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# A minimal volume arithmetic cusped complex hyperbolic orbifold

BY TIEHONG ZHAO

*Institut de Mathématiques, Université Pierre et Marie Curie,*
4, *Place Jussieu, F*-75252 *Paris, France.*
*e-mail:* zhao@math.jussieu.fr

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*Abstract*

The sister of Eisenstein–Picard modular group is described explicitly in [**10**], whose quotient is a noncompact arithmetic complex hyperbolic 2-orbifold of minimal volume (see [**16**]). We give a construction of a fundamental domain for this group. A presentation of that lattice can be obtained from that construction, which relates to one of Mostow's lattices.

## 1. *Introduction*

The explicit construction of fundamental domains for lattices in complex hyperbolic space is particularly difficult. Lattices in rank one symmetric spaces have been studied for a long time and very few examples of lattices have been constructed. For a general survey on lattices in PU(2,1) we refer the reader to [**11**]. Fundamental domains were obtained by Mostow in order to show the existence of non-arithmetic lattices in PU(2,1), see [**9**]. Following Mostow's work, other explicit constructions of fundamental domains for lattices in PU(2,1) were obtained in [**1**, **2**, **12**].

Let $\mathcal{O}_d$ be the ring of integers in the imaginary quadratic number field $\mathbb{Q}(i\sqrt{d})$ where $d$ is a positive square-free integer. If $d \equiv 1, 2 \pmod 4$ then $\mathcal{O}_d = \mathbb{Z}[i\sqrt{d}]$ and if $d \equiv 3 \pmod 4$ then $\mathcal{O}_d = \mathbb{Z}[(1 + i\sqrt{d})/2]$. The subgroup of PU(2,1) with entries in $\mathcal{O}_d$ is called *Picard modular group* for $\mathcal{O}_d$ and written PU(2,1,$\mathcal{O}_d$). The general family of arithmetic groups PU(2,1,$\mathcal{O}_d$) gives the simplest lattices known and they are due to Picard [**13**, **14**]. The case of $\mathcal{O}_3 = \mathbb{Z}[\omega]$, where $\omega$ is a cube root of unity was treated in [**2**]. A description of the fundamental domain in the case $\mathcal{O}_1 = \mathbb{Z}[i]$ was obtained in [**3**, **4**, **5**]. In [**3**], Falbel, Francsics and Parker describe a fundamental domain for the group PU(2, 1; $\mathcal{O}_1$) and analyse the combinatorics of the fundamental domain to obtain a presentation of the group.

In this paper, we consider the sister of Eisenstein–Picard modular group PU(2, 1, $\mathbb{Z}[\omega]$) which is defined explicitly in [**10**]. It is the subgroup of PU($\mathbb{Q}(i\sqrt{3})$) that, written in $3 \times 3$ complex matrices $(z_{jk})_{1\leqslant j,k\leqslant 3}$, have $z_{11}$, $z_{12}$, $z_{13}(i\sqrt{3})$, $z_{21}/(i\sqrt{3})$, $z_{22}$, $z_{23}$, $z_{31}/(i\sqrt{3})$, $z_{32}/(i\sqrt{3})$, and $z_{33}$ all in $\mathcal{O}_3$. Following [**10**] we will call it $G_2$. Recently, Stover [**16**] has studied volumes of Picard modular surfaces. One of his main results is that there are exactly two of these with minimal volume, namely the Eisenstein–Picard modular surface and its sister, which corresponds to the group $G_2$ which we study here. Neither Parker nor Stover give an explicit fundamental domain or a presentation, that is the aim of this paper.

The special feature shared by some other Picard modular groups PU(2, 1; $\mathcal{O}_d$) is that the quotient $\mathbf{H}^2_{\mathbb{C}}/G_2$ has only one cusp. We construct a fundamental domain from the Ford

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domain instead of the Dirichlet domain. The Ford domain is the intersection of the exteriors of isometric spheres of all elements not fixing infinity, that is also a fundamental domain for the coset space of $(G_2)_\infty$ (the stabiliser of the point at infinity). In order to construct a fundamental domain, we must intersect the Ford domain with a suitable fundamental domain for $(G_2)_\infty$. Using the complex hyperbolic version of Poincaré's polyhedron theorem, we show their intersection gives a fundamental domain, from which we obtain a presentation. The main difference between the fundamental domain we construct and the one for PU(2, 1, $\mathbb{Z}[\omega]$) is that our domain comprises two compact sides with a side pairing map. We also compute, from the combinatorics of the fundamental domain, the known value of the volume of the quotient orbifold [**10**].

## 2. *Complex hyperbolic space and isometric spheres*

### 2·1. *The Siegel domain*

In this section we give the necessary background material on complex hyperbolic space. For further details we refer the reader to [**7**].

Let $\mathbb{C}^{2,1}$ denote the complex vector space equipped with the Hermitian form defined by

$$\langle \mathbf{z}, \mathbf{w} \rangle = z_1\overline{w}_3 + z_2\overline{w}_2 + z_3\overline{w}_1,$$

where $\mathbf{z}$ and $\mathbf{w}$ are the column vectors $[z_1, z_2, z_3]^t$ and $[w_1, w_2, w_3]^t$ respectively. The projective model of complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$ is defined to be the collection of negative lines in $\mathbb{C}^{2,1}$, namely, those points $\mathbf{z}$ satisfying $\langle \mathbf{z}, \mathbf{z} \rangle < 0$.

Recall that the boundary of the *Siegel domain* $\mathfrak{S}$ is identified with the one-point compactification of the Heisenberg group. The Heisenberg group $\mathfrak{N}$ is $\mathbb{C} \times \mathbb{R}$ with the group law

$$(z_1, t_1)\Diamond(z_2, t_2) = (z_1 + z_2, t_1 + t_2 + 2\mathrm{Im}(z_1\overline{z}_2)).$$

Complex hyperbolic space is parametrised in horospherical coordinates by

$$(z, t, u) \longrightarrow \begin{bmatrix} (-|z|^2 - u + it)/2 \\ z \\ 1 \end{bmatrix} \tag{2·1}$$

and the point at infinity being

$$q_\infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then $\mathfrak{S} = \mathfrak{N} \times \mathbb{R}_+$ and $\partial\mathfrak{S} = (\mathfrak{N} \times \{0\}) \bigcup \{q_\infty\}$.

### 2·2. *Complex hyperbolic isometries*

Let U(2, 1) be the group of matrices that are unitary with respect to the form $\langle ., . \rangle$. The group of holomorphic isometries of complex hyperbolic space is the projective unitary group PU(2, 1) = U(2, 1)/U(1), with a natural identification $\mathrm{U}(1) = \{e^{i\theta} I, \theta \in [0, 2\pi)\}$. We now describe the action of the stabiliser of $q_\infty$ on the Heisenberg group.

The Heisenberg group acts on itself by *Heisenberg translations*. For $(\tau, v) \in \mathfrak{N}$, this is

$$T_{(\tau,v)} : (z, t) \longmapsto (z + \tau, t + v + 2\mathrm{Im}(\tau\overline{z})) = (\tau, v)\Diamond(z, t).$$

Heisenberg translation by $(0, v)$ for any $t \in \mathbb{R}$ is called *vertical translation* by $v$.

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The unitary group U(1) acts on the Heisenberg group by *Heisenberg rotations*. For $e^{i\theta} \in$ U(1), the rotation fixing $q_0 = (0, 0, 0)$ is given by

$$R_\theta : (z, t) \longmapsto (e^{i\theta} z, t).$$

All other Heisenberg rotations may be obtained from these by conjugating by a Heisenberg translation.

For $\lambda \in \mathbb{R}_+$, *Heisenberg dilation* by $\lambda$ fixing $q_\infty$ and $q_0 = (0, 0, 0) \in \partial\mathbf{H}^2_{\mathbb{C}}$ is given by

$$D_\lambda : (z, t) \longmapsto (\lambda z, \lambda^2 t).$$

All other Heisenberg dilations fixing $q_\infty$ may be obtained by conjugating by a Heisenberg translation.

The stabiliser of $q_\infty$ in PU(2, 1) is generated by all Heisenberg translations, rotations and dilations. However, only Heisenberg translations and rotations are isometric with respect to various natural metrics on $\mathfrak{R}$. For this reason the group generated by all Heisenberg translations and rotations, which is the semidirect product $U(1) \ltimes \mathfrak{R}$, is called the *Heisenberg isometry group* Isom($\mathfrak{R}$). The nontrivial central elements of the Heisenberg isometry group are precisely the vertical translations. In particular, each element of Isom($\mathfrak{R}$) preserves every horosphere.

There is a canonical projection from $\mathfrak{R}$ to $\mathbb{C}$ called *vertical projection* and denoted by $\Pi$, given by $\Pi : (z, t) \longmapsto z$. Using the exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{R} \xrightarrow{\Pi} \mathbb{C} \longrightarrow 0,$$

we obtain the exact sequence (see Scott [**15**, page 467])

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathrm{Isom}(\mathfrak{R}) \xrightarrow{\Pi_*} \mathrm{Isom}(\mathbb{C}) \longrightarrow 1. \tag{2·2}$$

Here Isom($\mathbb{C}$) is the group of orientation preserving Euclidean isometries of $\mathbb{C}$.

Observe the elements in Isom($\mathbb{C}$) can be represented by matrices in GL(2, $\mathbb{C}$) of the form

$$\begin{bmatrix} e^{i\theta} & z_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} e^{i\theta} z + z_0 \\ 1 \end{bmatrix}.$$

Therefore, the map $\Pi_*$ can be given by

$$\Pi_* : \begin{bmatrix} 1 & -\bar{z}_0 e^{i\theta} & (-|z_0|^2 + it_0)/2 \\ 0 & e^{i\theta} & z_0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} e^{i\theta} & z_0 \\ 0 & 1 \end{bmatrix}. \tag{2·3}$$

It is clear that

$$\mathrm{Ker}(\Pi_*) = \left\{ \begin{bmatrix} 1 & 0 & it_0/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : t_0 \in \mathbb{R} \right\}$$

is the group of vertical translations fixing $q_\infty$.

### 2·3. *Isometric spheres*

Given an element $A \in$ PU(2, 1) such that $A(q_\infty) \neq q_\infty$, we define the isometric sphere of $A$ to be the hypersurface

$$\left\{ \mathbf{z} \in \mathbf{H}^2_{\mathbb{C}} : |\langle \mathbf{z}, q_\infty \rangle| = |\langle \mathbf{z}, A^{-1}(q_\infty) \rangle| \right\}.$$

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For example, the isometric sphere of

$$I_1 = \begin{bmatrix} 0 & 0 & i/\sqrt{3} \\ 0 & -\omega & 0 \\ i\sqrt{3} & 0 & 0 \end{bmatrix}$$

is

$$\mathcal{S}_0 = \{(z, t, u) : \left||z|^2 + u - it\right| = 2/\sqrt{3}\} \tag{2·4}$$

in horospherical coordinates or

$$\mathcal{S}_0 = \left\{[z_1, z_2, z_3] \in \mathbf{H}^2_{\mathbb{C}} : \sqrt{3}|z_1| = |z_3|\right\} \tag{2·5}$$

in homogeneous coordinates.

All other isometric spheres are images of $\mathcal{S}_0$ by Heisenberg dilations, rotations and translations. Thus the isometric sphere with radius $r$ and centre $(z_0, t_0, 0)$ is given by

$$\left\{(z, t, u) : \left||z - z_0|^2 + u + it - it_0 + 2i\mathrm{Im}(z\bar{z}_0)\right| = r^2\right\}.$$

If $A$ has the matrix form

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix},$$

then $A(q_\infty) \neq q_\infty$ if and only if $g \neq 0$. The isometric sphere of $A$ has radius $r = \sqrt{2/|g|}$ and centre $A^{-1}(q_\infty)$, which in horospherical coordinates is

$$(z_0, t_0, 0) = (\overline{h}/\overline{g}, 2\mathrm{Im}(\overline{j}/\overline{g}), 0).$$

Isometric spheres are examples of bisectors. Mostow [**9**] showed that a bisector is the preimage of a geodesic, called spine, under orthogonal projection onto the unique complex line containing it. The fibres of this projection are complex lines called the slices of the bisector. Goldman [**7**] showed that a bisector is the union of all totally real Larangian planes containing the spine. Such Lagrangian planes are called the meridians. Together the slices and meridians give geographical coordinates on the bisector (see [**2**]). In order to parametrise $\mathcal{S}_0$ we write

$$|z|^2 + u - it = \frac{2}{\sqrt{3}} e^{i\theta}$$

for $\theta \in [-\pi/2, \pi/2]$ and

$$z = re^{i\alpha + i\theta/2}$$

for $r \in [-\sqrt{2\cos\theta/\sqrt{3}}, \sqrt{2\cos\theta/\sqrt{3}}]$ and $\alpha \in [-\pi/2, \pi/2)$.

In geographical coordinates $\mathcal{S}_0$ is parametrised by

$$\left\{ \begin{bmatrix} -e^{i\theta}/\sqrt{3} \\ re^{i\alpha + i\theta/2} \\ 1 \end{bmatrix} : \theta \in [-\pi/2, \pi/2], \alpha \in [-\pi/2, \pi/2), r \in \left[-\sqrt{\frac{2\cos\theta}{\sqrt{3}}}, \sqrt{\frac{2\cos\theta}{\sqrt{3}}}\right] \right\}. \tag{2·6}$$

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In horospherical coordinates, the point of $\mathcal{S}_0$ with geographical coordinates $(r, \theta, \alpha)$ is given by $(re^{i\alpha+i\theta/2}, -2\sin\theta/\sqrt{3}, 2\cos\theta/\sqrt{3}-r^2)$. The spine, slices and meridians of $\mathcal{S}_0$ are given in the next proposition in terms of geographical coordinates.

PROPOSITION 2·1. (cf. [**2**]). *The isometric sphere of* $I_1$, $\mathcal{S}_0$, *with coordinates given by* (2·6) *is a bisector. Moreover*;
(i) *the spine of* $\mathcal{S}_0$ *is given by* $r = 0$;
(ii) *the slices of* $\mathcal{S}_0$ *are given by* $\theta = \theta_0$ *for fixed* $\theta_0 \in [-\pi/2, \pi/2]$;
(iii) *the meridians of* $\mathcal{S}_0$ *are given by* $\alpha = \alpha_0$ *for fixed* $\alpha_0 \in [\pi/2, \pi/2)$.

## 3. *The group* $G_2$

In this section we describe some general features of the group $G_2$. In particular, we find a set of generators and describe a fundamental domain for the stabiliser of infinity.

Let $\omega$ denote the cube root of unity $(-1 + i\sqrt{3})/2$. The group $G_2$ comprises all matrices in U(2,1) (preserving the Hermitian form given in Section 2·1) of the form

$$A = \begin{bmatrix} a & b & ci/\sqrt{3} \\ di\sqrt{3} & e & f \\ gi\sqrt{3} & hi\sqrt{3} & j \end{bmatrix}$$

where $a, b, \ldots, h, j$ are all elements of $\mathbb{Z}[\omega]$.

### 3·1. *The stabiliser* $(G_2)_\infty$ *of* $q_\infty$

First, we want to analyse $(G_2)_\infty$, the stabiliser of $q_\infty$. Every element of $(G_2)_\infty$ is upper triangular, and its diagonal entries are units in $\mathbb{Z}[\omega]$. Therefore, $(G_2)_\infty$ contains no dilations and so is a subgroup of Isom($\mathfrak{R}$); and fits into the exact sequence (2) as

$$0 \longrightarrow \mathbb{R} \bigcap (G_2)_\infty \longrightarrow (G_2)_\infty \xrightarrow{\Pi_*} \Pi_*((G_2)_\infty) \longrightarrow 1.$$

We can find explicitly the kernel and image of $(G_2)_\infty$ as in [**2**].

PROPOSITION 3·1. *The stabiliser* $(G_2)_\infty$ *of* $q_\infty$ *in* $G_2$ *satisfies*

$$0 \longrightarrow \frac{2}{\sqrt{3}}\mathbb{Z} \longrightarrow (G_2)_\infty \xrightarrow{\Pi_*} \triangle(2,3,6) \longrightarrow 1,$$

*where* $\triangle(2,3,6)$ *denotes the triangle group comprising orientation-preserving symmetries of* $\mathbb{Z}[\omega]$.

*Proof.* From the explicit construction (2·3) of $\Pi_*$, we see that for $A \in (G_2)_\infty$, $\Pi_*(A)$ is not dependent on the entry $z_{13}$ of $A$. As in the proof of [**2**], we obtain that $\Pi_*((G_2)_\infty)$ is the triangle group $\triangle(2,3,6)$. Likewise, the kernel of $\Pi_*$ is easily seen to consist of those vertical translations in $(G_2)_\infty$, that is , Heisenberg translation by $(0, 2n/\sqrt{3}) \in \mathfrak{R}$ for $n \in \mathbb{Z}$.

The following proposition gives the generators of $(G_2)_\infty$.

PROPOSITION 3·2. $(G_2)_\infty$ *is generated by*

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\overline{\omega} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 1 & -\overline{\omega}i/\sqrt{3} \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

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*Proof.* The triangle group $\triangle(2,3,6)$ is generated by

$$\Pi_*(R) = R_* : z \longrightarrow -\overline{\omega}z, \qquad \Pi_*(S) = S_* : z \longrightarrow -z+1.$$

Hence we only need to show that $R$ and $S$ generate $\mathbb{R} \bigcap (G_2)_\infty \simeq (2/\sqrt{3})\mathbb{Z}$. Observe that

$$S^2 = \begin{bmatrix} 1 & 0 & i/\sqrt{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = T. \tag{3·1}$$

This is precisely the generator of $(2/\sqrt{3})\mathbb{Z} \bigcap (G_2)_\infty$. Therefore $(G_2)_\infty$ is generated by $R$, $S$ and $T$.

We first construct a fundamental domain for the parabolic subgroup $(G_2)_\infty$ acting on the Heisenberg group. We want to describe the action of $R$ and $S$ on each horosphere. Recall that complex hyperbolic space can be parametrised in horospherical coordinates $(z,t,u) \in \mathfrak{R} \times \mathbb{R}^+$ by:

$$(z,t,u) \longrightarrow \begin{bmatrix} (-|z|^2 - u + it)/2 \\ z \\ 1 \end{bmatrix}.$$

Then, using the matrices of $R$ and $S$, we obtain the following action of $R$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\overline{\omega} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (-|z|^2 - u + it)/2 \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} (-|z|^2 - u + it)/2 \\ -\overline{\omega}z \\ 1 \end{bmatrix}$$

and $S$,

$$\begin{aligned} &\begin{bmatrix} 1 & 1 & -\overline{\omega}i/\sqrt{3} \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (-|z|^2 - u + it)/2 \\ z \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} (-|z|^2 - u + it)/2 + z - \overline{\omega}i/\sqrt{3} \\ -z+1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \left(-|-z+1|^2 - u + i\left[t + 2\mathrm{Im}(z) + 1/\sqrt{3}\right]\right)/2 \\ -z+1 \\ 1 \end{bmatrix}. \end{aligned}$$

We also introduce two elliptic elements fixing $q_\infty$ used in later:

$$RS^{-1} = \begin{bmatrix} 1 & 1 & \omega i/\sqrt{3} \\ 0 & \overline{\omega} & -\overline{\omega} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad R^{-1}S = \begin{bmatrix} 1 & 1 & -\overline{\omega}i/\sqrt{3} \\ 0 & \omega & -\omega \\ 0 & 0 & 1 \end{bmatrix}. \tag{3·2}$$

Therefore, we have the properties of those matrices:
(i) $R$ has order 6 and rotates the complex line spanned by 0 and $\infty$ by $\pi/3$, that is, it acts on horospherical coordinates by $R : (z,t,u) \mapsto (-\overline{\omega}z, t, u)$.

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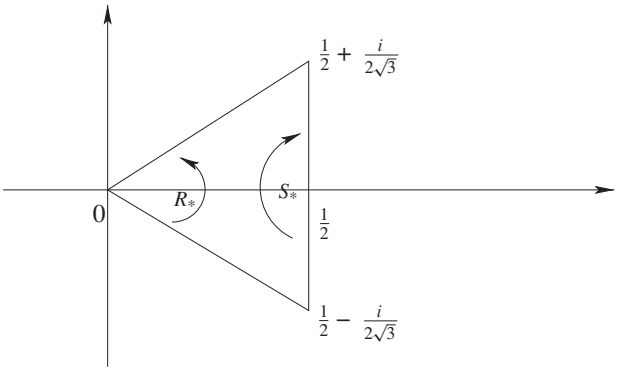



Fig. 1. Fundamental domain for $\Delta(2, 3, 6)$ in $\mathbb{C}$. The map $R_*$ rotates by $\pi/3$ about the origin and the map $S_*$ rotates by $\pi$ about 1/2.

(ii) $T$ is the shortest vertical translation. Its action on horospherical coordinates is given by $T : (z, t, u) \mapsto (z, t + 2/\sqrt{3}, u)$.
(iii) $S$ has a screw parabolic map with axis the complex line through $z = 1/2$. It acts on horospherical coordinates by $S : (z, t, u) \mapsto (-z + 1, t + 2\mathrm{Im}(z) + 1/\sqrt{3}, u)$.
(iv) $RS^{-1}$ has order 3 and rotates the complex line through $z = 1/2 + i/2\sqrt{3}$ by $2\pi/3$. It acts on horospherical coordinates by $RS^{-1} : (z, t, u) \mapsto (\overline{\omega}z - \overline{\omega}, t + 2\mathrm{Im}(\omega z) - 1/\sqrt{3}, u)$.
(v) $R^{-1}S$ also has order 3 and rotates the complex line through $z = 1/2 - i/2\sqrt{3}$ by $2\pi/3$. It acts on horospherical coordinates by $R^{-1}S : (z, t, u) \mapsto (\omega z - \omega, t + 2\mathrm{Im}(z) + 1/\sqrt{3}, u)$.

These actions preserve each horosphere, that is, the set of points where $u$ is constant. Thus we may drop the dependence on $u$, and we obtain the action on $\mathfrak{R} = \mathbb{C} \times \mathbb{R}$.

We now construct a fundamental domain for the action of $(G_2)_\infty$ on $\mathfrak{R}$. We know that $\Pi_*((G_2)_\infty = \Delta(2, 3, 6)$ is a triangle group of $\mathbb{Z}[\omega]$. A fundamental domain for this group is the triangle in $\mathbb{C}$ with vertices at $0, 1/2 - i/2\sqrt{3}$ and $1/2 + i/2\sqrt{3}$; see Figure 1. Side pairing maps are given by

$$R_* = \begin{bmatrix} -\overline{\omega} & 0 \\ 0 & 1 \end{bmatrix}, \qquad S_* = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The first of these is a rotation of order 6 fixing 0 and the second is a rotation of order 2 fixing $1/2$.

A fundamental domain for $(G_2)_\infty$ can be constructed by the intersection of preimages of this triangle under vertical projection $\Pi$ and a fundamental domain for $\ker(\Pi_*)$. The inverse image of the triangle under $\Pi$ is an infinite prism. The kernel of $\Pi_*$ is the infinite cycle group generated by $T$, the vertical translation by $(0, 2/\sqrt{3})$. So a fundamental domain for this group is the set of points where $-1/\sqrt{3} < t < 1/\sqrt{3}$. Hence a fundamental domain for $(G_2)_\infty$ is the prism in $\mathfrak{R}$ with vertices at $(0, \pm 1/\sqrt{3})$, $(1/2 - i/2\sqrt{3}, \pm 1/\sqrt{3})$, $(1/2 + i/2\sqrt{3}, \pm 1/\sqrt{3})$; see Figure 2. The side pairings act on points $(z, t) \in \mathfrak{R}$ as follows:

$$R(z, t) = (-\overline{\omega}z, t), \quad S(z, t) = (-z + 1, t + 2\mathrm{Im}(z) + 1/\sqrt{3}), \quad T(z, t) = (z, t + 2/\sqrt{3}).$$

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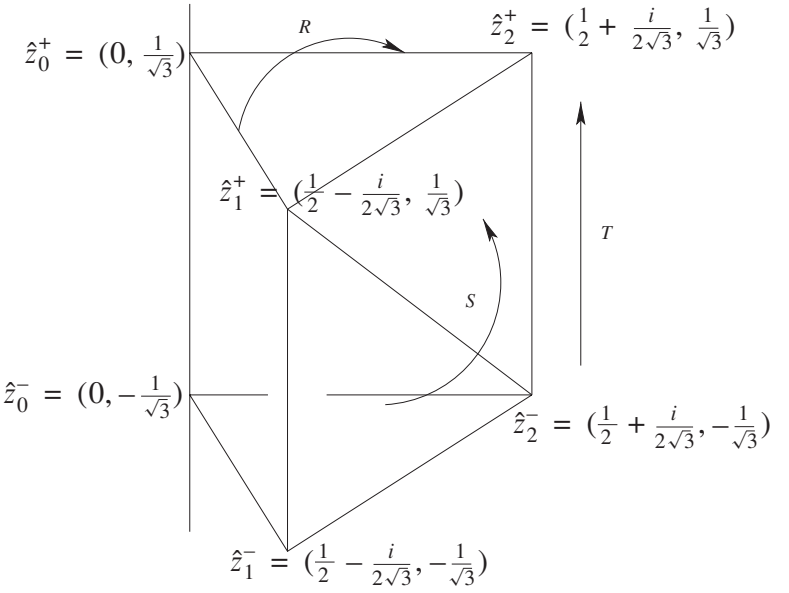



Fig. 2. Fundamental domain for $(G_2)_\infty$ in the Heisenberg group: the map $R$ rotates through $\pi/3$ about $z = 0$; the map $S$ is a Heisenberg rotations through $\pi$ about $z = 1/2$; the map $T$ translates $t$ by $2/\sqrt{3}$.

We can see the action of $S$ on the triangle $(\hat{z}_1^+, \hat{z}_1^-, \hat{z}_2^-)$ is the composition of a reflection on the edge $\hat{z}_1^+\hat{z}_2^-$ and a clockwise rotation. We list the side pairing maps in the following table:

$$\begin{aligned} R &: (\hat{z}_0^+, \hat{z}_0^-, \hat{z}_1^-, \hat{z}_1^+) \longmapsto (\hat{z}_0^+, \hat{z}_0^-, \hat{z}_2^-, \hat{z}_2^+), \\ S &: (\hat{z}_1^+, \hat{z}_1^-, \hat{z}_2^-) \longmapsto (\hat{z}_2^+, \hat{z}_2^-, \hat{z}_1^+), \\ T &: (\hat{z}_0^-, \hat{z}_1^-, \hat{z}_2^-) \longmapsto (\hat{z}_0^+, \hat{z}_1^+, \hat{z}_2^+). \end{aligned}$$

Denoting all the edges by the ordered pairs of their endpoints, we can also consider the edge cycles given by these side-pairing are

$$\begin{aligned} &(\hat{z}_0^+, \hat{z}_0^-) \xrightarrow{R} (\hat{z}_0^+, \hat{z}_0^-), \\ &(\hat{z}_1^+, \hat{z}_1^-) \xrightarrow{S} (\hat{z}_2^+, \hat{z}_2^-) \xrightarrow{R^{-1}} (\hat{z}_1^+, \hat{z}_1^-), \\ &(\hat{z}_1^-, \hat{z}_2^-) \xrightarrow{S} (\hat{z}_2^-, \hat{z}_1^+) \xrightarrow{S} (\hat{z}_1^+, \hat{z}_2^+) \xrightarrow{T^{-1}} (\hat{z}_1^-, \hat{z}_2^-), \\ &(\hat{z}_0^-, \hat{z}_1^-) \xrightarrow{R} (\hat{z}_0^-, \hat{z}_2^-) \xrightarrow{T} (\hat{z}_0^+, \hat{z}_2^+) \xrightarrow{R^{-1}} (\hat{z}_0^+, \hat{z}_1^+) \xrightarrow{T^{-1}} (\hat{z}_0^-, \hat{z}_1^-). \end{aligned}$$

This has used all the edges of fundamental domain. The first of these cycles gives the relation $R^6 = I$, the second gives the relation $(R^{-1}S)^3 = I$, the third gives the relation $S^2 = T$ and the last gives the relation $T^{-1}R^{-1}TR = I$. These relations also follow from the properties of matrices (3·1) and (3·2).

This enables us to give the following proposition.

PROPOSITION 3·3. *A fundamental domain for $(G_2)_\infty$ is the prism with vertices (in Heisenberg coordinates) $(z, t) = (0, \pm 1/\sqrt{3})$, $(1/2 - i/2\sqrt{3}, \pm 1/\sqrt{3})$, $(1/2 + i/2\sqrt{3}, \pm 1/\sqrt{3})$.*

*A presentation is given by*

$$(G_2)_\infty = \langle R, S, T | R^6 = (R^{-1}S)^3 = [R, T] = identity, \quad S^2 = T \rangle.$$

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3·2. *Generators of $G_2$*

First recall the map

$$I_1 = \begin{bmatrix} 0 & 0 & i/\sqrt{3} \\ 0 & -\omega & 0 \\ i\sqrt{3} & 0 & 0 \end{bmatrix},$$

it has the isometric sphere $\mathcal{S}_0$ which we equip with geographical coordinates. The action of $I_1$ on $\mathcal{S}_0$ is given by

$$\begin{bmatrix} 0 & 0 & i/\sqrt{3} \\ 0 & -\omega & 0 \\ i\sqrt{3} & 0 & 0 \end{bmatrix} \begin{bmatrix} -e^{i\theta}/\sqrt{3} \\ re^{i\alpha+i\theta/2} \\ 1 \end{bmatrix} = \begin{bmatrix} i/\sqrt{3} \\ re^{i\alpha+i\theta/2-i\pi/3} \\ -ie^{i\theta} \end{bmatrix} = -ie^{i\theta} \begin{bmatrix} -e^{-i\theta}/\sqrt{3} \\ re^{i(\alpha+\pi/6)-i\theta/2} \\ 1 \end{bmatrix}.$$

We see that $I_1$ maps $\mathcal{S}_0$ to itself, sending the point with coordinates $(r, \theta, \alpha)$ to the point with coordinates $(r, -\theta, \alpha + \pi/6)$ when $-\pi/2 \leqslant \alpha < \pi/3$ or the point with coordinates $(-r, -\theta, \alpha - 5\pi/6)$ when $\pi/3 \leqslant \alpha < \pi/2$. $I_1$ is an elliptic element of order 12. Moreover, $I_1$ swaps the inside and the outside of $\mathcal{S}_0$.

PROPOSITION 3·4. *Let $(G_2)_\infty$ be the stabiliser of $\infty$ in $G_2$. Then there exists a fundamental domain for the action of $(G_2)_\infty$ on $\partial \boldsymbol{H}^2_{\mathbb{C}}$ whose interior lies inside the ball in $\partial \boldsymbol{H}^2_{\mathbb{C}}$ whose boundary is $\partial \mathcal{S}_0$.*

*Proof.* The same results are proved for the Eisenstein–Picard group in [**2**] and Gauss–Pircard group in [**3**]. As in the proof of Proposition 3·3, the prism with vertices (in Heisenberg coordinates) $(0, \pm 1/\sqrt{3})$, $(1/2 - i/2\sqrt{3}, \pm 1/\sqrt{3})$, $(1/2 + i/2\sqrt{3}, \pm 1/\sqrt{3})$ is a fundamental domain for $(G_2)_\infty$. All these vertices lie inside $\partial \mathcal{S}_0$ in Heisenberg group. Since $\partial \mathcal{S}_0$ is (affinely) convex, the whole prism lies inside $\partial \mathcal{S}_0$.

We now show that adjoining $I_1$ to $(G_2)_\infty$ gives the full group $G_2$. In order to prove that, we should show that $\langle R, S, T, I_1 \rangle$ has only one cusp. The fact that $G_2$ has only one cusp is already known we refer the reader to [**10**].

PROPOSITION 3·5. *The group $G_2$ is generated by $T$, $R$, $S$ and $I_1$.*

*Proof.* (cf. [**2**, **3**]). The same results for the Eisenstein–Picard group and Gauss–Picard group. Since there is a fundamental domain for $\langle R, S, T, I_1 \rangle$ lying outside the isometric sphere of $I_1$ and inside the fundamental domain for $\langle R, S, T \rangle$, the Proposition 3·4 implies $\langle R, S, T, I_1 \rangle$ has only one cusp. The fact that $G_2$ has the same cusp as the group generated by $T$, $R$, $S$ and $I_1$ implies that they are the same.

## 4. *Construction of a prism*

In this section, we will construct a prism $\mathbf{P}_0$ by the intersection of isometric spheres. In fact, the prism $\mathbf{P}_0$ is the compact part of the boundary of a fundamental domain for $G_2$. To have a global view of the prism we refer the readers to Figures 3 and 6. The prism contains two compact sides of the fundamental domain and the other sides are cones based at the faces of this prism with cone point the ideal vertex.

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We begin by investigating the intersection of $\mathcal{S}_0$ with its neighbouring isometric spheres.

4·1. *The intersection of $\mathcal{S}_0$ and its neighbours*

Here we will use some isometric spheres and their intersections to define the compact sides. The bisectors we need are the following:

$$\mathcal{S}_0,\ T(\mathcal{S}_0),\ T^{-1}(\mathcal{S}_0),\ S(\mathcal{S}_0),\ S^{-1}(\mathcal{S}_0),\ RS^{-1}(\mathcal{S}_0),\ R^{-1}S(\mathcal{S}_0)$$

which are, respectively, the isometric spheres of

$$I_1,\ \ TI_1T^{-1},\ \ T^{-1}I_1T,\ \ SI_1S^{-1},\ \ S^{-1}I_1S,\ \ RS^{-1}I_1SR^{-1},\ \ R^{-1}SI_1S^{-1}R.$$

In particular, it suffices to determine

$$T^{\pm}(\mathcal{S}_0) \bigcap \mathcal{S}_0, \quad S^{\pm}(\mathcal{S}_0) \bigcap \mathcal{S}_0.$$

The other intersections are easily obtained from these. For example,

$$R^{-1}S(\mathcal{S}_0) \bigcap \mathcal{S}_0 = R^{-1}(S(\mathcal{S}_0) \bigcap \mathcal{S}_0).$$

We need to investigate the intersection of these bisectors with $\mathcal{S}_0$. A direct computation shows that $T(\mathcal{S}_0)$, $T^{-1}(\mathcal{S}_0)$, $S(\mathcal{S}_0)$, $S^{-1}(\mathcal{S}_0)$ are given by

$$T(\mathcal{S}_0) = \left\{(z,t,u) : \left||z|^2 + u - it + 2i/\sqrt{3}\right| = 2/\sqrt{3}\right\}, \tag{4·1}$$

$$T^{-1}(\mathcal{S}_0) = \left\{(z,t,u) : \left||z|^2 + u - it - 2i/\sqrt{3}\right| = 2/\sqrt{3}\right\}, \tag{4·2}$$

$$S(\mathcal{S}_0) = \left\{(z,t,u) : \left||z|^2 - 2z + u + 1 - it + i/\sqrt{3}\right| = 2/\sqrt{3}\right\}, \tag{4·3}$$

$$S^{-1}(\mathcal{S}_0) = \left\{(z,t,u) : \left||z|^2 - 2z + u + 1 - it - i/\sqrt{3}\right| = 2/\sqrt{3}\right\}. \tag{4·4}$$

We start to describe the intersection of these bisectors. Note that $T$, $T^{-1}$ and $S$, $S^{-1}$ will be denoted by $T^{\pm}$ and $S^{\pm}$ respectively in Lemmas 4·1 and 4·2.

LEMMA 4·1. *In geographical coordinates, a point $(r,\theta,\alpha)$ of $\mathcal{S}_0$ with $\theta \in [-\pi/6, \pi/6]$ does not intersect the interior of $T(\mathcal{S}_0)$ and $T^{-1}(\mathcal{S}_0)$. Furthermore, $T^{\pm}(\mathcal{S}_0) \bigcap \mathcal{S}_0$ is a common slice of the bisectors $T^{\pm}(\mathcal{S}_0)$ and $\mathcal{S}_0$ which corresponds to $\theta = \mp\pi/6$.*

*Proof.* Recall that a point of $\mathcal{S}_0$ is given by

$$\begin{bmatrix} -e^{i\theta}/\sqrt{3} \\ re^{i\alpha+i\theta/2} \\ 1 \end{bmatrix}.$$

In horospherical coordinates, it is given by $(re^{i\alpha+i\theta/2}, -2\sin\theta/\sqrt{3}, 2\cos\theta/\sqrt{3} - r^2)$, where $\theta \in [-\pi/2, \pi/2]$, $\alpha \in [-\pi/2, \pi/2)$, $r \in [-\sqrt{(2\cos\theta)/\sqrt{3}}, \sqrt{(2\cos\theta)/\sqrt{3}}]$. Therefore, a point of $\mathcal{S}_0$ does not intersect the interior of $T(\mathcal{S}_0)$ if and only if

$$\left| r^2 + \frac{2}{\sqrt{3}}\cos\theta - r^2 + i\frac{2}{\sqrt{3}}\sin\theta + i\frac{2}{\sqrt{3}} \right| \geqslant \frac{2}{\sqrt{3}}.$$

A simple computation implies that $\sin\theta \geqslant -1/2$, i.e. $\theta \in [-\pi/6, \pi/2]$. Similarly, a point of $\mathcal{S}_0$ does not intersect the interior of $T^{-1}(\mathcal{S}_0)$ if and only if $\theta \in [-\pi/2, \pi/6]$. Therefore, $T^{\pm}(\mathcal{S}_0) \bigcap \mathcal{S}_0$ are respectively the slices $\theta = \mp\pi/6$. This proves the result.

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LEMMA 4·2. *$S^{\pm}(\mathcal{S}_0) \bigcap \mathcal{S}_0$ is a topological disc.*

*Proof.* In fact in horospherical coordinates $(z, t, u)$ of the bisector $\mathcal{S}_0$, the intersection is foliated by arcs of circles, each contained in a slice of height $t$. We show it explicitly as follows. Recall that the coordinates $(z, t)$, with $|z|^4 + t^2 < 4/3$, parametrise the bisector $\mathcal{S}_0$ whose equation, in horospherical coordinates, are $(|z|^2 + u)^2 + t^2 = 4/3$. As vectors in $\mathbb{C}^{2,1}$ we obtain therefore

$$\begin{bmatrix} (-|z|^2 - u + it)/2 \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} (-\sqrt{4/3 - t^2} + it)/2 \\ z \\ 1 \end{bmatrix}.$$

Then we have

$$\begin{aligned} S \cdot & \begin{bmatrix} (-\sqrt{4/3 - t^2} + it)/2 \\ z \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & -\overline{\omega} i/\sqrt{3} \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (-\sqrt{4/3 - t^2} + it)/2 \\ z \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} (-\sqrt{4/3 - t^2} + it)/2 + z - \overline{\omega} i/\sqrt{3} \\ z \\ 1 \end{bmatrix}. \end{aligned}$$

The condition that such a point lies in $\mathcal{S}_0$ is that the first entry has absolute value $1/\sqrt{3}$, that is,

$$\left| \frac{-\sqrt{4/3 - t^2} + it}{2} + z - \frac{\overline{\omega} i}{\sqrt{3}} \right| = \frac{1}{\sqrt{3}}.$$

For each fixed height (with $-2/\sqrt{3} < t < 2/\sqrt{3}$), the intersection lies on the circle centered at $(\sqrt{4/3 - t^2} - it)/2 + \overline{\omega} i/\sqrt{3}$ with radius $1/\sqrt{3}$. The condition for the existence of an intersection between this circle and the slice of height $t$ is that the sum of the radii of these two circles, $1/\sqrt{3} + (4/3 - t^2)^{1/4}$, be greater than the distance between the centers, that is, $|(\sqrt{4/3 - t^2} - it)/2 + \overline{\omega} i/\sqrt{3}|$. That gives the equation

$$\left[ \frac{1}{\sqrt{3}} + \left( \frac{4}{3} - t^2 \right)^{1/4} \right]^2 > \left| \frac{\overline{\omega} i}{\sqrt{3}} + \frac{\sqrt{4/3 - t^2} - it}{2} \right|^2$$

whose solution is

$$-\frac{2}{\sqrt{3}} < t < \frac{(291\sqrt{3} + 504)^{1/3}}{3} + \frac{1}{(291\sqrt{3} + 504)^{1/3}} - \frac{4\sqrt{3}}{3} < \frac{2}{\sqrt{3}}.$$

This shows that the intersection is foliated by a one-parameter family of arcs of circles and therefore is a topological disc. Observe that $S^{-1}(\mathcal{S}_0) \bigcap \mathcal{S}_0 = S^{-1}(\mathcal{S}_0 \bigcap S(\mathcal{S}_0)$ implies that is also a topological disc.

We describe this intersection in more details below.

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4·2. *The vertices*

In this section we define the vertices of the prism, all of them will correspond to vertices of the fundamental domain for $(G_2)_\infty$ in the Heisenberg group. In fact, the vertical projection $(z, t, u) \to (z, t)$ which projects the geodesic passing through $(z, t, u)$ to $q_\infty$ to its end point $(z, t)$ maps the vertices of this prism to the vertices of fundamental domain for $(G_2)_\infty$.

We start to define the vertices of the prism as points of $\mathbb{C}^{2,1}$ and discuss their horospherical and geographical coordinates. There are three vertices lying on the common slice of $\mathcal{S}_0$ and $T(\mathcal{S}_0)$, which are denoted by $z_j^+$, $j = 0, 1, 2$. Let $z_0^+$ define the intersection of the spine of $\mathcal{S}_0$ and the common slice $\theta = -\pi/6$ (that is height $t = 1/\sqrt{3}$ in Heisenberg group $\mathfrak{R}$). The other two vertices are the intersection of four bisectors, that are given by

$$z_1^+ = \mathcal{S}_0 \bigcap T(\mathcal{S}_0) \bigcap S^{-1}(\mathcal{S}_0) \bigcap R^{-1}S(\mathcal{S}_0),$$
$$z_2^+ = \mathcal{S}_0 \bigcap T(\mathcal{S}_0) \bigcap S(\mathcal{S}_0) \bigcap RS^{-1}(\mathcal{S}_0).$$

Similarly, three vertices lie on the common slice of $\mathcal{S}_0$ and $T^{-1}(\mathcal{S}_0)$, which are denoted by $z_j^-$, $j = 0, 1, 2$. Let $z_0^-$ define the intersection of the spine of $\mathcal{S}_0$ and the common slice $\theta = \pi/6$ (that is height $t = -1/\sqrt{3}$ in $\mathfrak{R}$). The other two vertices are given by

$$z_1^- = \mathcal{S}_0 \bigcap T^{-1}(\mathcal{S}_0) \bigcap S^{-1}(\mathcal{S}_0) \bigcap R^{-1}S(\mathcal{S}_0),$$
$$z_2^- = \mathcal{S}_0 \bigcap T^{-1}(\mathcal{S}_0) \bigcap S(\mathcal{S}_0) \bigcap RS^{-1}(\mathcal{S}_0).$$

We now list all the vertices as points of $\mathbb{C}^{2,1}$:
the vertices on the common slice of $\mathcal{S}_0$ and $T(\mathcal{S}_0)$ are

$$z_0^+ = \begin{bmatrix} -\frac{1}{2} + \frac{i}{2\sqrt{3}} \\ 0 \\ 1 \end{bmatrix}, \quad z_1^+ = \begin{bmatrix} -\frac{1}{2} + \frac{i}{2\sqrt{3}} \\ \frac{1}{2} - \frac{i}{2\sqrt{3}} \\ 1 \end{bmatrix}, \quad z_2^+ = \begin{bmatrix} -\frac{1}{2} + \frac{i}{2\sqrt{3}} \\ \frac{1}{2} + \frac{i}{2\sqrt{3}} \\ 1 \end{bmatrix}.$$

The vertices on the common slice of $\mathcal{S}_0$ and $T^{-1}(\mathcal{S}_0)$ are

$$z_0^- = \begin{bmatrix} -\frac{1}{2} - \frac{i}{2\sqrt{3}} \\ 0 \\ 1 \end{bmatrix}, \quad z_1^- = \begin{bmatrix} -\frac{1}{2} - \frac{i}{2\sqrt{3}} \\ \frac{1}{2} - \frac{i}{2\sqrt{3}} \\ 1 \end{bmatrix}, \quad z_2^- = \begin{bmatrix} -\frac{1}{2} - \frac{i}{2\sqrt{3}} \\ \frac{1}{2} + \frac{i}{2\sqrt{3}} \\ 1 \end{bmatrix}.$$

In horospherical and geographical coordinates of these vertices $z_j^\pm$ points are given by

| | $z$ | $t$ | $u$ | $r$ | $\theta$ | $\alpha$ |
|---|---|---|---|---|---|---|
| $z_0^+$ | 0 | $1/\sqrt{3}$ | 1 | 0 | $-\pi/6$ | |
| $z_1^+$ | $1/2 - i/2\sqrt{3}$ | $1/\sqrt{3}$ | 2/3 | $1/\sqrt{3}$ | $-\pi/6$ | $-\pi/12$ |
| $z_2^+$ | $1/2 + i/2\sqrt{3}$ | $1/\sqrt{3}$ | 2/3 | $1/\sqrt{3}$ | $-\pi/6$ | $\pi/4$ |
| $z_0^-$ | 0 | $-1/\sqrt{3}$ | 1 | 0 | $\pi/6$ | |
| $z_1^-$ | $1/2 - i/2\sqrt{3}$ | $-1/\sqrt{3}$ | 2/3 | $1/\sqrt{3}$ | $\pi/6$ | $-\pi/4$ |
| $z_2^-$ | $1/2 + i/2\sqrt{3}$ | $-1/\sqrt{3}$ | 2/3 | $1/\sqrt{3}$ | $\pi/6$ | $\pi/12$ |

By computation we verify that $z_1^+, z_2^- \in S(\mathcal{S}_0) \bigcap S^{-1}(\mathcal{S}_0)$.

4·3. *The edges*

We first investigate the intersection of $\mathcal{S}_0$ and $S(\mathcal{S}_0)$, $S^{-1}(\mathcal{S}_0)$ more closely than before. We compute it explicitly in geographical coordinates and show the following result.

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LEMMA 4·3. *A point $(r, \theta, \alpha)$ of $\mathcal{S}_0$ in geographical coordinates with $-\pi/12 \leqslant \alpha \leqslant \pi/4$ does not intersect the interior of $S(\mathcal{S}_0)$, provided that*

$$r \leqslant \frac{1}{\sqrt{3}}\left[2\cos\left(\frac{\theta}{2}-\frac{\pi}{12}\right)\cos\left(\alpha-\frac{\pi}{12}\right)-\sqrt{1-4\cos^2\left(\frac{\theta}{2}-\frac{\pi}{12}\right)\sin^2\left(\alpha-\frac{\pi}{12}\right)}\right]$$

*with equality if and only if the point lies in $\mathcal{S}_0 \cap S(\mathcal{S}_0)$.*

*Proof.* A point of $\mathbf{H}^2_{\mathbb{C}}$ does not lie in the interior of $S(\mathcal{S}_0)$ (cf. 4·3) if satisfies

$$\frac{2}{\sqrt{3}} \leqslant \left||z|^2 - 2z + u + 1 - it + \frac{i}{\sqrt{3}}\right|. \tag{4·5}$$

If such a point lies on $\mathcal{S}_0$, we can write it in geographical coordinates. Substituting in (4·5) gives

$$1 \leqslant \left|-\sqrt{3}re^{i(\alpha+\theta/2)} + e^{i\theta} + e^{i\pi/6}\right| = \left|\sqrt{3}re^{i(\alpha-\pi/12)} - 2\cos\left(\frac{\theta}{2}-\frac{\pi}{12}\right)\right|. \tag{4·6}$$

We have equality in (4·6) if and only if the point lies in $\mathcal{S}_0 \cap S(\mathcal{S}_0)$. Expanding out the right-hand side of (4·6), we see that is equivalent to

$$3r^2 - 4\sqrt{3}r\cos\left(\frac{\theta}{2}-\frac{\pi}{12}\right)\cos\left(\alpha-\frac{\pi}{12}\right) + 4\cos^2\left(\frac{\theta}{2}-\frac{\pi}{12}\right) - 1 \geqslant 0. \tag{4·7}$$

By solving this inequality, we know that all points of $\mathcal{S}_0$ do not intersect the interior of $S(\mathcal{S}_0)$ if satisfy with

$$r \leqslant \frac{1}{\sqrt{3}}\left[2\cos\left(\frac{\theta}{2}-\frac{\pi}{12}\right)\cos\left(\alpha-\frac{\pi}{12}\right)-\sqrt{1-4\cos^2\left(\frac{\theta}{2}-\frac{\pi}{12}\right)\sin^2\left(\alpha-\frac{\pi}{12}\right)}\right]$$

or

$$r \geqslant \frac{1}{\sqrt{3}}\left[2\cos\left(\frac{\theta}{2}-\frac{\pi}{12}\right)\cos\left(\alpha-\frac{\pi}{12}\right)+\sqrt{1-4\cos^2\left(\frac{\theta}{2}-\frac{\pi}{12}\right)\sin^2\left(\alpha-\frac{\pi}{12}\right)}\right].$$

We claim that the second of these solutions is always greater than $\sqrt{2\cos(\theta)/\sqrt{3}}$ as $-\pi/12 \leqslant \alpha \leqslant \pi/4$ and so does not correspond to a point of $\mathcal{S}_0$. In order to see this, observe that $-\pi/12 \leqslant \alpha \leqslant \pi/4$ implies $2\cos(\alpha-\pi/12) \geqslant \sqrt{3}$ and $4\sin^2(\alpha-\pi/12) \leqslant 1$. Thus

$$\begin{aligned}
&\frac{1}{\sqrt{3}}\left[2\cos\left(\frac{\theta}{2}-\frac{\pi}{12}\right)\cos\left(\alpha-\frac{\pi}{12}\right)+\sqrt{1-4\cos^2\left(\frac{\theta}{2}-\frac{\pi}{12}\right)\sin^2\left(\alpha-\frac{\pi}{12}\right)}\right]\\
&\quad\geqslant \frac{1}{\sqrt{3}}\left[\sqrt{3}\cos\left(\frac{\theta}{2}-\frac{\pi}{12}\right)-\sin\left(\frac{\theta}{2}-\frac{\pi}{12}\right)\right]\\
&\quad= \frac{2}{\sqrt{3}}\cos\left(\frac{\theta}{2}+\frac{\pi}{12}\right)\\
&\quad= \sqrt{\frac{2}{\sqrt{3}}\left[\frac{\cos(\theta+\pi/6)+1}{\sqrt{3}}\right]}\\
&\quad\geqslant \sqrt{\frac{2\cos\theta}{\sqrt{3}}}.
\end{aligned}$$

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In fact,

$$\frac{\cos(\theta+\pi/6)+1}{\sqrt{3}}-\cos\theta=\frac{1}{\sqrt{3}}\left[\frac{\sqrt{3}}{2}\cos\theta-\frac{1}{2}\sin\theta+1-\sqrt{3}\cos\theta\right]$$
$$=\frac{1-\sin(\theta+\pi/3)}{\sqrt{3}}\geqslant 0.$$

Note that when $\theta=\pi/6$ and $\alpha=\pi/12$ or $\pi/4$, then we have equality in both inequalities above. But in this case $1=4\cos^2(\theta/2-\pi/12)\sin^2(\alpha-\pi/12)$, so the quadratic equation (4·7) has a repeated root.

Similarly, we can also obtain the following results.

LEMMA 4·4. *A point $(r,\theta,\alpha)$ of $\mathcal{S}_0$ in geographical coordinates with $-\pi/4\leqslant\alpha\leqslant\pi/12$ does not intersect the interior of $S^{-1}(\mathcal{S}_0)$, provided that*

$$r\leqslant\frac{1}{\sqrt{3}}\left[2\cos\left(\frac{\theta}{2}+\frac{\pi}{12}\right)\cos\left(\alpha+\frac{\pi}{12}\right)-\sqrt{1-4\cos^2\left(\frac{\theta}{2}+\frac{\pi}{12}\right)\sin^2\left(\alpha+\frac{\pi}{12}\right)}\right]$$

*with equality if and only if the point lies in $\mathcal{S}_0\cap S^{-1}(\mathcal{S}_0)$.*

We should understand the intersection of $\mathcal{S}_0$ and $R^{-1}S(\mathcal{S}_0)$, $RS^{-1}(\mathcal{S}_0)$ and observe that $R^{-1}S(\mathcal{S}_0)\cap\mathcal{S}_0=R^{-1}(S(\mathcal{S}_0)\cap\mathcal{S}_0)$, $RS^{-1}(\mathcal{S}_0)\cap\mathcal{S}_0=R(S^{-1}(\mathcal{S}_0)\cap\mathcal{S}_0)$. The following lemmas can be derived from Lemma 4·3 and Lemma 4·4.

LEMMA 4·5. *A point $(r,\theta,\alpha)$ of $\mathcal{S}_0$ in geographical coordinates with $-\pi/4\leqslant\alpha\leqslant-\pi/12$ does not intersect the interior of $R^{-1}S(\mathcal{S}_0)$, provided that*

$$r\leqslant\frac{1}{\sqrt{3}}\left[2\cos\left(\frac{\theta}{2}-\frac{\pi}{12}\right)\cos\left(\alpha+\frac{\pi}{4}\right)-\sqrt{1-4\cos^2\left(\frac{\theta}{2}-\frac{\pi}{12}\right)\sin^2\left(\alpha+\frac{\pi}{4}\right)}\right]$$

*with equality if and only if the point lies in $\mathcal{S}_0\cap R^{-1}S(\mathcal{S}_0)$.*

LEMMA 4·6. *A point $(r,\theta,\alpha)$ of $\mathcal{S}_0$ in geographical coordinates with $-\pi/12\leqslant\alpha\leqslant5\pi/12$ does not intersect the interior of $RS^{-1}(\mathcal{S}_0)$, provided that*

$$r\leqslant\frac{1}{\sqrt{3}}\left[2\cos\left(\frac{\theta}{2}+\frac{\pi}{12}\right)\cos(\alpha-\frac{\pi}{4})-\sqrt{1-4\cos^2\left(\frac{\theta}{2}+\frac{\pi}{12}\right)\sin^2\left(\alpha-\frac{\pi}{4}\right)}\right]$$

*with equality if and only if the point lies in $\mathcal{S}_0\cap RS^{-1}(\mathcal{S}_0)$.*

We can now characterize the edges of the prism not containing $q_\infty$. Some edges are obtained by intersecting three bisectors. Some edges are contained in geodesic arcs by construction. We now list them in the following lemma.

LEMMA 4·7.

(i) *The edge joining $z_0^+$ and $z_0^-$ is contained in the spine of $\mathcal{S}_0$.*

(ii) *The edge joining $z_0^\pm$ and $z_j^\pm$ for $j=1,2$ is a geodesic arc.*

(iii) *The edge joining $z_1^+$ and $z_2^+$ is given by points in geographical coordinates*

$$\theta=-\frac{\pi}{6},\quad r=\cos\left(\alpha-\frac{\pi}{12}\right)-\sqrt{\frac{1}{3}-\sin^2\left(\alpha-\frac{\pi}{12}\right)},\quad -\frac{\pi}{12}\leqslant\alpha\leqslant\frac{\pi}{4}.$$

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(iv) *The edge joining $z_1^-$ and $z_2^-$ is given by points in geographical coordinates*

$$\theta = \frac{\pi}{6}, \quad r = \cos\left(\alpha + \frac{\pi}{12}\right) - \sqrt{\frac{1}{3} - \sin^2\left(\alpha + \frac{\pi}{12}\right)}, \quad -\frac{\pi}{4} \leqslant \alpha \leqslant \frac{\pi}{12}.$$

(v) *The edge joining $z_1^+$ and $z_1^-$ is given by points in geographical coordinates*

$$r = \frac{1}{\sqrt{3}}, \quad \alpha = -\frac{\theta}{2} - \frac{\pi}{6} \quad \text{and} \quad -\frac{\pi}{6} \leqslant \theta \leqslant \frac{\pi}{6}.$$

*Notice that the complex line $z = re^{i\alpha + i\theta/2} = 1/2 - i/2\sqrt{3}$ contains this edge.*

(vi) *The edge joining $z_2^+$ and $z_2^-$ is given by points in geographical coordinates*

$$r = \frac{1}{\sqrt{3}}, \quad \alpha = \frac{\pi}{6} - \frac{\theta}{2} \quad \text{and} \quad -\frac{\pi}{6} \leqslant \theta \leqslant \frac{\pi}{6}.$$

*Notice that this edge lies in the complex line $z = 1/2 - i/2\sqrt{3}$.*

(vii) *The edge joining $z_1^+$ and $z_2^-$ is given by points in geographical coordinates*

$$r = \frac{1}{\sqrt{3}}, \quad \alpha = \frac{\theta}{2} \quad \text{and} \quad -\frac{\pi}{6} \leqslant \theta \leqslant \frac{\pi}{6}.$$

*Proof.* Parts (i) and (ii) follow by construction.

We prove (iii) and then (iv) follows similarly. The edge joining $z_1^+$ and $z_2^+$ is defined to be the common intersection of the bisectors $\mathcal{S}_0$, $T(\mathcal{S}_0)$, $S(\mathcal{S}_0)$. Being in the first two of these implies that $\theta = -\pi/6$ following from Lemma 4·1. Substituting in Lemma 4·3 and requiring equality gives

$$\begin{aligned} r &= \frac{1}{\sqrt{3}}\left[\sqrt{3}\cos\left(\alpha - \frac{\pi}{12}\right) - \sqrt{1 - 3\sin^2\left(\alpha - \frac{\pi}{12}\right)}\right] \\ &= \cos\left(\alpha - \frac{\pi}{12}\right) - \sqrt{\frac{1}{3} - \sin^2\left(\alpha - \frac{\pi}{12}\right)}. \end{aligned}$$

We know that $\alpha = -\pi/12$ at $z_1^+$ and $\alpha = \pi/4$ at $z_2^+$. Moreover, $r = 1/\sqrt{3}$ at both $z_1^+$ and $z_2^+$.

We now prove (vi) and then (v), (vii) follow similarly. The edge joining $z_2^+$ and $z_2^-$ is defined to be the common intersection of the bisectors $\mathcal{S}_0$, $S(\mathcal{S}_0)$, $RS^{-1}(\mathcal{S}_0)$. As in the proof of Lemma 4.3, the intersection of $\mathcal{S}_0$ and $S(\mathcal{S}_0)$ implies that

$$3r^2 - 4\sqrt{3}r\cos\left(\frac{\theta}{2} - \frac{\pi}{12}\right)\cos\left(\alpha - \frac{\pi}{12}\right) + 4\cos^2\left(\frac{\theta}{2} - \frac{\pi}{12}\right) - 1 = 0. \tag{4·8}$$

Similarly, the intersection of $RS^{-1}(\mathcal{S}_0)$ with $\mathcal{S}_0$ implies another equality

$$3r^2 - 4\sqrt{3}r\cos\left(\frac{\theta}{2} + \frac{\pi}{12}\right)\cos\left(\alpha - \frac{\pi}{4}\right) + 4\cos^2\left(\frac{\theta}{2} + \frac{\pi}{12}\right) - 1 = 0. \tag{4·9}$$

First we write $\phi = \alpha - \pi/6$. Then subtracting the equations (4·8) and (4·9) gives

$$\sqrt{3}r\sin\left(\frac{\theta}{2} - \phi\right) = \sin\theta. \tag{4·10}$$

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Furthermore, adding the equations (4·8) and (4·9) derives

$$\begin{aligned} 0 &= 3r^2 - 2\sqrt{3}r\cos\left(\tfrac{\theta}{2}+\phi\right) - 3r\cos\left(\tfrac{\theta}{2}-\phi\right) + \sqrt{3}\cos\theta + 1 \\ &= 3r^2 - \sqrt{3}r(2\cos\theta+\sqrt{3})\cos\left(\tfrac{\theta}{2}-\phi\right) - 2\sqrt{3}r\sin\theta\sin\left(\tfrac{\theta}{2}-\phi\right) + \sqrt{3}\cos\theta + 1. \end{aligned} \tag{4·11}$$

Combining with the equation (4·10), we can simplify the equation (4·11) that

$$\sqrt{3}r\left(2\cos\theta+\sqrt{3}\right)\cos\left(\frac{\theta}{2}-\phi\right) = 3r^2 - 1 + \left(2\cos\theta+\sqrt{3}\right)\cos\theta. \tag{4·12}$$

Making use of the equalities (4·10) and (4·12), we see that

$$\begin{aligned} 3r^2\left(2\cos\theta+\sqrt{3}\right)^2 &= 3r^2\left(2\cos\theta+\sqrt{3}\right)^2\cos^2\left(\frac{\theta}{2}-\phi\right) \\ &\quad + 3r^2\left(2\cos\theta+\sqrt{3}\right)^2\sin^2\left(\frac{\theta}{2}-\phi\right) \\ &= \left[3r^2 - 1 + \left(2\cos\theta+\sqrt{3}\right)\cos\theta\right]^2 + \left(2\cos\theta+\sqrt{3}\right)^2(1-\cos^2\theta) \\ &= (3r^2-1)^2 + 2(3r^2-1)^2\left(2\cos\theta+\sqrt{3}\right)\cos\theta + \left(2\cos\theta+\sqrt{3}\right)^2. \end{aligned}$$

Taking all terms to the right of the equation and simplifying, we can obtain

$$0 = (3r^2-1)\left(3r^2 - 2\sqrt{3}\cos\theta - 4\right).$$

Thus, we can get the solution that

$$3r^2 = 1 \quad \text{or} \quad 3r^2 = 2\sqrt{3}\cos\theta + 4,$$

of which the latter is impossible since $3r^2 \leqslant 2\sqrt{3}\cos\theta$. So $r = 1/\sqrt{3}$. Substituting $r = 1/\sqrt{3}$ in the equation (4.10), it is easy to get $\phi = -\theta/2$, that is, $\alpha = \pi/6 - \theta/2$. This proves the result.

*Remark* 4·8. The edges $z_1^+z_1^-$, $z_1^+z_2^-$ and $z_2^+z_2^-$ lie on the surface of cylinder which satisfies the equations $x^2+y^2 = 1/3$ and $-\pi/6 \leqslant z \leqslant \pi/6$ in the Euclidian coordinates. Notice that two faces $(z_1^+, z_1^-, z_2^-)$ and $(z_1^+, z_2^-, z_2^+)$ can not lie on the surface of cylinder.

### 4·4. *Compact sides*

In this section, we will choose two compact sides for the fundamental domain contained in the isometric sphere $\mathcal{S}_0$. We denote the two compact sides by $\mathbf{F}_7$ and $\mathbf{F}_8$. The noncompact sides will be denoted by $\mathbf{F}_i$, $1 \leqslant i \leqslant 6$.

Definition 4·9. In geographical coordinates from (2·6), the compact sides are the points in $\mathcal{S}_0$ given by

$$\begin{aligned} &\mathbf{F}_7 : 0 \leqslant r \leqslant \rho_1(\theta,\alpha),\ -\pi/6 \leqslant \theta \leqslant \pi/6,\ -\theta/2 - \pi/6 \leqslant \alpha \leqslant \theta/2, \\ &\mathbf{F}_8 : 0 \leqslant r \leqslant \rho_2(\theta,\alpha),\ -\pi/6 \leqslant \theta \leqslant \pi/6,\ \theta/2 \leqslant \alpha \leqslant \pi/6 - \theta/2, \end{aligned}$$

where

$$\rho_1(\theta,\alpha) = \frac{1}{\sqrt{3}}\left[2\cos\left(\frac{\theta}{2}+\frac{\pi}{12}\right)\cos\left(\alpha+\frac{\pi}{12}\right) - \sqrt{1-4\cos^2\left(\frac{\theta}{2}+\frac{\pi}{12}\right)\sin^2\left(\alpha+\frac{\pi}{12}\right)}\right],$$

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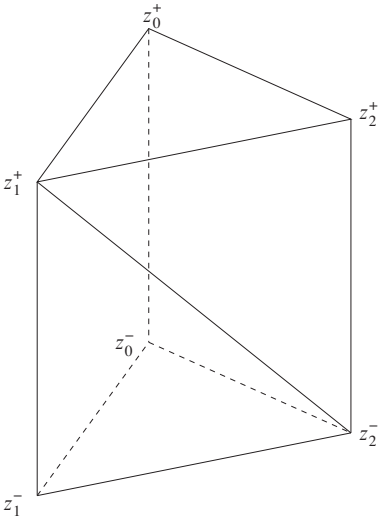
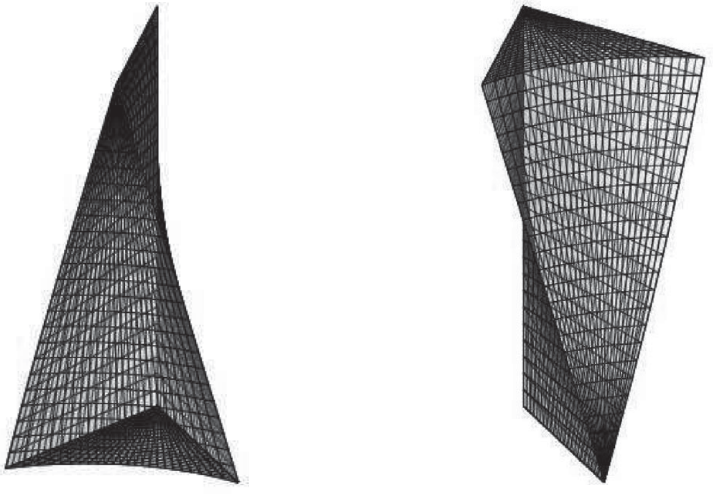



Fig. 3. A schematic view of 1-skeleton of the prism $\mathbf{P}_0$. The compact sides are contained in the isometric sphere $\mathcal{S}_0$ and the noncompact sides of the fundamental domain are cones over the faces of the prism with cone point $\infty$.



(a) $\mathbf{F}_7$ (b) $\mathbf{F}_8$

Fig. 4. A realistic view of the two compact sides contained in the isometric sphere $\mathcal{S}_0$.

$$\rho_2(\theta,\alpha)=\frac{1}{\sqrt{3}}\left[2\cos\left(\frac{\theta}{2}-\frac{\pi}{12}\right)\cos\left(\alpha-\frac{\pi}{12}\right)-\sqrt{1-4\cos^2\left(\frac{\theta}{2}-\frac{\pi}{12}\right)\sin^2\left(\alpha-\frac{\pi}{12}\right)}\right].$$

A realistic view of $\mathbf{F}_7$ and $\mathbf{F}_8$ is given in Figure 4. We next describe explicitly all the faces of $\mathbf{F}_7$ and $\mathbf{F}_8$.

(i) The face $\mathcal{F}_1 = (z_0^+, z_0^-, z_1^-, z_1^+)$ of $\mathbf{F}_7$ is union of geodesics arcs with endpoints in edges $(z_0^+ z_0^-)$ and $(z_1^+, z_1^-)$ on each slice. Therefore, its points are parametrised by

$$0 \leqslant r \leqslant \frac{1}{\sqrt{3}}, \quad \alpha = -\frac{\theta}{2} - \frac{\pi}{6} \quad \text{and} \quad -\frac{\pi}{6} \leqslant \theta \leqslant \frac{\pi}{6}.$$

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We remark this is the same as the face $(\hat{z}_0^+, \hat{z}_0^-, \hat{z}_1^-, \hat{z}_1^+)$ of the prism in Figure 2, which corresponds to $\arg(z) = -\pi/6$.

(ii) The common face $\mathcal{F}' = (z_0^+, z_1^+, z_2^-, z_0^-)$ of $\mathbf{F}_7$ and $\mathbf{F}_8$ is the union of geodesics arcs with endpoints in edges $(z_0^+ z_0^-)$ and $(z_1^+, z_2^-)$ on each slice. Therefore, its points are parametrised by

$$0 \leqslant r \leqslant \frac{1}{\sqrt{3}}, \quad \alpha = \frac{\theta}{2} \quad \text{and} \quad -\frac{\pi}{6} \leqslant \theta \leqslant \frac{\pi}{6}.$$

(iii) The face $\mathcal{F}_2 = (z_0^+, z_0^-, z_2^-, z_2^+)$ of $\mathbf{F}_8$ is union of geodesics arcs with endpoints in edges $(z_0^+ z_0^-)$ and $(z_2^+, z_2^-)$ on each slice. Therefore, its points are parametrised by

$$0 \leqslant r \leqslant \frac{1}{\sqrt{3}}, \quad \alpha = \frac{\pi}{6} - \frac{\theta}{2} \quad \text{and} \quad -\frac{\pi}{6} \leqslant \theta \leqslant \frac{\pi}{6}.$$

We remark this is the same as the face $(\hat{z}_0^+, \hat{z}_0^-, \hat{z}_2^-, \hat{z}_2^+)$ of the prism in Figure 2, which corresponds to $\arg(z) = \pi/6$.

(iv) The face $\mathcal{F}_3 = (z_0^-, z_1^-, z_2^-)$ of $\mathbf{F}_7$ is its intersection with the bisector $T^{-1}(\mathcal{S}_0)$ given by $\theta = \pi/6$. Therefore, its points are parametrised by $-\pi/4 \leqslant \alpha \leqslant \pi/12$ and

$$0 \leqslant r \leqslant \cos\left(\alpha + \frac{\pi}{12}\right) - \sqrt{\frac{1}{3} - \sin^2\left(\alpha + \frac{\pi}{12}\right)}.$$

(v) The face $\mathcal{F}_4 = (z_0^+, z_1^+, z_2^+)$ of $\mathbf{F}_8$ is its intersection with the bisector $T(\mathcal{S}_0)$ given by $\theta = -\pi/6$. Therefore, its points are parametrised by $-\pi/12 \leqslant \alpha \leqslant \pi/4$ and

$$0 \leqslant r \leqslant \cos\left(\alpha - \frac{\pi}{12}\right) - \sqrt{\frac{1}{3} - \sin^2\left(\alpha - \frac{\pi}{12}\right)}.$$

(vi) The face $\mathcal{F}_5 = (z_1^+, z_1^-, z_2^-)$ of $\mathbf{F}_7$ is its intersection with the bisector $S^{-1}(\mathcal{S}_0)$ given by $-\pi/6 \leqslant \theta \leqslant \pi/6$, $-\theta/2 - \pi/6 \leqslant \alpha \leqslant \theta/2$ and

$$r = \frac{1}{\sqrt{3}}\left[2\cos\left(\frac{\theta}{2} + \frac{\pi}{12}\right)\cos\left(\alpha + \frac{\pi}{12}\right) - \sqrt{1 - 4\cos^2\left(\frac{\theta}{2} + \frac{\pi}{12}\right)\sin^2\left(\alpha + \frac{\pi}{12}\right)}\right].$$

(vii) The face $\mathcal{F}_6 = (z_1^+, z_2^-, z_2^+)$ of $\mathbf{F}_8$ is its intersection with the bisector $S(\mathcal{S}_0)$ given by $-\pi/6 \leqslant \theta \leqslant \pi/6$, $\theta/2 \leqslant \alpha \leqslant \pi/6 - \theta/2$ and

$$r = \frac{1}{\sqrt{3}}\left[2\cos\left(\frac{\theta}{2} - \frac{\pi}{12}\right)\cos\left(\alpha - \frac{\pi}{12}\right) - \sqrt{1 - 4\cos^2\left(\frac{\theta}{2} - \frac{\pi}{12}\right)\sin^2\left(\alpha - \frac{\pi}{12}\right)}\right].$$

We remark that the major modifications for the fundamental domain of $(G_2)_\infty$ occurred in the faces $\mathcal{F}_5$ and $\mathcal{F}_6$, see Figure 5. The reason is that the faces $\mathcal{F}_5$ and $\mathcal{F}_6$ are the intersection of two isometric spheres rather than purely the interseciton of $\mathcal{S}_0$ and the infinite geodesic cone over two faces $(\hat{z}_1^+, \hat{z}_1^-, \hat{z}_2^-)$ and $(\hat{z}_1^+, \hat{z}_2^-, \hat{z}_2^+)$ in Figure 2.

### 4·5. *The basic prism*

We now are ready to construct the prism $\mathbf{P}_0$. In the previous section, we have constructed the fundamental domain for $(G_2)_\infty$. The fundamental domain for $G_2$ is the intersection of the outside of the isometric sphere $\mathcal{S}_0$ of $I_1$ with the fundamental domain of $(G_2)_\infty$ we have already constructed.

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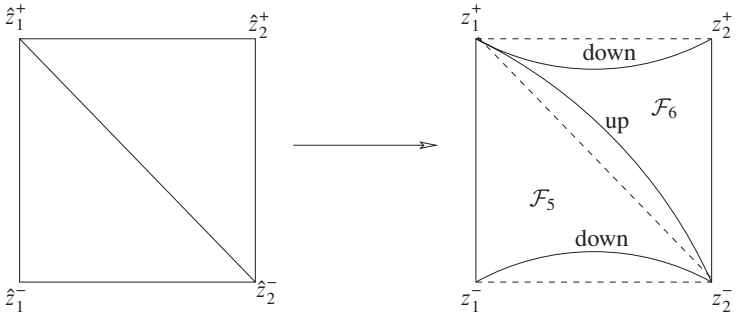



Fig. 5. A schematic top view of the modifications on the edges of $\mathcal{F}_5$ and $\mathcal{F}_6$. The faces $\mathcal{F}_5$ and $\mathcal{F}_6$ are not affine but negative cambers, the words "up" and "down" are denoted by excurvation and incurvation respectively.

In order to make the boundary of our prism lie outside any other isometric sphere, we should make suitable modifications to the fundamental domain of $(G_2)_\infty$. The modifications consist of using the intersection of $\mathcal{S}_0$ and its neighbouring isometric spheres stated in Figure 5. The vertices of the fundamental domain are the same as those for the intersection of $\mathcal{S}_0$ with the prism we have already constructed for $(G_2)_\infty$, that is also the intersection of three neighbouring bisectors with $\mathcal{S}_0$, listed in Section 4·2. There are some geodesic edges, one of whose is contained in the spine of isometric sphere $\mathcal{S}_0$, the others are each contained in one of two slices of bisector $\mathcal{S}_0$. Some generic edges are the intersection of three bisectors constructed in Section 4·3.

The two dimensional faces containing $q_\infty$ are foliated by geodesics starting at the ideal point $q_\infty$ and arriving at the corresponding edges. To determine the remaining compact faces and sides, we observe that all the finite edges are contained in the isometric sphere $\mathcal{S}_0$. Two of the faces are contained in complex lines, two of the faces are foliated by a family of geodesic arcs with endpoints at two edges, and the two remaining faces are defined as intersections of $\mathcal{S}_0$ with appropriate images of $\mathcal{S}_0$ under elements of $(G_2)_\infty$.

The two compact sides are contained in $\mathcal{S}_0$. The other six sides are cones based at the faces of the prism with the cone point the ideal vertex $q_\infty$, all of which are listed in Section 4·4.

By constructions, we give the definition of the basic prism $\mathbf{P}_0$.

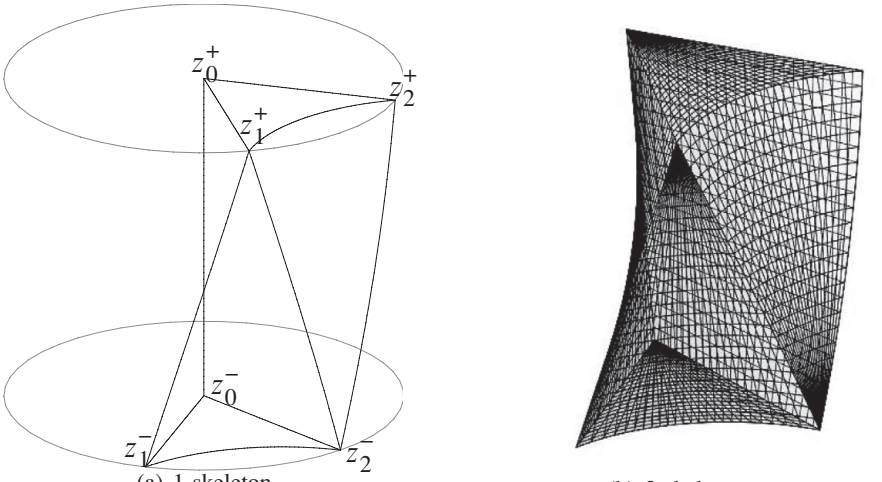
Definition 4·10. In geographical coordinates, the prism $\mathbf{P}_0$ consists of those points of $\mathcal{S}_0$ for which $-\pi/6 \leqslant \theta \leqslant \pi/6$,

$$0 \leqslant r \leqslant \begin{cases} \rho_1(\theta, \alpha) \text{ for } -\theta/2 - \pi/6 \leqslant \alpha \leqslant \theta/2, \\ \rho_2(\theta, \alpha) \text{ for } \theta/2 \leqslant \alpha \leqslant \pi/6 - \theta/2, \end{cases}$$

where $\rho_1(\theta, \alpha)$ and $\rho_2(\theta, \alpha)$ are defined as before.

Following from Definitions 4·9 and 4·10, we can see that the basic prism $\mathbf{P}_0$ is the union of two compact sides $\mathbf{F}_7$ and $\mathbf{F}_8$. (Compare with the standard fundamental domain for $\mathrm{PSL}(2, \mathbb{Z})$, that is an ideal triangle whose boundary consists of two vertical lines and an arc of Euclidean circle with radius 1 centred at the origin. This arc is the union of two compact sides of the fundamental domain of $\mathrm{PSL}(2, \mathbb{Z})$ with the side pairing $z \mapsto -1/z$). The boundary of the prism consists of all the faces of the two compact sides except for the common face $\mathcal{F}'$ that is in the interior of $\mathbf{P}_0$. The schematic view is given in Figure 3 and a realistic view is given in Figure 6.

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(a) 1-skeleton (b) 3-skeleton

Fig. 6. A realistic view of the basic prism $\mathbf{P_0}$ inside the isometric sphere $\mathcal{S}_0$.

## 5. *The side pairing maps*

In the previous section we have constructed two compact sides $\mathbf{F}_7$ and $\mathbf{F}_8$ of the fundamental domain. The other sides are cones from the faces of our prism to the point $q_\infty$ defined in Section 5·2. A side pairing map is an element of PU(2,1) that sends one of these sides to another (possibly the same). In this section, we will describe all the side pairing maps in $G_2$.

### 5·1. *Compact side pairing map*

We consider the map $I_1$

$$\begin{bmatrix} 0 & 0 & i/\sqrt{3} \\ 0 & -\omega & 0 \\ i\sqrt{3} & 0 & 0 \end{bmatrix}$$

acting on $\mathcal{S}_0$ given by

$$I_1 : (r, \theta, \alpha) \longrightarrow (r, -\theta, \alpha + \pi/6).$$

By definition, we know that the compact side $\mathbf{F}_7$ is a domain in $\mathcal{S}_0$ bounded by four faces $\mathcal{F}_1$, $\mathcal{F}'$, $\mathcal{F}_3$, $\mathcal{F}_5$ and the compact side $\mathbf{F}_8$ is a domain in $\mathcal{S}_0$ bounded by four faces $\mathcal{F}'$, $\mathcal{F}_2$, $\mathcal{F}_4$, $\mathcal{F}_6$. Observe that the map

$$I_1 : (r, \theta, \alpha) \longrightarrow (r, -\theta, \alpha + \pi/6), \qquad I_1^2 = R : (r, \theta, \alpha) \longrightarrow (r, \theta, \alpha + \pi/3).$$

We can verify that

$$\mathcal{F}' = I_1(\mathcal{F}_1), \quad \mathcal{F}_4 = I_1(\mathcal{F}_3),$$
$$\mathcal{F}_6 = I_1(\mathcal{F}_5), \quad \mathcal{F}_2 = R(\mathcal{F}_1) = I_1^2(\mathcal{F}_1) = I_1(\mathcal{F}').$$

For example, we take a point $(r, \theta, \alpha)$ of $\mathcal{F}_1$, that is satisfied the set of inequalities

$$0 \leqslant r \leqslant \frac{1}{\sqrt{3}}, \quad \alpha = -\frac{\theta}{2} - \frac{\pi}{6} \quad \text{and} \quad -\frac{\pi}{6} \leqslant \theta \leqslant \frac{\pi}{6}.$$

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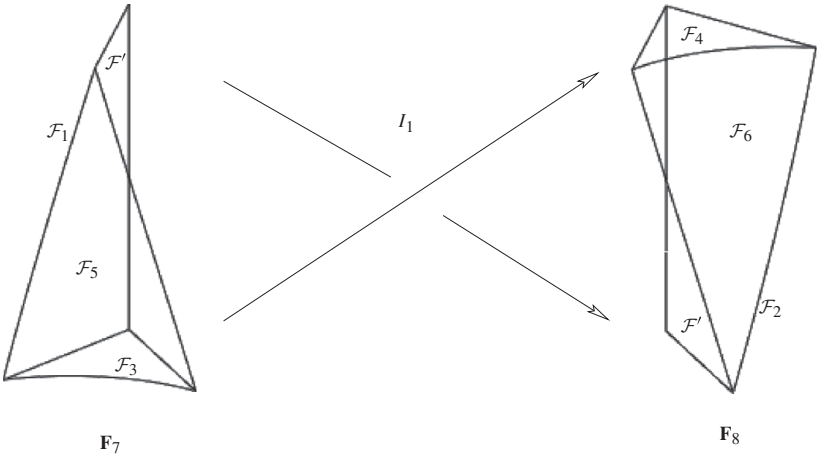



Fig. 7. Two compact sides in coordinates $(r, \theta, \alpha)$ inside the isometric sphere $\mathcal{S}_0$ and its side pairing $I_1$ maps $\mathbf{F}_7$ to $\mathbf{F}_8$ upside down and with a rotation of $\pi/6$.

It is clearly that the point $I_1(r, \theta, \alpha)$ satisfies the equations

$$0 \leqslant r \leqslant \frac{1}{\sqrt{3}}, \quad \alpha = \frac{\theta}{2} \quad \text{and} \quad -\frac{\pi}{6} \leqslant \theta \leqslant \frac{\pi}{6},$$

that is, the point $I_1(r, \theta, \alpha)$ lies on the face $\mathcal{F}'$. Thus we can obtain $\mathbf{F}_8 = I_1(\mathbf{F}_7)$, namely, $I_1$ is the side pairing map from $\mathbf{F}_7$ to $\mathbf{F}_8$. The realistic view of these two compact sides and its side pairing is given by Figure 7.

### 5·2. *Noncompact sides and side pairing maps*

As the action of elements of $(G_2)_\infty$ preserves each horosphere, then we can give a natural extension of faces of the prism $\mathbf{P}_0$. We define two pyramids $\mathbf{F}_1, \mathbf{F}_2$ and four tetrahedra $\mathbf{F}_3, \mathbf{F}_4, \mathbf{F}_5, \mathbf{F}_6$. Each of these is the geodesic cone from $q_\infty$ over the union of faces $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5$ and $\mathcal{F}_6$ of $\mathbf{P}_0$. To be precise, for $j = 1, \cdots, 6$ the side $\mathbf{F}_j$ is defined to be the union of geodesic arcs based at the face $\mathcal{F}_j$ with cone point $q_\infty$. By construction, the intersection of $\mathbf{P}_0$ with each of the pyramids $\mathbf{F}_1, \mathbf{F}_2$ and tetrahedra $\mathbf{F}_3, \mathbf{F}_4, \mathbf{F}_5, \mathbf{F}_6$ is nothing other than the corresponding face of $\mathbf{P}_0$.

We define the 4-dimensional polyhedron $\mathbf{D}$ to be the geodesic cone from $q_\infty$ over the prism $\mathbf{P}_0$. Moreover, $\mathbf{D}$ has eight three-dimensional sides, namely $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_4$, $\mathbf{F}_5$, $\mathbf{F}_6$, $\mathbf{F}_7$, $\mathbf{F}_8$. Later we will show $\mathbf{D}$ is a fundamental domain for $G_2$.

This enables to give the following proposition.

PROPOSITION 5·1. *$R^{-1}(\boldsymbol{D}) \bigcap \boldsymbol{D} = \mathbf{F}_1$, and $R$ maps $\mathbf{F}_1$ to $\mathbf{F}_2$; $T^{-1}(\boldsymbol{D}) \bigcap \mathbf{D} = \mathbf{F}_3$, and $T$ maps $\mathbf{F}_3$ to $\mathbf{F}_4$, likewise, $S^{-1}(\mathbf{D}) \bigcap \mathbf{D} = \mathbf{F}_5$, $S$ maps $\mathbf{F}_5$ to $\mathbf{F}_6$.*

*Proof.* By constructions these sides are corresponding to associated faces of $\mathbf{P}_0$, and so this directly follows from the matrices $R$, $S$ and $T$ in Proposition 3·2.

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We now have already described all the side pairing maps. In what follows, we summarise the side pairing maps in terms of their action on the vertices.

$$\begin{aligned}
R &: (\infty, z_0^+, z_0^-, z_1^-, z_1^+) \longmapsto (\infty, z_0^+, z_0^-, z_2^-, z_2^+),\\
S &: (\infty, z_1^+, z_1^-, z_2^-) \longmapsto (\infty, z_2^+, z_2^-, z_1^+),\\
T &: (\infty, z_0^-, z_1^-, z_2^-) \longmapsto (\infty, z_0^+, z_1^+, z_2^+),\\
I_1 &: (z_0^+, z_0^-, z_1^+, z_1^-, z_2^-) \longmapsto (z_0^-, z_0^+, z_2^-, z_1^+, z_2^+).
\end{aligned}$$

5·3. *The face cycles*

In this section we describe all the 2-dimensional faces and find the associated face cycles and cycle transformations, which give rise to a presentation by Poincaré's polyhedron theorem.

5·3·1. *Faces in complex lines*

Observe that the faces $\mathcal{F}_3$ and $\mathcal{F}_4$ are each contained in a complex line, that is, a slice of $\mathcal{S}_0$. These two complex lines are paired by $T$ and each is preserved by $R$. Moreover, one face is the image of the other under $T$ or $T^{-1}$. In order to obtain cycle transformation we should find other side pairings for these two faces. We know these two faces are each contained in one of compact sides $\mathbf{F}_7$ and $\mathbf{F}_8$ and they are the associated to side pairing map $I_1$. Therefore, this cycle can be described as the following table, which each column has the vertices of each face and the generator in the fist row indicates that this column is the image of the previous one under this map.

$$\begin{array}{ccccc}
 & I_1 & & T^{-1} & \\
z_0^- & \longmapsto & z_0^+ & \longmapsto & z_0^- \\
z_1^- & \longmapsto & z_1^+ & \longmapsto & z_1^- \\
z_2^- & \longmapsto & z_2^+ & \longmapsto & z_2^-.
\end{array}$$

The face $\mathcal{F}_3$ is fixed by $T^{-1}I_1$, which is a complex reflection of order 3. Therefore the cycle relation is $(T^{-1}I_1)^3 = I$.

5·3·2. *Faces in Giraud discs*

The faces $\mathcal{F}_5, \mathcal{F}_6$ with vertices the ordered triple $(z_1^+, z_1^-, z_2^-)$, $(z_2^+, z_2^-, z_1^+)$ respectively are neither contained in complex line nor in Lagrangian planes, which are contained in a *Giraud disc* (such surfaces were described in [**6**]). The map $S^{-1}I_1$ maps $\mathcal{F}_5$ to itself but acts on this face as a rotation of order 3, which is a regular elliptic element. They form a face cycle as described in the following table.

$$\begin{array}{ccccccccccc}
 & I_1 & & S^{-1} & & I_1 & & S^{-1} & & I_1 & & S^{-1} & \\
z_1^+ & \longmapsto & z_2^- & \longmapsto & z_1^- & \longmapsto & z_1^+ & \longmapsto & z_2^- & \longmapsto & z_2^+ & \longmapsto & z_1^+ \\
z_1^- & \longmapsto & z_1^+ & \longmapsto & z_2^- & \longmapsto & z_2^+ & \longmapsto & z_1^+ & \longmapsto & z_2^- & \longmapsto & z_1^- \\
z_2^- & \longmapsto & z_2^+ & \longmapsto & z_1^+ & \longmapsto & z_2^- & \longmapsto & z_1^- & \longmapsto & z_1^+ & \longmapsto & z_2^-
\end{array}$$

The associated face cycle transformation is $(S^{-1}I_1)^3$.

5·3·3. *The generic quadrilateral faces*

The faces $\mathcal{F}_1, \mathcal{F}', \mathcal{F}_2$ with vertices the ordered quadruples respectively $(z_0^+, z_1^+, z_1^-, z_0^-)$, $(z_0^-, z_2^-, z_1^+, z_0^+)$, $(z_0^+, z_2^+, z_2^-, z_0^-)$ are foliated by a family of geodesic arcs that form a single

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face cycle. This is described in the following table.

$$\begin{array}{cccc} & I_1 & I_1 & R^{-1} \\ z_0^+ & \longmapsto z_0^- & \longmapsto z_0^+ & \longmapsto z_0^+ \\ z_1^+ & \longmapsto z_2^- & \longmapsto z_2^+ & \longmapsto z_1^+ \\ z_1^- & \longmapsto z_1^+ & \longmapsto z_2^- & \longmapsto z_1^- \\ z_0^- & \longmapsto z_0^+ & \longmapsto z_0^- & \longmapsto z_0^- \end{array}$$

The associated cycle transformation is $R^{-1}I_1^2$.

5·3·4. *The infinite faces*

We have already seen the infinite faces and the associated face cycles, which correspond to edges and edge cycles we considered when we analysed $(G_2)_\infty$. We now list them again.

There are three face cycles associated to geodesic faces. More precisely, for $j = 0, 1, 2$ the points $q_\infty$, $z_j^+$ and $z_j^-$ define a face contained in a complex line and the points $q_\infty$, $z_0^\pm$ and $z_j^\pm$ for $j = 1, 2$ define a face contained in a Lagrangian plane.

$$\begin{array}{ccc} & R & \\ q_\infty & \longmapsto q_\infty & \\ z_0^+ & \longmapsto z_0^+ & \\ z_0^- & \longmapsto z_0^- & \end{array} \qquad \begin{array}{cccc} & S & R^{-1} & \\ q_\infty & \longmapsto q_\infty & \longmapsto q_\infty & \\ z_1^+ & \longmapsto z_2^+ & \longmapsto z_1^+ & \\ z_1^- & \longmapsto z_2^- & \longmapsto z_1^- & \end{array}$$

$$\begin{array}{ccccc} & R & T & R^{-1} & T^{-1} \\ q_\infty & \longmapsto q_\infty & \longmapsto q_\infty & \longmapsto q_\infty & \longmapsto q_\infty \\ z_0^- & \longmapsto z_0^- & \longmapsto z_0^+ & \longmapsto z_0^+ & \longmapsto z_0^- \\ z_1^- & \longmapsto z_2^- & \longmapsto z_2^+ & \longmapsto z_1^+ & \longmapsto z_1^- \end{array}$$

The face cycles transformations are $R$, $R^{-1}S$ and $T^{-1}R^{-1}TR$ respectively, which give rise to $R^6 = I$ and $(R^{-1}S)^3 = I$ and the last may be rewritten as $[T, R] = I$.

The remaining face cycle is associated to generic faces. Indeed, the face $(q_\infty, z_1^\pm, z_2^\pm)$ or $(q_\infty, z_1^+, z_2^-)$ with the ordered triple vertices is really a generic face.

$$\begin{array}{cccc} & S & S & T^{-1} \\ q_\infty & \longmapsto q_\infty & \longmapsto q_\infty & \longmapsto q_\infty \\ z_1^- & \longmapsto z_2^- & \longmapsto z_1^+ & \longmapsto z_1^- \\ z_2^- & \longmapsto z_1^+ & \longmapsto z_2^+ & \longmapsto z_2^- \end{array}$$

The associated cycle transformation is $T^{-1}S^2$, which may be rewritten as $T = S^2$.

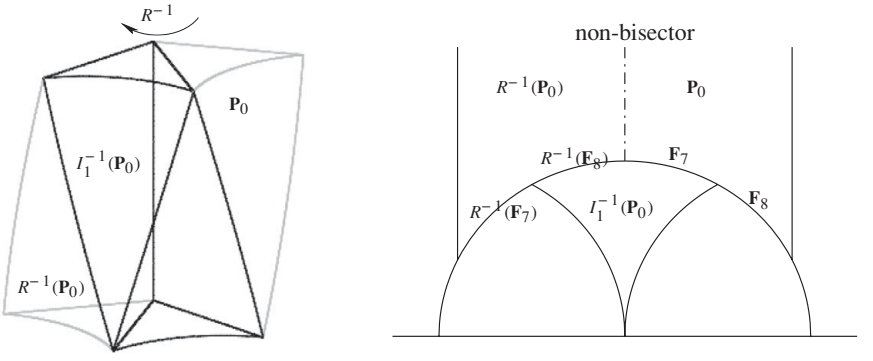
## 6. *The main theorems*

In this section we prove that the 4-dimensional polyhedron **D** constructed above is indeed a fundamental domain for $G_2$ and find a presentation for the group $G_2$.

### 6·1. ***D*** *is a fundamental domain*

In this subsection, we use Poincaré polyhedron theorem to conclude **D** is a fundamental domain for $G_2$. The techniques are the same as [**2**] and [**3**]. We just apply these arguments on this polyhedron **D**.

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(a) Three-dimensional view

(b) Two-dimensional view

Fig. 8. The images of $\mathbf{D}$ cover a neighbourhood of the face $\mathcal{F}_1$.

THEOREM 6·1. *The polyhedron $\boldsymbol{D}$ is a fundamental domain for $G_2$.*

*Proof.* We have given $\mathbf{D}$ the structure of a prism with side pairing maps. For each two-dimensional face of $\mathbf{D}$, we have found all the face cycles given by the side-pairing maps. In what follows, we want to verify that the images of $\mathbf{D}$ cover a neighbourhood of the interior of each two-dimensional face.

The faces containing $q_\infty$ are cones over its edges of $\mathbf{P}_0$. As the infinite faces are sent to other infinite faces by maps in $(G_2)_\infty$, then the face cycles from faces containing $q_\infty$ are corresponding to edge cycles from $\mathbf{P}_0$ that are listed in Section 5·3·4. Therefore, the images of $\mathbf{D}$ under $(G_2)_\infty$ cover any horoball not intersecting $\mathcal{S}_0$ from the construction of fundamental domain for $(G_2)_\infty$.

For all the compact faces, we need only take one face in each face cycle in order to verify the tessellation around the associated face. Now consider the face $\mathcal{F}_1$ with vertices the ordered quadruple $(z_0^+, z_1^+, z_1^-, z_0^-)$. The face cycle is

$$(z_0^+, z_1^+, z_1^-, z_0^-) \xrightarrow{I_1} (z_0^-, z_2^-, z_1^+, z_0^+) \xrightarrow{I_1} (z_0^+, z_2^+, z_2^-, z_0^-) \xrightarrow{R^{-1}} (z_0^+, z_1^+, z_1^-, z_0^-).$$

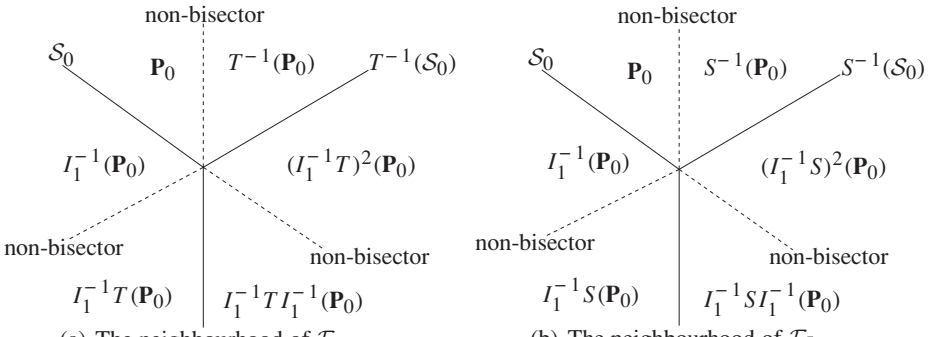
Therefore, $R^{-1}I_1^2$ is the identity on $\mathcal{F}_1$, that is also the identity in $G_2$. We want to show that $\mathbf{D}$, $I_1^{-1}(\mathbf{D})$, $I_1^{-2}(\mathbf{D}) = R^{-1}(\mathbf{D})$ cover a neighbourhood of $\mathcal{F}_1$. In fact, the map $R$ is a rotation of $\mathcal{S}_0$ about its spine and preserves $\mathcal{S}_0$. Therefore, $R^{-1}(\mathbf{P}_0)$ is also contained in $\mathcal{S}_0$. The image of $\mathbf{D}$ under $R^{-1}$ is the geodesic cone of $R^{-1}(\mathbf{P}_0)$. Hence $\mathbf{D} \cup R^{-1}(\mathbf{D})$ covers that part of a neighbourhood of $\mathcal{F}_1$ exterior to $\mathcal{S}_0$. Observe that the face $\mathcal{F}_1$ is contained in $I_1^{-1}(\mathbf{F}_8) = \mathbf{F}_7$ and $I_1^{-1}(\mathbf{F}_7) = I_1^{-2}(\mathbf{F}_8) = R^{-1}(\mathbf{F}_8)$, in other words, $I_1^{-1}(\mathbf{P}_0)$ has a common side with each of $\mathbf{P}_0$ and $R^{-1}(\mathbf{P}_0)$, namely, $\mathbf{P}_0 \cap I_1^{-1}(\mathbf{P}_0) = \mathbf{F}_7$ and $R^{-1}(\mathbf{P}_0) \cap I_1^{-1}(\mathbf{P}_0) = R^{-1}(\mathbf{F}_8)$. Moreover the map $I_1$ swaps the exterior and the interior of $\mathcal{S}_0$. Thus we conclude that $\mathbf{D} \cup I_1^{-1}(\mathbf{D}) \cup R^{-1}(\mathbf{D})$ covers a neighbourhood of $\mathcal{F}_1$, see Figure 8 for a schematic view.

Next we consider the face $\mathcal{F}_3$ with vertices the ordered triples $(z_0^-, z_1^-, z_2^-)$. The face cycles are

$$(z_0^-, z_1^-, z_2^-) \xrightarrow{I_1} (z_0^+, z_1^+, z_2^+) \xrightarrow{T^{-1}} (z_0^-, z_1^-, z_2^-).$$

Therefore $T^{-1}I_1$ is the identity on $\mathcal{F}_3$ and a rotation about complex line containing $(z_0^-, z_1^-, z_2^-)$. In fact, we can easily verify $(T^{-1}I_1)^3$ is identity in $G_2$. We will show that a

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(a) The neighbourhood of $\mathcal{F}_3$ (b) The neighbourhood of $\mathcal{F}_5$

Fig. 9. The images of $\mathbf{D}$ cover a neighbourhood of the faces $\mathcal{F}_3$ and $\mathcal{F}_5$.

neighbourhood of the face $\mathcal{F}_3$ is tessellated by some images of $\mathbf{D}$. To see this, first observe that $T^{-1}$ is a vertical translation in the Heiseinberg group $\mathfrak{R}$, therefore, $\mathbf{D} \cup T^{-1}(\mathbf{D})$ covers a neighbourhood of $\mathcal{F}_3$ exterior to two isometric spheres $\mathcal{S}_0$ and $T^{-1}(\mathcal{S}_0)$. The result will be completed by a similar argument ([**3**, section 7·8]). This argument may be applied to the faces that are in the intersection of two distinct isometric spheres of the Ford domain. This also shows a neighbourhood of $\mathcal{F}_5$ is covered by some images of $\mathbf{D}$ since the face $\mathcal{F}_5$ is in the intersection of two isometric spheres of $I_1$ and $S^{-1}I_1S$.

From the geometrical point of view, we state that the face $\mathcal{F}_3$ is covered by $\mathbf{D}$, $I_1^{-1}(\mathbf{D})$, $I_1^{-1}T(\mathbf{D})$, $I_1^{-1}TI_1^{-1}(\mathbf{D})$, $(I_1^{-1}T)^2(\mathbf{D})$ and $(I_1^{-1}T)^2I_1^{-1}(\mathbf{D}) = T^{-1}(\mathbf{D})$ and the face $\mathcal{F}_5$ is tessellated by $\mathbf{D}$, $I_1^{-1}(\mathbf{D})$, $I_1^{-1}S(\mathbf{D})$, $I_1^{-1}SI_1^{-1}(\mathbf{D})$, $(I_1^{-1}S)^2(\mathbf{D})$ and $(I_1^{-1}S)^2I_1^{-1}(\mathbf{D}) = S^{-1}(\mathbf{D})$ by the same argument as above. We draw two-dimensional pictures to show more clearly, see Figure 9.

Notice that there is a difference between the two cases, that is not apparent from the two dimensional picture. Namely, the map $T^{-1}I_1$ is a complex reflection of order 3 and fixes each point of the face $\mathcal{F}_1$, whereas $S^{-1}I_1$ is a regular elliptic of order 3 and acts on the face $\mathcal{F}_5$ as a rotation.

By Poincaré's polyhedron theorem, we conclude that $\mathbf{D}$ is a fundamental domain for the group generated by the side pairing maps and the presentation is given by the cycle relations. This completes the proof by the Proposition 3·5.

### 6·2. *A presentation for* $G_2$

In this subsection, we also use Poincaré's polyhedron theorem on $\mathbf{D}$ to derive a presentation for $G_2$. We know the generators of $G_2$ are side-pairing maps for $\mathbf{D}$ given in Section 5·2.

For each two-dimensional face of $\mathbf{D}$, we have already found all the cycles given by the side-pairing maps listed in Section 5·3. Thus the stabiliser of infinity $(G_2)_\infty$ has the following presentation

$$(G_2)_\infty = \langle R, S, T | R^6 = (R^{-1}S)^3 = [R, T] = S^2T^{-1} = identity \rangle.$$

For the compact faces, we have given the cycles in Sections 5·3·1, 5·3·2, 5·3·3. Therefore, we have obtained the additional relations

$$(T^{-1}I_1)^3 = (S^{-1}I_1)^3 = R^{-1}I_1^2 = identity.$$

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Thus a presentation for $G_2$ is given by

$$\langle R, S, T, I_1 | R^6 = (R^{-1}S)^3 = [R, T] = S^2T^{-1} = (T^{-1}I_1)^3 = (S^{-1}I_1)^3 = R^{-1}I_1^2 = id\rangle.$$

We want to give a briefer presentation by eliminating several generators. Suppose that $A_2 = T^{-1}I_1$ and $A_3 = S^{-1}I_1$, then we can get the relations $S = A_3A_2^{-1}$, $R = I_1^2 = (A_3A_2^{-1}A_3)^2$ which can be used to eliminate the generators $R$, $S$, $T$ and $I_1$. From these relations we obtain the following theorem:

THEOREM 6·2. *The maps $A_2 = T^{-1}I_1$ and $A_3 = S^{-1}I_1$ generate $G_2$. Moreover, a presentation on these generators is*

$$\langle A_2, A_3 | A_2^3 = A_3^3 = (A_3A_2^{-1}A_3)^{12} = [(A_3A_2^{-1}A_3)^2, (A_3A_2^{-1})^2] = identity\rangle.$$

*Proof.* We begin by showing that the relations involving $A_2$ and $A_3$ follow from the relations involving $R$, $S$, $T$ and $I_1$. First, the relations $A_2^3 = A_3^3 = 1$ follow directly from the definition of $A_2$ and $A_3$. Secondly, observe that the relations

$$A_3A_2^{-1}A_3 = S^{-1}I_1I_1^{-1}TS^{-1}I_1 = I_1$$

and

$$A_3A_2^{-1} = S^{-1}I_1I_1^{-1}T = S, \quad T = S^2 = (A_3A_2^{-1})^2.$$

Thus $R = I_1^2$, $R^6 = 1$ and $[R, T]$=1 imply that

$$(A_3A_2^{-1}A_3)^{12} = [(A_3A_2^{-1}A_3)^2, (A_3A_2^{-1})^2] = 1.$$

Using $R = (A_3A_2^{-1}A_3)^2$, $S = A_3A_2^{-1}$, we obtain

$$I_1 = SA_3 = A_3A_2^{-1}A_3$$

and

$$T = I_1A_2^{-1} = A_3A_2^{-1}A_3A_2^{-1} = (A_3A_2^{-1})^2.$$

Hence, $\langle R, S, T, I_1\rangle = \langle A_2, A_3\rangle$.

Finally, we show that the relations involving $R$, $S$, $T$ and $I_1$ are a consequence of those involving $A_2$, $A_3$. First, it is obvious that

$$(T^{-1}I_1)^3 = A_2^3 = 1, \quad (S^{-1}I_1)^3 = A_3^3 = 1, \quad R^6 = (A_3A_2^{-1}A_3)^{12} = 1$$

and

$$R = (A_3A_2^{-1}A_3)^2 = I_1^2, \quad T = (A_3A_2^{-1})^2 = S^2, \quad [R, T] = [(A_3A_2^{-1}A_3)^2, (A_3A_2^{-1})^2] = 1.$$

Finally,

$$\begin{aligned}(R^{-1}S)^3 &= [(A_3A_2^{-1}A_3)^{-2}A_3A_2^{-1}]^3\\ &= (A_3^{-1}A_2A_3^{-1}A_2^{-1}A_2A_3^{-1}A_2A_2^{-1})^3\\ &= (A_3^{-1}A_2A_3)^3\\ &= A_3^{-1}A_2^3A_3\\ &= 1.\end{aligned}$$

This completes the proof.

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6·3. *The volume of the orbifold* $\boldsymbol{H}^2_{\mathbb{C}}/G_2$

We calculate the volume of $\mathbf{H}^2_{\mathbb{C}}/G_2$ using the complex hyperbolic Gauss–Bonnet formula. The similar case for Gauss–Picard group has been done, see [**3**]. We state the Gauss–Bonnet theorem for our convenience.

$$Volume(M) = \frac{8\pi^2}{3}\chi(M), \qquad \chi(M) = e_0 - e_1 + e_2 - e_3 + e_4,$$

where $e_i$ is the numbers of i-cells weighted by the order of the stabilisers.

The volume of complex hyperbolic 2-orbifold $\mathbf{H}^2_{\mathbb{C}}/G_2$ was calculated firstly by John Parker [**10**] as $\pi/27$. Note that Stover [**16**] states that there are exactly two noncompact arithmetic complex hyperbolic 2-orbifold of minimal volume $\pi/27$, whose fundamental groups are PU(2, 1; $\mathbb{Z}[\omega]$) and its sister $G_2$. Observe, however, that [**10**] also uses the Gauss–Bonnet theorem to calculate the volume. From [**8**, table 1, pages 228 and 229], Holzapfel shows that $\chi(\mathbf{H}^2_{\mathbb{C}}/\mathrm{PSU}(2, 1; \mathbb{Z}[\omega])) = 1/24$, which implies that $\chi(\mathbf{H}^2_{\mathbb{C}}/\mathrm{PU}(2, 1; \mathbb{Z}[\omega])) = 1/72$. This is also shows the Euler characteristic of $\mathbf{H}^2_{\mathbb{C}}/G_2$ is $1/72$ by the fact that the groups PU(2, 1; $\mathbb{Z}[\omega]$)) and $G_2$ have a common subgroup of index 4.

By the combinatorics of **D**, we also show the volume of the orbifold $\mathbf{H}^2_{\mathbb{C}}/G_2$.

PROPOSITION 6·3. *The volume of the orbifold* $\boldsymbol{H}^2_{\mathbb{C}}/G_2$ *is* $\pi/27$.

*Proof.* According to the Gauss–Bonnet theorem, we need only show the Euler characteristic of $\mathbf{H}^2_{\mathbb{C}}/G_2$ is $1/72$. It suffices to find the stabiliser of $i$-dimensional faces of **D** and calculate their order.

(1) Vertices:

| *cycle* | *stabiliser* | *order* |
|---|---|---|
| $z_0^+, z_0^-$ | $\langle T^{-1}I_1, R\rangle$ | 18 |
| $z_1^+, z_1^-, z_2^+, z_2^-$ | $\langle I_1T^{-1}, S^{-1}R\rangle$ | 24 |

The orders of the stabilisers are found as follows. Since $R = I_1^2$ and $[R, T] = I$, $T^{-1}I_1$ and $R$ commute. Therefore, the group $\langle T^{-1}I_1, R\rangle$ is the product of two cyclic groups $\langle T^{-1}I_1\rangle$ and $\langle R\rangle$, and then it has order 18 since $(T^{-1}I_1)^3 = I$ and $R$ has order 6. Next, it is easy to check that $(I_1T^{-1})(S^{-1}R)^{-1}$ has order 4, and that its square is in the centre of $\langle I_1T^{-1}, S^{-1}R\rangle$. Therefore, $\langle I_1T^{-1}, S^{-1}R\rangle$ is a degree 2 extension of an (orientation preserving) (2, 3, 3) triangle group (that is a tetrahedral group) and so it has order $2 \times 12 = 24$ (compare with [**11**, proposition 5·1]).

We conclude that

$$e_0 = \frac{1}{18} + \frac{1}{24} = \frac{7}{72}.$$

(2) Edges:

| *cycle* | *stabiliser* | *order* |
|---|---|---|
| $(z_0^+, z_0^-)$ | $\langle I_1\rangle$ | 12 |
| $(z_0^+, z_1^+), (z_0^+, z_2^+), (z_0^-, z_1^-), (z_0^-, z_2^-)$ | $\langle T^{-1}I_1\rangle$ | 3 |
| $(z_1^+, z_1^-), (z_2^+, z_2^-), (z_1^+, z_2^+), (z_1^+, z_2^-), (z_1^-, z_2^-)$ | $\langle R^{-1}S\rangle$ | 3 |
| $(z_0^+, \infty), (z_0^-, \infty)$ | $\langle R\rangle$ | 6 |
| $(z_1^+, \infty), (z_1^-, \infty), (z_2^+, \infty), (z_2^-, \infty)$ | $\langle R^{-1}S\rangle$ | 3 |

These stabilisers are all cyclic groups and so their orders are the same as those generators, seen in the previous section.

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We conclude that

$$e_1 = \frac{1}{12} + \frac{1}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{3} = \frac{5}{4}.$$

(3) 2-faces:

| *cycle* | *stabiliser* | *order* |
|---|---|---|
| $(z_0^+, z_1^+, z_2^+), (z_0^-, z_1^-, z_2^-)$ | $\langle T^{-1}I_1\rangle$ | 3 |
| $(z_1^+, z_1^-, z_2^-), (z_2^+, z_2^-, z_1^+)$ | $\langle S^{-1}I_1\rangle$ | 3 |
| $(z_0^+, z_0^-, z_1^-, z_1^+), (z_0^+, z_1^+, z_2^-, z_0^-), (z_0^+, z_0^-, z_2^-, z_2^+)$ | $\langle Id\rangle$ | 1 |
| $(z_0^+, z_0^-, \infty)$ | $\langle R\rangle$ | 6 |
| $(z_1^+, z_1^-, \infty), (z_2^+, z_2^-, \infty)$ | $\langle R^{-1}S\rangle$ | 3 |
| $(z_1^+, z_2^+, \infty), (z_1^+, z_2^-, \infty), (z_1^-, z_2^-, \infty)$ | $\langle Id\rangle$ | 1 |
| $(z_0^+, z_1^+, \infty), (z_0^+, z_2^+, \infty), (z_0^-, z_1^-, \infty), (z_0^-, z_2^-, \infty)$ | $\langle Id\rangle$ | 1 |

These stabilisers are also cyclic and so their orders are obvious. We conclude that

$$e_2 = \frac{1}{3} + \frac{1}{3} + 1 + \frac{1}{6} + \frac{1}{3} + 1 + 1 = \frac{25}{6}.$$

(4) 3-faces:

| *cycle* | *stabiliser* | *order* |
|---|---|---|
| $(\infty, z_0^+, z_1^+, z_2^+), (\infty, z_0^-, z_1^-, z_2^-)$ | $\langle Id\rangle$ | 1 |
| $(\infty, z_1^+, z_1^-, z_2^-), (\infty, z_2^+, z_2^-, z_1^+)$ | $\langle Id\rangle$ | 1 |
| $(\infty, z_0^+, z_0^-, z_1^-, z_1^+), (\infty, z_0^+, z_0^-, z_2^-, z_2^+)$ | $\langle Id\rangle$ | 1 |
| $(z_0^+, z_1^+, z_0^-, z_1^-, z_2^-), (z_0^-, z_2^-, z_0^+, z_1^+, z_2^+)$ | $\langle Id\rangle$ | 1 |

We conclude that

$$e_3 = 4.$$

Finally $e_4 = 1$.

We compute now the orbifold Euler number to be

$$\chi(M) = e_0 - e_1 + e_2 - e_3 + e_4 = \frac{7}{72} - \frac{5}{4} + \frac{25}{6} - 4 + 1 = \frac{1}{72}.$$

6·4. *Relation with Mostow's group*

Mostow [**9**] considered a family of complex reflection groups with the angle $2\pi/p$ where $p$ is one of 3, 4, 5. He described these groups by a Coxeter diagram and a phase shift $\phi = \exp(\pi it/3)$ where $t = 1/p + 2/k - 1/2$. He then showed the existence of non-arithmetic lattices in PU(2, 1) by the construction of an explicit fundamental domain for groups defined by certain values of $k$ and gave a presentation for each of these groups. Mostow's groups can be divided into two types (see [**11**, pages 28,35]). The first type consists of complex reflection groups where $1/p+1/k \geqslant 1/2$, $p \leqslant 6$ and those of the second type are the groups where $1/p + 1/k < 1/2$, $p \leqslant 6$. The Eisenstein–Picard modular group is a Mostow's group of the first type with $p = 6$ and $k = 2$; (see [**11**] and [**2**, corollary 5·13]). We show now that the sister of Eisenstein–Picard modular group is a Mostow's group of the first type with $p = 3$ and $k = 6$.

As in Theorem 6·2, if preserving the generator $R$, we may rewrite this presentation as

$$G_2 = \langle A_2, A_3, R | A_2^3 = A_3^3 = R^6 = [R, (A_3A_2^{-1})^2] = 1, R = (A_3A_2^{-1}A_3)^2\rangle.$$

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Using the notation of [**11**], we state a Mostow's group of the first type with $p = 3$ and $k = 6$ as follows

$$\Gamma = \langle J, R_1, A_1 | J^3 = R_1^3 = A_1^6 = 1, A_1 = (JR_1^{-1}J)^2, A_1R_1 = R_1A_1 \rangle.$$

We make a explicit connection between the group $G_2$ and $\Gamma$ in the following proposition.

PROPOSITION 6·4. *There is an isomorphism $\varphi$ from $G_2$ to $\Gamma$ given by $\varphi(A_3) = J$, $\varphi(A_2) = R_1$ and $\varphi(R) = A_1$.*

*Proof.* First, observe that the orders of generators of $G_2$ and $\Gamma$ are the same if giving a homomorphism satisfies with $\varphi(A_3) = J$, $\varphi(A_2) = R_1$ and $\varphi(R) = A_1$. So it need only to show other relations of $G_2$ and $\Gamma$ are equivalent.

In fact,

$$A_1 = \varphi(R) = \varphi((A_3A_2^{-1}A_3)^2) = (\varphi(A_3)\varphi(A_2)^{-1}\varphi(A_3))^2 = (JR_1^{-1}J)^2$$

and $[R, (A_3A_2^{-1})^2] = 1$ yields $A_1(JR_1^{-1})^2A_1^{-1}(JR_1^{-1})^{-2} = 1$. Moreover,

$$\begin{aligned} 1 &= (JR_1^{-1})^{-2}A_1(JR_1^{-1})^2A_1^{-1} \\ &= R_1(J^{-1}R_1J^{-1})A_1(JR_1^{-1}J)R_1^{-1}A_1^{-1} \\ &= R_1A_1R_1^{-1}A_1^{-1} \end{aligned}$$

following from $A_1 = (JR_1^{-1}J)^2$, that is, $A_1R_1 = R_1A_1$. Obviously, the above steps are reversible, this completes the proof.

*Remark* 6·5. For the general case of Mostow's groups of the first type with the values of $p, k$, the orbifold Euler characteristic has been calculated in [**11**], that is,

$$\chi(\mathbf{H}^2_{\mathbb{C}}/\Gamma) = \frac{1}{2}\left[2\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{k}\right]^2.$$

Therefore, we can show again the Euler characteristic

$$\chi(M) = \chi(\mathbf{H}^2_{\mathbb{C}}/G_2) = \chi(\mathbf{H}^2_{\mathbb{C}}/\Gamma) = \frac{1}{72}$$

for $p = 3$ and $k = 6$ following from Proposition 6·4.

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