilon]$, then $t - \theta + \varepsilon \leqslant 0$ and so $\alpha(t) \geqslant \theta - \varepsilon + 1(t - \theta + \varepsilon) = t$, as $0 < \lambda < 1$, which implies

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that $\alpha(t) > \mu(t)$. If $t \in [\theta - \varepsilon, \tau]$, then $\alpha(t) - (t - \varepsilon) = \theta - t + \lambda(t - \theta + \varepsilon) = \varepsilon(\tau - t)/(\tau - \theta + \varepsilon) \geqslant 0$, and so $\alpha(t) \geqslant t - \varepsilon \geqslant \mu(t)$. If $t > \tau$, then, by (3.2), $\alpha(t) \geqslant \theta - \varepsilon + (\tau - \theta) = \tau - \varepsilon > \mu(t)$. Therefore α is an affine majorant of μ and hence $\alpha(\theta) \geqslant \omega(\theta)$. As $\alpha(\theta) = \theta - \varepsilon + \lambda\varepsilon < \theta$, one has $\omega(\theta) < \theta$, and, from the contradiction, the claim (3.1) follows.

Since the uniformly continuous function φ has modulus of continuity $\mu(t) \leqslant \omega(t)$, $t \in \mathbb{R}^+$, and ω is concave, then by Grunbaum and Zarantonello's theorem [**15**] (see also Benyamini and Lindenstrauss [**1**, p. 18]) φ has a uniformly continuous extension $\psi : \mathbb{H} \longrightarrow \mathbb{K}$ with modulus of continuity $\overline{\mu}(t) \leqslant \omega(t)$, $t \in R^+$. For arbitrary $x, y \in \mathbb{H}$, $x \neq y$, set $t = \|x - y\|$. In view of (3.1), one has $\overline{\mu}(t) < t$, which implies that $\|\psi(x) - \psi(y)\| < \|x - y\|$, completing the proof. □

REMARK 3.1. A contractive map $\varphi : D \longrightarrow \mathbb{K}$ may have no contractive extension if D is not closed. In fact, let $D = \{(x, y) \in \mathbb{R}^2 \mid x \neq 0,\ x^2 + 4y^2 < 2\}$, and let $\varphi : D \longrightarrow \mathbb{R}$ be defined by $\varphi(x, y) = (1 - x^2)y$. For $x, y \in D$, we have $\|\nabla\varphi\|^2 = x^2(x^2 + 4y^2 - 2) + 1$ and thus $\|\nabla\varphi\| < 1$, which implies that φ is contractive on D. If we extend φ to the closure of D, we have $\varphi(0, y) = y$, showing that the extension is not contractive on $\overline{D}$. Moreover, it can be easily shown that a locally contractive mapping $\varphi : D \longrightarrow \mathbb{K}$ may have no locally contractive extension to all of $\mathbb{H}$.

The next theorem shows that certain contractive mappings have contractive extensions with some additional properties, which will be useful in the sequel.

THEOREM 3.2. *Let Ω be a nonempty open bounded subset of $\mathbb{R}^p$ and let $\varphi : \partial\Omega \longrightarrow \mathbb{R}^q$ be contractive on $\partial\Omega$. Then there exist two sequences $\{\Omega_n\}$, $\{\lambda_n\}$, where $\Omega_n \subset \Omega$ is open nonempty, $\lambda_n \in [0, 1)$, and there exists a contractive map $\psi : \overline{\Omega} \longrightarrow \mathbb{R}^q$ satisfying the following properties.*

(i) $\overline{\Omega}_n \subset \Omega_{n+1}$ *for all* $n \in \mathbb{N}$, $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$.
(ii) $d(\partial\Omega_n, \partial\Omega_{n+1}) = \rho_n > 0$, $n \in \mathbb{N}$, *and* $\lim_{n\to\infty} \rho_n = 0$.
(iii) $|\psi(x) - \psi(y)| \leqslant \lambda_n |x - y|$ *for every* $x, y \in \overline{\Omega}_n$, $n \in \mathbb{N}$.
(iv) $\psi(x) = \varphi(x)$ *for every* $x \in \partial\Omega$.

Proof. By virtue of Theorem 3.1, φ has a contractive extension $g : \overline{\Omega} \longrightarrow \mathbb{R}^q$, and thus

$$|g(x) - g(y)| < |x - y| \quad \text{for every } x, y \in \overline{\Omega},\ x \neq y, \tag{3.3}$$

and $g(x) = \varphi(x)$ for every $x \in \partial\Omega$.

Let $\{\Omega_n\}$ be the sequence of nonempty open sets $\Omega_n \subset \Omega$ given by Proposition 2.3. For $n \in \mathbb{N}$, set

$$\beta_n = \max\left\{ \frac{|g(x) - g(y)|}{|x - y|} \,\middle|\, x \in \partial\Omega_n,\ y \in \partial\Omega_{n+1} \right\}.$$

Clearly $0 \leqslant \beta_n < 1$ for all $n \in \mathbb{N}$, as $\partial\Omega_n$ and $\partial\Omega_{n+1}$ are compact disjoint sets. For each $n \in \mathbb{N}$ fix $\gamma_n > 0$ so that

$$\tilde{\gamma}_n = \beta_n + \frac{2\gamma_n}{\rho_n} < 1,$$

and observe that γ_n vanishes as $n \to \infty$ for ρ_n does so. Put, for convenience, $\Omega_0 = \{a\}$ for some $a \in \Omega_1$, and define $w_0 : \Omega_0 \longrightarrow \mathbb{R}^q$ by $w_0(a) = g(a)$. Denote by

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$\Gamma_1 = \{x_i^1\}_{i=1}^{d_1} \subset \partial\Omega_1$ a finite γ_1-net of $\partial\Omega_1$, and define

$$\tilde{w}_1 : \Omega_0 \cup \Gamma_1 \longrightarrow \mathbb{R}^q \quad \text{by } \tilde{w}_1(x) = g(x).$$

As $\tilde{w}_1$ is strictly contractive with some constant $\lambda_1 < 1$, then by the Kirszbraun–Valentine theorem, $\tilde{w}_1$ has an extension $w_1 : \overline{\Omega}_1 \longrightarrow \mathbb{R}^q$ strictly contractive on $\overline{\Omega}_1$ with the same Lipschitz constant λ_1.

Let $x \in \partial\Omega_1$ and $y \in \partial\Omega_2$ be arbitrary and take $x_i^1 \in \Gamma_1$ so that $|x - x_i^1| < \gamma_1$. We have

$$\begin{aligned} |w_1(x) - g(y)| &\leqslant \big|w_1(x) - w_1\big(x_i^1\big)\big| + \big|w_1\big(x_i^1\big) - g(x)\big| + |g(x) - g(y)| \\ &\leqslant \lambda_1\big|x - x_i^1\big| + \big|g\big(x_i^1\big) - g(x)\big| + \beta_1|x - y| \\ &\leqslant 2\big|x - x_i^1\big| + \beta_1|x - y| \leqslant \left(\beta_1 + \frac{2\gamma_1}{\rho_1}\right)|x - y| = \tilde{\lambda}_1|x - y|. \end{aligned}$$

Claim. For each $n \in \mathbb{N}$ there exist a function $w_n : \overline{\Omega}_n \longrightarrow \mathbb{R}^q$ and constants $\lambda_n, \tilde{\lambda}_n \in [0, 1)$ such that the following hold.

$$(\mathrm{P}_n) \begin{cases} \text{(j)} & w_n \text{ is strictly contractive on } \overline{\Omega}_n \text{ with constant } \lambda_n. \\ \text{(jj)} & \text{The restriction of } w_n \text{ to } \overline{\Omega}_{n-1} \text{ is equal to } w_{n-1}. \\ \text{(jjj)} & |w_n(x) - g(y)| \leqslant \tilde{\lambda}_n|x - y| \text{ for every } x \in \partial\Omega_n,\ y \in \partial\Omega_{n+1}. \end{cases}$$

We have already shown that w_1 satisfies (P_1). Arguing by induction, suppose that $w_{n-1} : \overline{\Omega}_{n-1} \longrightarrow \mathbb{R}^q$ has been defined and satisfies (P_{n-1}). To construct $w_n : \overline{\Omega}_n \longrightarrow \mathbb{R}^q$, denote by $\Gamma_n = \{x_i\}_{i=1}^{d_n} \subset \partial\Omega_n$ a finite γ_n-net of $\partial\Omega_n$. Since g is contractive, there exists $\alpha_n < 1$ such that $|g(x_i^n) - g(x_j^n)| \leqslant \alpha_n|x_i^n - x_j^n|$ for all $i, j = 1, \ldots, d_n$. Set

$$\lambda_n = \max\{\alpha_n, \lambda_{n-1}, \tilde{\lambda}_{n-1}\},$$

and observe that $\lambda_n < 1$. Define $\tilde{w}_n : \overline{\Omega}_{n-1} \cup \Gamma_n \longrightarrow \mathbb{R}^q$ by

$$\tilde{w}_n(x) = \begin{cases} w_{n-1}(x) & x \in \overline{\Omega}_{n-1} \\ g(x) & x \in \Gamma_n. \end{cases} \tag{3.4}$$

The function $\tilde{w}_n$ is strictly contractive with constant $\lambda_n < 1$. In fact, if $x \in \overline{\Omega}_{n-1}$ and $y \in \Gamma_n$ are arbitrary (if $x, y \in \overline{\Omega}_{n-1}$ or $x, y \in \Gamma_n$ there is nothing to prove), then, by taking $z \in [x, y] \cap \partial\Omega_{n-1}$, we have

$$\begin{aligned} |\tilde{w}_n(x) - \tilde{w}_n(y)| &\leqslant |\tilde{w}_n(x) - \tilde{w}_n(z)| + |\tilde{w}_n(z) - \tilde{w}_n(y)| \\ &\leqslant |w_{n-1}(x) - w_{n-1}(z)| + |w_{n-1}(z) - g(y)| \\ &\leqslant \lambda_{n-1}|x - z| + \tilde{\lambda}_{n-1}|z - y| \leqslant \lambda_n|x - y|. \end{aligned}$$

Thus by the Kirszbraun–Valentine theorem, $\tilde{w}_n$ has an extension $w_n : \overline{\Omega}_n \longrightarrow \mathbb{R}^q$ strictly contractive on $\overline{\Omega}_n$ with the same constant λ_n. By construction, w_n satisfies (P_n)(j) and (jj). To show (jjj), let $x \in \partial\Omega_n$, $y \in \partial\Omega_{n+1}$, and take $x_i^n \in \Gamma_n$ with $|x - x_i^n| < \gamma_n$. Then

$$\begin{aligned} |w_n(x) - g(y)| &\leqslant \big|w_n(x) - w_n\big(x_i^n\big)\big| + \big|w_n\big(x_i^n\big) - g(x)\big| + |g(x) - g(y)| \\ &\leqslant \lambda_n\big|x - x_i^n\big| + \big|g\big(x_i^n\big) - g(x)\big| + \beta_n|x - y| \\ &\leqslant 2|x - x_i^n| + \beta_n|x - y| \leqslant \left(\beta_n + \frac{2\gamma_n}{\rho_n}\right)|x - y| = \tilde{\lambda}_n|x - y|, \end{aligned}$$

and thus (jjj) also holds, whence w_n satisfies (P_n), proving the claim.

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Now define $\psi : \overline{\Omega} \longrightarrow \mathbb{R}^q$ by

$$\psi(x) = \begin{cases} w_n(x) & x \in \overline{\Omega}_n \\ \varphi(x) & x \in \partial\Omega. \end{cases} \tag{3.5}$$

This definition is meaningful because $\{\overline{\Omega}_n\}$ is monotone increasing, by inclusion, with $\bigcup_{n\in\mathbb{N}} \overline{\Omega}_n = \Omega$, and each w_n restricted to $\overline{\Omega}_{n-1}$ agrees with w_{n-1}.

Clearly ψ satisfies (iii) and (iv). It remains to show that ψ is contractive on $\overline{\Omega}$. Let $x, y \in \overline{\Omega}$ be arbitrary and suppose that $x \in \Omega$, $y \in \partial\Omega$ (the cases $x, y \in \Omega$ or $x, y \in \partial\Omega$ are trivial). Since for $n \to \infty$ we have $h(\partial\Omega_n, \partial\Omega) \to 0$, by Proposition 2.3(iii), and $\gamma_n \to 0$, there exists a sequence $\{y_n\}$, with $y_n \in \Gamma_n \subset \partial\Omega_n$, which converges to y. Take n_0 so that $x \in \Omega_n$ for all $n > n_0$. Then, by (3.5), $|\psi(x) - \psi(y_n)| = |w_n(x) - w_n(y_n)| \leqslant \lambda_n |x - y_n|$, for all $n > n_0$. Moreover, $\psi(y_n) = w_n(y_n) = g(y_n)$ and $\psi(y) = g(y)$, by (3.5) and (3.4), and hence, in view of (3.3), $|\psi(y_n) - \psi(y)| = |g(y_n) - g(y)| < |y_n - y|$. Therefore $|\psi(x) - \psi(y)| \leqslant \lambda_n |x - y_n| + |y_n - y|$ and thus $|\psi(x) - \psi(y)| \leqslant |x - y|$. To show that the latter inequality is strict, fix n so that $x \in \Omega_n$, and let $z \in (x, y) \cap \partial\Omega_n$. We have $|\psi(x) - \psi(z)| \leqslant \lambda_n |x - z|$ as ψ and w_n agree on $\overline{\Omega}_n$, while $|\psi(z) - \psi(y)| < |z - y|$, and thus $|\psi(x) - \psi(y)| \leqslant \lambda_n |x - z| + |z - y| < |x - y|$. This completes the proof. □

COROLLARY 3.1. *Let $\varphi : U \longrightarrow \mathbb{R}^q$ be contractive on $U = B[a, s] \subset \mathbb{R}^p$, $s > 0$, and let $\varepsilon > 0$. Then there exists a contractive map $\overline{\varphi} : U \longrightarrow \mathbb{R}^q$ such that the following hold.*

(i) $\|\overline{\varphi} - \varphi\|_U < \varepsilon$.

(ii) $\overline{\varphi}(x) = \varphi(x)$ *for every* $x \in \partial U$.

(iii) *The restriction of $\overline{\varphi}$ to any ball $V = B[a, r]$, $0 < r < s$, is strictly contractive with constant $\lambda(V) < 1$.*

Proof. Let $\psi : U \longrightarrow \mathbb{R}^q$ be given by Theorem 3.2. Then, for $t \in (0, 1]$ small enough, the map $\overline{\varphi} = (1 - t)\varphi + t\psi$ has the required properties. □

From the Baire category viewpoint, $\mathcal{K}(\overline{\Omega})$ is nearly all of $\mathcal{N}(\overline{\Omega})$, as is shown by the following proposition, which is a variant of a result of Reich and Zaslavski [**22**, **23**].

PROPOSITION 3.1. *$\mathcal{K}(\overline{\Omega})$ is a residual dense subset of $\mathcal{N}(\overline{\Omega})$.*

Proof. Set, for convenience, $\mathcal{K} = \mathcal{K}(\overline{\Omega})$, $\mathcal{N} = \mathcal{N}(\overline{\Omega})$. Define $R : \mathcal{N} \longrightarrow \mathbb{R}^+$ by $R(\varphi) = \sup A_\varphi$, where

$$A_\varphi = \{|x - y| \,|\, x, y \in \overline{\Omega}, (x, y) \subset \Omega \text{ and } |\varphi(x) - \varphi(y)| = |x - y|\}.$$

It is evident that $0 \in A_\varphi$ and $R(\varphi) \leqslant \operatorname{diam}(\Omega) < +\infty$.

We claim that $\varphi \in \mathcal{K}$ if and only if $R(\varphi) = 0$. Suppose that $\varphi \in \mathcal{K}$ and let $x, y \in \overline{\Omega}$, with $x \neq y$ and $(x, y) \in \Omega$, be arbitrary. Proposition 2.1(i) implies that $|\varphi(x) - \varphi(y)| < |x - y|$; hence $|x - y| \notin A_\varphi$, and thus $A_\varphi = \{0\}$ and $R(\varphi) = 0$. Conversely, suppose that $R(\varphi) = 0$, and thus $A_\varphi = \{0\}$, and let $x, y \in \overline{\Omega}$ be as above. Since $\varphi \in \mathcal{N}$, Proposition 2.1(ii) gives $|\varphi(x) - \varphi(y)| \leqslant |x - y|$, where the equality does not occur because it would imply that $A_\varphi \neq \{0\}$. Then, by Proposition 2.1(i), $\varphi \in \mathcal{K}$ and the claim is proved.

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Set $\mathcal{Z} = \mathcal{N} \setminus \mathcal{K}$. For $k \in \mathbb{N}$, define

$$\mathcal{Z}_k = \left\{ \varphi \in \mathcal{N} \;\middle|\; \frac{\operatorname{diam}\Omega}{k+1} \leqslant R(\varphi) \leqslant \frac{\operatorname{diam}\Omega}{k} \right\},$$

and observe that $\mathcal{Z} = \bigcup_{k\in\mathbb{N}} \mathcal{Z}_k$ as $\varphi \in \mathcal{Z}$ if and only if $R(\varphi) > 0$.

For each $k \in \mathbb{N}$, $\mathcal{Z}_k$ is nowhere dense in $\mathcal{N}$. It suffices to show that, given $\varphi_0 \in \mathcal{N}$ and $\varepsilon > 0$, there exist $\tilde{\varphi} \in \mathcal{N}$ and $\delta > 0$ such that

$$B_{\mathcal{N}}(\tilde{\varphi}, \delta) \subset B_{\mathcal{N}}(\varphi_0, \varepsilon) \cap (\mathcal{N} \setminus \mathcal{Z}_k). \tag{3.6}$$

By Proposition 2.2, take $\overline{\varphi} \in \mathcal{K}$, with $\|\overline{\varphi} - \varphi_0\|_{\overline{\Omega}} < \varepsilon/4$, and set $\tilde{\varphi} = \lambda\overline{\varphi}$, where $1 - \varepsilon/4(\|\overline{\varphi}\|_{\overline{\Omega}} + 1) < \lambda < 1$. Clearly $\tilde{\varphi}$ is locally strictly contractive with uniform constant λ, and $\|\tilde{\varphi} - \varphi_0\|_{\overline{\Omega}} < \varepsilon/2$, as $\|\tilde{\varphi} - \overline{\varphi}\|_{\overline{\Omega}} < \varepsilon/4$.

Let $0 < \eta < \operatorname{diam}(\Omega)/(k+1)$ and fix δ, with $0 < \delta < \min\{(1-\lambda)\eta/2, \varepsilon/2\}$. Then

$$R(\varphi) \;<\; \eta \quad \text{for every } \varphi \in B_{\mathcal{N}}(\tilde{\varphi}, \delta). \tag{3.7}$$

In fact, let $\varphi \in B_{\mathcal{N}}(\tilde{\varphi}, \delta)$ and suppose that $R(\varphi) > 0$ (the case $R(\varphi) = 0$ is trivial). From the definition of $R(\varphi)$, given $\sigma > 0$, there exist $x, y \in \overline{\Omega}$, with $(x, y) \subset \Omega$, such that $|\varphi(x) - \varphi(y)| = |x - y| > R(\varphi) - \sigma$. On the other hand, $|\tilde{\varphi}(x) - \tilde{\varphi}(y)| \leqslant \lambda|x - y|$, by Proposition 2.1(ii). Since

$$\begin{aligned} |x - y| = |\varphi(x) - \varphi(y)| &\leqslant |\varphi(x) - \tilde{\varphi}(x)| + |\tilde{\varphi}(x) - \tilde{\varphi}(y)| + |\tilde{\varphi}(y) - \varphi(y)| \\ &< 2\delta + \lambda|x - y|, \end{aligned}$$

it follows that $|x - y| < 2\delta/(1-\lambda)$, whence $R(\varphi) < 2\delta/(1-\lambda) + \sigma$, which implies that $R(\varphi) \leqslant 2\delta/(1-\lambda) < \eta$, proving (3.7). As $\eta < \operatorname{diam}(\Omega)/(k+1)$, (3.7) implies that $B_{\mathcal{N}}(\tilde{\varphi}, \delta) \subset (\mathcal{N} \setminus \mathcal{Z}_k)$. Moreover, $B_{\mathcal{N}}(\tilde{\varphi}, \delta) \subset B_{\mathcal{N}}(\varphi_0, \varepsilon)$, for $\|\overline{\varphi} - \varphi_0\|_{\overline{\Omega}} < \varepsilon/2$ and $\delta < \varepsilon/2$. Thus (3.6) holds, and $\mathcal{Z}_k$ is nowhere dense in $\mathcal{N}$.

Therefore $\mathcal{N} \setminus \mathcal{K}$ is of the Baire first category in $\mathcal{N}$. As $\mathcal{N}$ is complete, $\mathcal{K}$ is a residual dense subset of $\mathcal{N}$, completing the proof. □

REMARK 3.2. By a similar argument, one can show that the set of the contractive maps $\varphi : \partial\Omega \longrightarrow \mathbb{R}^q$ is a residual dense subset of the space of the nonexpansive maps $\varphi : \partial\Omega \longrightarrow \mathbb{R}^q$ endowed with the metric of uniform convergence.

PROPOSITION 3.2. *$\mathcal{S}(\overline{\Omega})$ is a Baire first category subset of $\mathcal{N}(\overline{\Omega})$.*

Proof. Fix a closed ball $B = B[a, r] \subset \Omega,\ r > 0$. For $k \in \mathbb{N}$, set

$$\mathcal{S}_k = \{\varphi \in \mathcal{S}(\overline{\Omega}) \,|\, \varphi \text{ is strictly contractive on } B \text{ with constant } k/(k+1)\}.$$

It is routine to show that each $\mathcal{S}_k$ is nowhere dense in $\mathcal{N}(\overline{\Omega})$. Moreover $\mathcal{S}(\overline{\Omega}) = \bigcup_{k\in\mathbb{N}} \mathcal{S}_k$ by Proposition 2.1(iii), and thus the statement holds, completing the proof. □

4. *Main lemmas*

In this section we prove two lemmas which play a crucial role in the proof of the first existence theorem for the vectorial Dirichlet problem (D). Lemma 4.1 is essentially known (see Dacorogna [**4**] and Dacorogna and Marcellini [**9**]). However, for the sake of completeness, the proof is included.

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LEMMA 4.1. *Let $\varepsilon > 0$, $0 < \sigma < 1$. Let $\alpha : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an affine function given by $\alpha(x) = Ax + b$, where A is an $n \times n$ real matrix with singular values $\sigma_j(A) < \sigma$, $j = 1, \dots, n$. Then there exists a piecewise affine function $\beta : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that the following hold.*

(i) $\sigma_j(\nabla\beta(x)) = \sigma$, $x \in \mathbb{R}^n$ *almost everywhere,* $j = 1, \dots, n$.
(ii) $\|\beta - \alpha\|_{\mathbb{R}^n} < \varepsilon$.

Proof. The construction of v is straightforward when $n = 1$ and, similarly, when A is diagonal. In the general case, by the singular value decomposition theorem [**16**, p. 144], there exist $n \times n$ orthogonal matrices U, V such that $UAV = D$, where D is diagonal with $D = \operatorname{diag}(\sigma_1(A), \dots, \sigma_n(A))$, and $\sigma_1(A), \dots, \sigma_n(A)$ are the singular values of A. Now set $\overline{u}(x) = (UuV)(x)$, $x \in \mathbb{R}^n$. We have $\nabla\overline{u}(x) = (U\nabla uV)(x) = (UAV)(x) = D$. Hence there exists a piecewice affine function $\overline{v} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $\nabla\overline{v}$ is diagonal with all entries equal to σ and $\|\overline{v} - \overline{u}\|_{\mathbb{R}^n} < \varepsilon$. Define $v : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $v(x) = (U^{-1}\overline{v}V^{-1})(x)$. As $\sigma_j(\nabla v(x)) = \sigma$, $x \in \mathbb{R}^n$ almost everywhere, it follows that v is piecewise affine and satisfies (i). Moreover, for every $x \in \mathbb{R}^n$, we have

$$|v(V(x)) - u(V(x))| = |U^{-1}(\overline{v}(x)) - U^{-1}(\overline{u}(x))| = |U^{-1}(\overline{v}(x) - \overline{u}(x))| \leqslant \|\overline{v} - \overline{u}\|_{\mathbb{R}^n} < \varepsilon,$$

and thus v also satisfies (ii), completing the proof. □

LEMMA 4.2. *Let $u : U \longrightarrow \mathbb{R}^n$ be contractive on the closed ball $U = B[a, s] \subset \mathbb{R}^n$, $s > 0$. Let $0 < \alpha < 1$ and $\varepsilon > 0$. Then there exists a contractive map $w : U \longrightarrow \mathbb{R}^n$ such that the following hold.*

(i) $w(x) = u(x)$, $x \in \partial U$.
(ii) $\int_U \sigma_j(\nabla w(x))\, dx > (1 - \alpha)m(U)$, $j = 1, \dots, n$.
(iii) $\|w - u\|_U < \varepsilon$.

Proof. By Corollary 3.1, there exists a contractive map $v : U \longrightarrow \mathbb{R}^n$ satisfying the following properties.

(a_1) $v(x) = u(x)$ for every $x \in \partial U$.
(a_2) The restriction of v to any ball $V = B[a, r]$, where $0 < r < s$, is strictly contractive with constant $\lambda(V) < 1$.
(a_3) $\|v - u\|_U < \varepsilon/2$.

Fix $r \in (0, s)$ so that

$$m(V) > \frac{1 - \alpha}{1 - \alpha/2} m(U), \tag{4.1}$$

and set $\lambda = \lambda(V)$.

Let $\Delta \subset \operatorname{int} V$ be the set of the interior points of V at which v is differentiable. By Rademacher's theorem, $m(\Delta) = m(V)$. Fix γ such that

$$0 < \gamma < \min\{\varepsilon, (1 - \lambda)(1 - \eta)/2\}, \quad \text{where } \eta = \left(\frac{1 - \alpha/4}{1 - \alpha/6}\right)^{1/n}. \tag{4.2}$$

For each $x \in \Delta$ there is a closed ball $B[x, r(x)] \subset \operatorname{int} V$ with $0 < r(x) < \varepsilon/8$ such that

$$|v(y) - v(x) - \langle \nabla v(x),\ y - x \rangle| \leqslant \gamma |y - x| \quad \text{for every } y \in B[x, r(x)]. \tag{4.3}$$

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The family $\{B[x,r]\,|\,x\in\Delta,\ 0<r<r(x)\}$ is a Vitali covering of Δ and hence, by Vitali's covering theorem [**24**, p. 109], there exists a finite subfamily $\{B[x_i,r_i]\}_{i=1}^p$ of pairwise disjoint balls such that

$$m\Big(\bigcup_{i=1}^p B[x_i,r_i]\Big)>\frac{1-\alpha/2}{1-\alpha/4}m(V). \tag{4.4}$$

Set $\tilde r_i=\eta r_i,\ i=1,\dots,p$. Clearly, by (4.2),

$$\lambda+\frac{2\gamma}{1-\eta}<1,\qquad m(B[x_i,\tilde r_i])=\frac{1-\alpha/4}{1-\alpha/6}m(B[x_i,r_i]),\qquad i=1,\dots,p. \tag{4.5}$$

Let $\tilde v_i: B[x_i,\tilde r_i]\longrightarrow\mathbb{R}^n$ be given by $\tilde v_i(x)=v(x_i)+\langle\nabla v(x_i),x-x_i\rangle,\ i=1,\dots,p$, and fix σ satisfying $\max\{\lambda,1-\alpha/6\}<\sigma<1$. Since v is strictly contractive on $B[x_i,r_i]$ with constant λ, we have $\|\nabla v(x_i)\|\leqslant\lambda$ and thus $\sigma_1(\nabla v(x_i))\leqslant\lambda<\sigma,\ i=1,\dots,p$. By Lemma 4.1 (with $\alpha=\tilde v_i$), for each $i=1,\dots,p$ there exists a piecewise affine function $v_i: B[x_i,\tilde r_i]\longrightarrow\mathbb{R}^n$ such that

$$\sigma_j(\nabla v_i(x))=\sigma,\quad x\in B[x_i,\tilde r_i]\text{ a.e., }\ j=1,\dots,n, \tag{4.6}$$

$$\|v_i-\tilde v_i\|_{B[x_i,\tilde r_i]}<\gamma r_i<\varepsilon/2, \tag{4.7}$$

where the last inequality holds, as $\gamma<1$ and $r_i<r(x_i)$. In particular, v_i is strictly contractive on $B[x_i,\tilde r_i]$ with constant σ.

For $i=1,\dots,p$, define $\tilde z_i: B[x_i,\tilde r_i]\cup\partial B[x_i,r_i]\longrightarrow\mathbb{R}^n$ by

$$\tilde z_i(x)=\begin{cases}v_i(x) & x\in B[x_i,\tilde r_i]\\ v(x) & x\in\partial B[x_i,r_i].\end{cases} \tag{4.8}$$

We claim that $\tilde z_i$ is strictly contractive with constant $\tilde\lambda=\max\{\sigma,\lambda+2\gamma/(1-\eta)\}<1$. Let $x\in B[x_i,\tilde r_i],\ y\in\partial B[x_i,r_i]$ be arbitrary (if $x,y\in B[x_i,\tilde r_i]$ or $x,y\in\partial B[x_i,r_i]$, then there is nothing to prove) and let $\overline{x}\in[x,y]\cap\partial B[x_i,\tilde r_i]$. Then, by (4.3) (with $x=x_i,\ y=\overline{x}$), $|v(\overline{x})-\tilde v_i(\overline{x})|\leqslant\gamma|\overline{x}-x_i|<\gamma r_i$. In view of the latter and (4.7), as v is strictly contractive with constant λ, it follows that

$$\begin{aligned}|\tilde z_i(\overline{x})-\tilde z_i(y)|&=|v_i(\overline{x})-v(y)|\\&\leqslant|v_i(\overline{x})-\tilde v_i(\overline{x})|+|\tilde v_i(\overline{x})-v(\overline{x})|+|v(\overline{x})-v(y)|\\&<\gamma r_i+\gamma r_i+\lambda|\overline{x}-y|\leqslant\left(\lambda+\frac{2\gamma r_i}{r_i-\tilde r_i}\right)|\overline{x}-y|\\&=\left(\lambda+\frac{2\gamma}{1-\eta}\right)|\overline{x}-y|.\end{aligned}$$

On the other hand, $|\tilde z_i(x)-\tilde z_i(\overline{x})|=|v_i(x)-v_i(\overline{x})|\leqslant\sigma|x-\overline{x}|$, whence

$$\begin{aligned}|\tilde z_i(x)-\tilde z_i(y)|&\leqslant|\tilde z_i(x)-\tilde z_i(\overline{x})|+|\tilde z_i(\overline{x})-\tilde z_i(y)|\\&\leqslant\sigma|x-\overline{x}|+\left(\lambda+\frac{2\gamma}{1-\eta}\right)|\overline{x}-y|\leqslant\tilde\lambda|x-y|,\end{aligned}$$

and the claim is proved. Then, by the Kirszbraun–Valentine theorem, $\tilde z_i$ has a strictly contractive extension $z_i: B[x_i,r_i]\longrightarrow\mathbb{R}^n$, with the same constant $\tilde\lambda$.

Now define $w: U\longrightarrow\mathbb{R}^n$ by

$$w(x)=\begin{cases}z_i(x) & x\in B[x_i,r_i],\ i=1,\dots,p\\ v(x) & x\in U\setminus\bigcup\limits_{i=1}^p B[x_i,r_i].\end{cases} \tag{4.9}$$

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The definition is meaningful, for the balls $B[x_i, r_i]$ are pairwise disjoint. Moreover w is contractive because v and the z_i are so and z_i agrees with v on $\partial B[x_i, r_i]$.

By construction, w satisfies (i). To show (ii), observe that $w = v_i$ on $B[x_i, \tilde{r}_i]$ by (4.8), (4.9). Thus, in view of (4.5), (4.6), for $j = 1, \dots, n$, one has

$$\int_{B[x_i,r_i]} \sigma_j(\nabla w(x))\,dx \geqslant \int_{B[x_i,\tilde{r}_i]} \sigma_j(\nabla v_i(x))\,dx = \sigma \frac{1-\alpha/4}{1-\alpha/6} m(B[x_i, r_i]) > \left(1 - \frac{\alpha}{4}\right) m(B[x_i, r_i])$$

as $\sigma > 1 - \alpha/6$. By the latter, (4.4) and (4.1), it follows that

$$\int_U \sigma_j(\nabla w(x))\,dx \geqslant \int_{\bigcup_{i=1}^p B[x_i,r_i]} \sigma_j(\nabla w(x))\,dx > \left(1 - \frac{\alpha}{4}\right) m\left(\bigcup_{i=1}^p B[x_i, r_i]\right) > (1 - \alpha/2)m(V) > (1 - \alpha)m(U),$$

for $j = 1, \dots, n$, and so (ii) also holds. It remains to check (iii). Let $x \in B[x_i, r_i]$ and denote by y a point of $\partial B[x_i, r_i]$. Since $w(y) = v(y)$ and $r_i < \varepsilon/4$, we have $|w(x) - v(x)| \leqslant |w(x) - w(y)| + |v(y) - v(x)| < \varepsilon/2$ and thus, by (a_3), $|w(x) - u(x)| < \varepsilon$. In view of (4.9) and (a_3), the latter inequality remains valid if $x \in U \setminus \bigcup_{i=1}^p B[x_i, r_i]$, whence $\|w - u\|_U < \varepsilon$, completing the proof. □

5. *Existence theorems*

Let Ω be a nonempty open bounded subset of $\mathbb{R}^n$, and let $\varphi : \partial\Omega \longrightarrow \mathbb{R}^n$ be continuous. In this section we shall establish two existence results for the Dirichlet problem

$$\begin{cases} \sigma_j(\nabla u(x)) = 1, & x \in \Omega \text{ a.e., } j = 1, \dots, n, \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{D}$$

We associate with (D) the convexified Dirichlet problem

$$\begin{cases} \sigma_j(\nabla u(x)) \leqslant 1, & x \in \Omega \text{ a.e., } j = 1, \dots, n, \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{C}$$

DEFINITION 5.1. By a *solution of* (D) (respectively (C)), we mean any continuous function $u : \overline{\Omega} \longrightarrow \mathbb{R}^n$, with $u \in W^{1,\infty}(\Omega, \mathbb{R}^n)$, satisfying (D) (respectively (C)).

We will denote by $\mathcal{M}_D$ (respectively $\mathcal{M}_C$) the solution set of (D) (respectively (C)). Moreover we suppose that $\mathcal{M}_C$ is endowed with the metric of uniform convergence.

REMARK 5.1. A function $u : \overline{\Omega} \longrightarrow \mathbb{R}^n$ is a solution of (C) if and only if $u \in \mathcal{N}(\overline{\Omega})$ and $u(x) = \varphi(x)$ for every $x \in \partial\Omega$. If $\mathcal{M}_C \neq \varnothing$, then $\mathcal{M}_C$ is a nonempty convex and complete metric space.

THEOREM 5.1. *Let Ω be a nonempty open bounded subset of $\mathbb{R}^n$ and let $\varphi : \overline{\Omega} \longrightarrow \mathbb{R}^n$ be continuous on $\overline{\Omega}$ and locally contractive on Ω. Then the solution set*

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$\mathcal{M}_{\mathrm{D}}$ of the Dirichlet problem (D) *is a residual dense subset of $\mathcal{M}_{\mathrm{C}}$. In particular, $\mathcal{M}_{\mathrm{D}}$ is nonempty.*

Proof. From the assumption, $\varphi \in \mathcal{K}(\overline{\Omega})$ is a solution of problem (C) and thus, by Remark 5.1, $\mathcal{M}_{\mathrm{C}}$ is a nonempty convex and complete metric space. For each $u \in \mathcal{M}_{\mathrm{C}}$ and $t \in [0,1)$, the function $(1-t)\varphi + tu$ lies in $\mathcal{K}(\overline{\Omega}) \cap \mathcal{M}_{\mathrm{C}}$ by Remark 5.1, whence $\mathcal{M}_{\mathrm{C}} = \operatorname{cl} \mathcal{M}_0$, where $\mathcal{M}_0 = \{u \in \mathcal{K}(\overline{\Omega}) \,|\, u = \varphi \text{ on } \partial\Omega\}$ and the closure is in the metric of $\mathcal{M}_{\mathrm{C}}$.

For $0 < \alpha < 1$, set

$$\mathcal{M}_\alpha = \left\{ u \in \mathcal{M}_{\mathrm{C}} \;\middle|\; \frac{1}{m(\Omega)} \int_\Omega \left(\frac{1}{n} \sum_{j=1}^{n} \sigma_j(\nabla u(x)) \right) dx > 1 - \alpha \right\}.$$

Claim 1. $\mathcal{M}_\alpha$ is open in $\mathcal{M}_{\mathrm{C}}$.

To show that $\mathcal{M}_{\mathrm{C}} \setminus \mathcal{M}_\alpha$ is closed, consider any sequence $\{u_k\} \subset \mathcal{M}_{\mathrm{C}} \setminus \mathcal{M}_\alpha$ converging to u in $\mathcal{M}_{\mathrm{C}}$. Denote by $M^{n\times n}$ the vector space of the $n \times n$ real matrices A with the usual norm $\|A\| = \sup\{|A(x)| \,|\, |x| \leqslant 1\}$. Since $\|\nabla u_k(x)\| \leqslant \sigma_1(\nabla u_k(x)) \leqslant 1$ for $x \in \Omega$ almost everywhere, the sequence $\{\nabla u_k\} \subset L^\infty(\Omega, M^{n\times n})$ is bounded, and thus by Alaoglu's theorem (see Brezis [**2**, p. 42]), a subsequence, say $\{\nabla u_k\}$, converges to some $\omega \in L^\infty(\Omega, M^{n\times n})$ in the weak* topology of $L^\infty(\Omega, M^{n\times n})$. By Mazur's theorem [**14**, p. 6], there exists a sequence of finite convex combinations $\{\nabla w_m\}$, where $w_m = \sum_{i=0}^{p_{k_m}} \lambda_i^{k_m} u_{k_m+i}$, $\{k_m\}$ is strictly increasing, $\lambda_i^{k_m} \geqslant 0$ and $\sum_{i=0}^{p_{k_m}} \lambda_i^{k_m} = 1$, which converges to ω in $L^\infty(\Omega, M^{n\times n})$. Since, in addition, w_m converges to u in $L^\infty(\Omega, M^{n\times n})$, it follows (see Brezis [**2**, p. 150]) that w_m converges to u in $W^{1,\infty}(\Omega, \mathbb{R}^n)$ and thus $\nabla u = \omega$.

As the map $A \longrightarrow \sum_{i=1}^{n} \sigma_i(A)$ is convex on $M^{n\times n}$ and $\{u_k\} \subset \mathcal{M}_{\mathrm{C}} \setminus \mathcal{M}_\alpha$, we have

$$\begin{aligned} &\frac{1}{m(\Omega)} \int_\Omega \left(\frac{1}{n} \sum_{j=1}^{n} \sigma_j(\nabla w_m(x)) \right) dx \\ &\qquad \leqslant \frac{1}{m(\Omega)} \sum_{i=0}^{p_{k_m}} \lambda_i^{k_m} \int_\Omega \left(\frac{1}{n} \sum_{j=1}^{n} \sigma_j(\nabla u_{k_m+i}(x)) \right) dx \leqslant 1 - \alpha. \end{aligned}$$

Since ∇w_m converges to ∇u in $L^\infty(\Omega, M^{n\times n})$, letting $n \to \infty$ gives

$$\frac{1}{m(\Omega)} \int_\Omega \left(\frac{1}{n} \sum_{j=1}^{n} \sigma_j(\nabla u(x)) \right) dx \leqslant 1 - \alpha,$$

and thus $u \in \mathcal{M}_{\mathrm{C}} \setminus \mathcal{M}_\alpha$. Therefore $\mathcal{M}_{\mathrm{C}} \setminus \mathcal{M}_\alpha$ is closed and Claim 1 holds.

Claim 2. $\mathcal{M}_\alpha$ is dense in $\mathcal{M}_{\mathrm{C}}$.

Since $\mathcal{M}_0$ is dense in $\mathcal{M}_{\mathrm{C}}$, it suffices to show that, given $u \in \mathcal{M}_0$ and $\varepsilon > 0$, there exists $v \in \mathcal{M}_\alpha$ such that $\|v - u\|_{\overline{\Omega}} < \varepsilon$. By Vitali's covering theorem, there exists a finite family of pairwise disjoint closed balls $U_i = B[x_i, r_i] \subset \Omega$, $r_i > 0$, $i = 1, \ldots, p$, such that

$$m\left(\bigcup_{i=1}^{p} U_i \right) > \frac{1-\alpha}{1-\alpha/2} m(\Omega). \tag{5.1}$$

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The restriction of u to each U_i is contractive, by Proposition 2.1(iii), whence, by Lemma 4.2, for each $i = 1, \dots, p$ there exists a contractive map $w_i : U_i \longrightarrow \mathbb{R}^n$ which is equal to u on ∂U_i and, moreover, satisfies

$$\|w_i - u\|_{U_i} < \varepsilon, \quad \int_{U_i} \sigma_j(\nabla w_i(x))\, dx > (1 - \alpha/2) m(U_i), \qquad j = 1, \dots, n. \tag{5.2}$$

Now define $v : \overline{\Omega} \longrightarrow \mathbb{R}^n$ by

$$v(x) = \begin{cases} w_i(x) & x \in U_i,\ i = 1, \dots, p \\ u(x) & x \in \overline{\Omega} \setminus \bigcup_{i=1}^{p} U_i. \end{cases} \tag{5.3}$$

The definition is meaningful, as the balls U_i are pairwise disjoint. The map v is locally contractive on Ω, because the w_i and u are locally contractive and each w_i equals u on ∂U_i. Moreover, for any $x \in \partial\Omega$ and $\varepsilon > 0$ small enough, we have $v = u$ on $B(x, \varepsilon) \cap \overline{\Omega}$, whence v is continuous on $\overline{\Omega}$, and thus $v \in \mathcal{N}(\overline{\Omega})$. As $v = \varphi$ on $\partial\Omega$, it follows that $v \in \mathcal{M}_{\mathrm{C}}$.

On the other hand, in view of (5.3), (5.2) and (5.1), we have

$$\begin{aligned} \int_\Omega \left(\frac{1}{n} \sum_{j=1}^{n} \sigma_j(\nabla v(x)) \right) dx &\geqslant \sum_{i=1}^{p} \int_{U_i} \left(\frac{1}{n} \sum_{j=1}^{n} \sigma_j(\nabla w_i(x)) \right) dx \\ &> \left(1 - \frac{\alpha}{2}\right) \sum_{i=1}^{p} m(U_i) > (1 - \alpha) m(\Omega). \end{aligned}$$

Therefore $v \in \mathcal{M}_\alpha$. Furthermore, $\|v - u\|_{\overline{\Omega}} < \varepsilon$, by (5.3) and (5.2), and thus Claim 2 holds.

Now set

$$\mathcal{M}^* = \bigcap_{k \in \mathbb{N}} \mathcal{M}_{\alpha_k}, \qquad \alpha_k = 1/2^k.$$

By Claims 1 and 2, each $\mathcal{M}_{\alpha_k}$ is open and dense in $\mathcal{M}_{\mathrm{C}}$, a nonempty complete metric space, whence $\mathcal{M}^*$ is a residual dense subset of $\mathcal{M}_{\mathrm{C}}$ and, in particular, nonempty.

We have $\mathcal{M}^* = \mathcal{M}_{\mathrm{C}}$. In fact, let $u \in \mathcal{M}^*$, and thus by Remark 5.1, $u = \varphi$ on $\partial\Omega$ and $u \in \mathcal{N}(\overline{\Omega})$. The latter implies that

$$\frac{1}{n} \sum_{j=1}^{n} \sigma_j(\nabla u(x)) \leqslant 1, \qquad x \in \Omega \text{ a.e.}$$

Since, in addition, $u \in \mathcal{M}_{\alpha_k}$ for every $k \in \mathbb{N}$, it follows that

$$\frac{1}{m(\Omega)} \int_\Omega \left(\frac{1}{n} \sum_{j=1}^{n} \sigma_j(\nabla u(x)) \right) dx = 1.$$

Therefore $\sigma_j(\nabla u(x)) = 1,\ x \in \Omega$ almost everywhere, $j = 1, \dots, n$, and thus $\mathcal{M}^* \subset \mathcal{M}_{\mathrm{C}}$. The opposite inclusion being obvious, we have $\mathcal{M}^* = \mathcal{M}_{\mathrm{C}}$, completing the proof. □

THEOREM 5.2. *Let Ω be a nonempty open bounded subset of $\mathbb{R}^n$, and let $\varphi : \partial\Omega \longrightarrow \mathbb{R}^n$ be contractive. Then the solution set $\mathcal{M}_{\mathrm{D}}$ of the Dirichlet problem* (D) *is a residual dense subset of $\mathcal{M}_{\mathrm{C}}$ and is, in particular, non-empty.*

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Proof. By Theorem 3.1, φ admits a continuous extension to all of Ω, say $\varphi : \overline{\Omega} \longrightarrow \mathbb{R}^n$, and thus by Theorem 5.1 the statement follows. □

REMARK 5.2. If in Theorem 5.1 φ is continuous on $\overline{\Omega}$ and locally nonexpansive on Ω, then the solution set $\mathcal{M}_{\mathrm{D}}$ can be empty, as is shown by the following example. Let $\Omega = (0,1)^2$ and let $\varphi : \Omega \longrightarrow \mathbb{R}^2$ be given by $\varphi : (x_1, x_2) \longrightarrow (x_1, 0)$. Then φ is nonexpansive and the Dirichlet problem (D) has no solution. Suppose, by contradiction, that $u \in W^{1,\infty}(\Omega, \mathbb{R}^2)$, $u = (u^1, u^2)$, is a solution of (D). Since almost everywhere in Ω the singular values of ∇u are equal to 1, each entry of ∇u has modulus at most 1. We claim that $u^1(x_1, x_2) = x_1$. For almost all $x_2 \in [0,1]$, we have

$$u^1(1, x_2) - u^1(0, x_2) = \int_0^1 u^1_{x_1}(t, x_2)\, dt.$$

As $u = \varphi$ on $\partial\Omega$, $\int_0^1 u^1_{x_1}(t, x_2)\, dt = 1$. In addition, $|u^1_{x_1}| \leqslant 1$, and hence, for almost all $x_2 \in [0,1]$, $u^1_{x_1}(t, x_2) = 1$ almost everywhere, whence $u^1(x_1, x_2) = x_1$ for all $x_1 \in [0,1]$, $x_2 \in [0,1]$ almost everywhere. By continuity, $u^1(x_1, x_2) = x_1$ for all $(x_1, x_2) \in \Omega$; hence $\nabla u^1(x_1, x_2) = (1, 0)$. Since $|\det \nabla u| = 1$, it follows that $|u^2_{x_2}(x_1, x_2)| = 1$ almost everywhere in Ω. Furthermore, $u^2_{x_1}(x_1, x_2) = 0$ almost everywhere in Ω, for otherwise, the largest singular value of the matrix ∇u would be larger than 1. Since $u^2 = \varphi^2 = 0$ on $\partial\Omega$, it follows that $u^2 = 0$ on Ω, a contradiction.

REMARK 5.3. Theorem 5.1 generalizes a result, proved in [**7**], under the assumption that $\varphi : \overline{\Omega} \longrightarrow \mathbb{R}^n$ is continuous and satisfies either condition (d_1) or (d_2) in the introduction. It is easy to see that the set of all such maps is contained in $\mathcal{S}(\overline{\Omega})$, which, by Proposition 3.2, is of the Baire first category in $\mathcal{N}(\overline{\Omega})$. On the other hand, Theorem 5.1 shows that problem (D) has solutions for each $\varphi \in \mathcal{K}(\overline{\Omega})$, where the set $\mathcal{K}(\overline{\Omega})$ contains $\mathcal{S}(\overline{\Omega})$ and is residual in $\mathcal{N}(\overline{\Omega})$. The previous example shows that for $\varphi \in \mathcal{N}(\overline{\Omega}) \setminus \mathcal{K}(\overline{\Omega})$, a Baire first category set, the problem (D) can have no solution at all.

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