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A 2(3-RRPS) parallel manipulator inspired by Gough–Stewart platform

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SUMMARY
This study addresses the kinematics of a six-degrees-of-freedom parallel manipulator whose moving platform is a regular triangular prism. The moving and fixed platforms are connected to each other by means of two identical parallel manipulators. Simple forward kinematics and reduced singular regions are the main benefits offered by the proposed parallel manipulator. The Input–Output equations of velocity and acceleration are systematically obtained by resorting to reciprocal-screw theory. A case study, which is verified with the aid of commercially available software, is included with the purpose to exemplify the application of the method of kinematic analysis.

KEYWORDS: Novel applications of robotics; Parallel manipulators; Gough/Stewart platform; Semiclosed form solution; Screw theory; Space kinematics.

1. Introduction

Almost six decades ago Gough, an automotive engineer at Dunlop Rubber Co., developed a variable-length-strut octahedral hexapod in order to respond to problems of aero-landing combined loads.[1,2] Two decades later Stewart, working for the Space and Weapons Research Establishment on aviation simulation at the Elliott Automation Co., proposed a flight simulator based on an eight-bar spatial linkage.[3] Nowadays, although the tire-testing machine built by Gough and the flight simulator proposed by Stewart are mechanically different, six-legged parallel manipulators are often referred as Gough–Stewart platforms. With reference to Fig. 1, a typical Gough–Stewart platform, or hexapod for short, consists of a moving platform connected via six adjustable-length limbs to the fixed platform through, respectively, spherical and universal joints. Due to its multiloop architecture, the forward displacement analysis (FDA) of the general Gough–Stewart platform, hereafter FDAGS, does not allow for closed-form solutions. In that way, the FDAGS was considered by Freudenstein as the "Everest of modern kinematics."

Raghavan[4] proved that the FDAGS yields 40 complex solutions which were then computed by using polynomial continuation methods. The implementation of polynomial equation systems was the mathematical tool employed by Innocenti[5] in order to approach the FDAGS. Such strategy was certified later by Rolland.[6] In that concern, it is noticeable that the formidable work done to solve the FDAGS, free of spurious solutions, culminated with the formulation of a 40th order univariate polynomial equation. On the other hand, owing to the notorious complexity to find closed-form solutions, the topology of the Gough–Stewart platform has been recursively modified yielding mechanisms with nearly closed-form solutions, e.g., the introduction of mechanisms with compound joints.[1–12] However, problems associated with the mechanical assembly such as physical interferences between kinematic pairs have been an obstacle to perform practical applications with these well-intentioned proposals. Another interesting strategy to avoid the complex FDAGS has been the introduction of hexapods whose moving platforms are regular polyhedrons.[13–16]

In this work, a 2(3-RRPS) parallel manipulator inspired by the general Gough–Stewart platform, called PrismCQ, is introduced. The rest of the contribution is organized as follows. In Section 2, the proposed parallel manipulator is outlined. The kinematic analyses of the parallel manipulator are carried out in Section 3. With this aim, an asymmetrical scheme of actuation is chosen. With regard to the finite kinematics, semiclosed form solutions are easily derived by resorting to simple closure equations established in accordance with the coordinates of three points embedded in the moving platform. On the other hand, the Input–Output (I/O) equations of velocity and acceleration are systematically obtained using reciprocal-screw theory. Furthermore, the singular configurations of the parallel manipulator are determined based on the I/O equation of velocity. Finally, in order to exemplify the application of the method of kinematic analysis, a case study, which is verified using commercially available software, is included at the end of Section 3.

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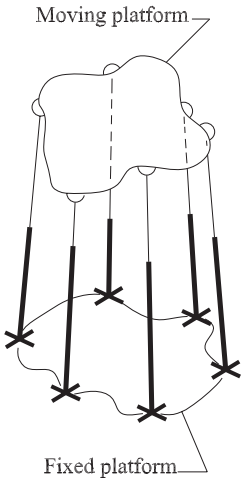


Fig. 1. The Gough–Stewart platform.

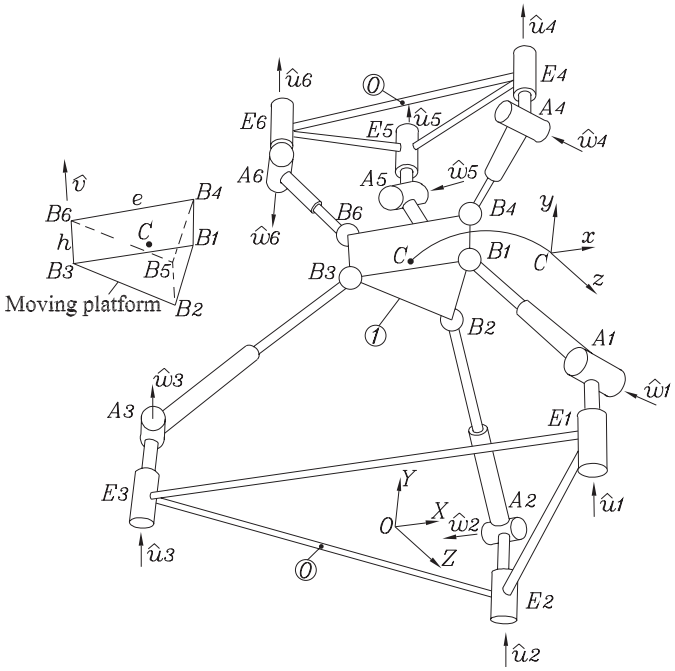


Fig. 2. The PrismCQ robot.

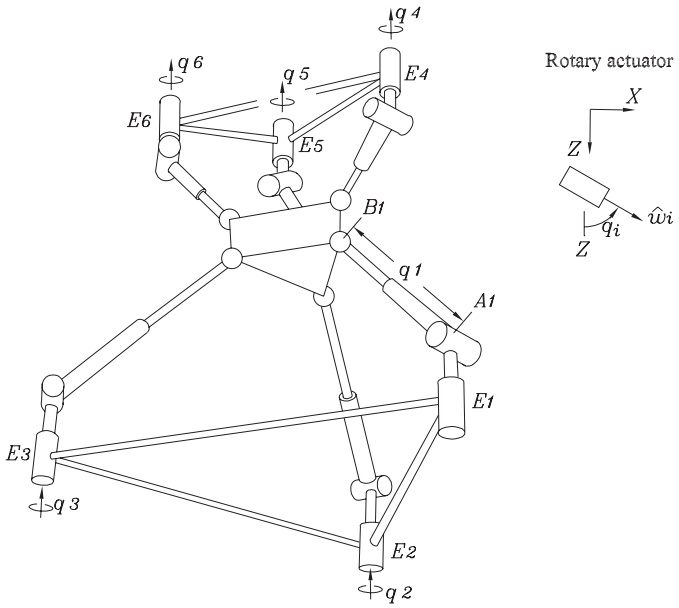


Fig. 3. Generalized coordinates of the parallel manipulator.

2. Description of the Proposed Parallel Manipulator

The proposed parallel manipulator, as well as most of the notation used throughout the contribution, is depicted in Fig. 2.

As shown in Fig. 2, the robot under study consists of two 3-RRPS parallel manipulators sharing a common moving platform, where R, P, and S stand for Revolute, Prismatic, and Spherical joints, respectively. The moving platform (1) is a triangular prism of side e and height h whose vertices B_i are located by vectors $\boldsymbol{B}_i$, unless otherwise specified, in the remainder of the contribution $i = 1, 2, \ldots, 6$. Whereas two parallel equilateral triangles, $\Delta E_1E_2E_3$ and $\Delta E_4E_5E_6$, play the role of the fixed platform (0). In order to describe the geometry of the parallel manipulator, the reference coordinate system XYZ is chosen in such a way that the origin is placed at the center O of the triangle $\Delta E_1E_2E_3$ with the Y-axis normal to the plane $E_1E_2E_3$. Likewise, a reference frame xyz is attached to the moving platform with the origin located at the center C of the triangular prism 1. The limbs are connected to the fixed platform through revolute joints whose axes are oriented in accordance with unit vectors $\hat{\boldsymbol{u}}_i$ along the Y-axis. These kinematic pairs have nominal positions E_i located by vectors $\boldsymbol{E}_i$. The remaining revolute joints are characterized by points A_i located by vectors $\boldsymbol{A}_i$ where the corresponding revolute axes are oriented in agreement with unit vectors $\hat{\boldsymbol{w}}_i$ lying on the XZ-plane, i.e., $\hat{\boldsymbol{u}}_i \bullet \hat{\boldsymbol{w}}_i = 0$ where the bullet $\bullet$ denotes the inner product of the three-dimensional vectorial algebra.

Overall, the proposed parallel manipulator exhibits the following features: (i) unlike usual serially connected parallel manipulators,[17,18] all the limbs are linked to both the moving and fixed platforms, (ii) errors are noncumulative over the moving platform, and (iii) the motors can be installed near to the fixed platform. Hence, depending on the selected scheme of actuation, applications offered for most parallel manipulators are available for the robot under study, e.g., if the actuators are the six prismatic joints then typical applications like multiaxis machine tools, flight simulators, medical devices, and so forth can be considered. Furthermore, by combining linear and rotary actuators, it is possible to improve the performance of the parallel manipulator due to the fact that singular configurations are avoided. This possibility is developed in Section 3.

3. Case Study

In this section, the kinematics of a parallel manipulator equipped with an asymmetrical scheme of actuation is investigated, in other words the set of generalized coordinates for the robot is a combination of linear and rotary independent motions. With reference to Fig. 3, the linear generalized coordinate q_1 comprises the distance between points A_1 and B_1 whereas, dealing with the remaining limbs, actuators for the revolute joints mounted on the fixed platform complete

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the set of six generalized coordinates. The rotary generalized coordinate $q_i (i = 2, 3, \ldots, 6)$ denotes the orientation of the unit vector $\hat{\boldsymbol{w}}_i$ measured CCW from the Z-axis.

3.1. Finite kinematics

This subsection approaches the displacement analyses of the parallel manipulator. The FDA consists of finding the pose, position and orientation, of the moving platform with respect to the fixed platform for a given set of generalized coordinates q_i. Due to the complexity of this issue, it is convenient to consider that the pose of a rigid body can be specified given the coordinates of three of its points. Hence, the coordinates of $B_1 = (X_1, Y_1, Z_1)$, $B_2 = (X_2, Y_2, Z_2)$, and $B_3 = (X_3, Y_3, Z_3)$ may be used to formulate the FDA of the parallel manipulator. To this end, the vectors $\boldsymbol{B}_i (i = 4, 5, 6)$ are expressed as

$$\boldsymbol{B}_{i+3} = \boldsymbol{B}_i + h\hat{\boldsymbol{v}} \quad i = 1, 2, 3, \tag{1}$$

where $\hat{\boldsymbol{v}}$ is a unit vector normal to the plane $B_1B_2B_3$ which is computed in accordance with the usual cross-product as

$$\hat{\boldsymbol{v}} = 2(\boldsymbol{B}_1 - \boldsymbol{B}_3) \times (\boldsymbol{B}_2 - \boldsymbol{B}_3)/\sqrt{3}e^2. \tag{2}$$

In what follows closure equations concerned with the FDA are easily established. A closure equation for the generalized coordinate q_1 is given by

$$(\boldsymbol{B}_1 - \boldsymbol{A}_1) \bullet (\boldsymbol{B}_1 - \boldsymbol{A}_1) = q_1^2. \tag{3}$$

Furthermore, consider that the relative vector between points A_i and B_i is constrained to

$$(\boldsymbol{A}_i - \boldsymbol{B}_i) \bullet \hat{\boldsymbol{w}}_i = 0 \quad i = 2, 3, \ldots, 6, \tag{4}$$

whereas three compatibility closure equations for the distance e may be expressed as

$$(\boldsymbol{B}_i - \boldsymbol{B}_j) \bullet (\boldsymbol{B}_i - \boldsymbol{B}_j) = e^2 \quad i, j = 1, 2, 3 \quad \text{mod}(3). \tag{5}$$

Expressions (3)–(5) form a system of five linear and four quadratic equations, called the *characteristic equations* of the manipulator, in the nine unknowns $X_i, Y_i, Z_i (i = 1, 2, 3)$. This system is reduced into a 8th order univariate polynomial equation by applying the Sylvester's dialytic method of elimination.[19–22] Hence, the moving platform can reach, excluding reflected solutions, at most eight poses with respect to the fixed platform. After, the center of the moving triangular prism, the point C located by the vector $\boldsymbol{C}$ is determined as

$$\boldsymbol{C} = \frac{1}{6}\sum_{i=1}^{6} \boldsymbol{B}_i, \tag{6}$$

whereas the rotation matrix $\mathbf{R}$ between the moving and fixed platforms is computed by applying the method introduced by Gallardo-Alvarado *et al.*[23] Finally, the pose of the moving platform is interpreted as the well-known 4×4 homogeneous transformation matrix $\mathbf{T}$ defined as

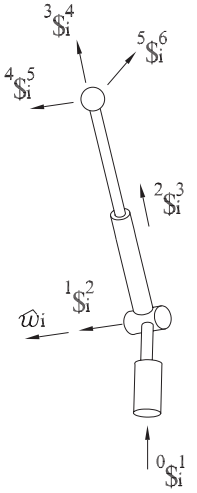


Fig. 4. Infinitesimal screws of a limb.

$$\mathbf{T} \equiv \begin{bmatrix} \mathbf{R} & \boldsymbol{C} \\ \mathbf{0} & 1 \end{bmatrix}. \tag{7}$$

On the other hand, the inverse displacement analysis of the robot consists of finding the generalized coordinates q_i given the pose of the moving platform with respect to the fixed platform, i.e., given matrix $\mathbf{T}$. With this aim, consider that the coordinates of vertex B_i, that are expressed in the reference frame XYZ, can be computed based on the following relationship:

$$\begin{bmatrix} B_i \\ 1 \end{bmatrix} = \mathbf{T} \begin{bmatrix} B_i^* \\ 1 \end{bmatrix}, \tag{8}$$

where B_i^* is the point B_i but expressed in the moving reference frame xyz. After, the generalized coordinate q_1 is computed by means of Eq. (3) while in order to compute the angles $q_i (i = 2, 3, \ldots, 6)$ from Eq. (4) it follows that

$$q_i = -\arctan\left[\frac{(\boldsymbol{A}_i - \boldsymbol{B}_i) \bullet \hat{\boldsymbol{k}}}{(\boldsymbol{A}_i - \boldsymbol{B}_i) \bullet \hat{\boldsymbol{i}}}\right] \quad i = 2, 3, \ldots, 6. \tag{9}$$

3.2. Infinitesimal kinematics

In this subsection, the I/O equations of velocity and acceleration as well as the singularity analysis of the parallel manipulator are approached by means of the theory of screws. With this aim, the models of the screws representing the kinematic pairs of a limb are depicted in Fig. 4.

Referring to Fig. 4, the velocity state of the moving platform with respect to the fixed platform, the six-dimensional vector $\boldsymbol{V}_O = \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{v}_O \end{bmatrix}$ may be expressed in a screw form through any of the limbs as

$${}_0\omega_1^i \; {}^0\$_i^1 + {}_1\omega_2^i \; {}^1\$_i^2 + \cdots + {}_5\omega_6^i \; {}^5\$_i^6 = \boldsymbol{V}_O, \tag{10}$$

where $\boldsymbol{\omega}$ is the angular velocity vector of the moving platform with respect to the fixed platform, while $\boldsymbol{v}_O$ is the linear

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(translational) velocity vector of a point embedded in the moving platform which is instantaneously coincident with the origin O of the reference frame. Point O is named the reference pole. Moreover, in accordance with the actuators of the parallel manipulator, note that ${}_2\omega_3^1 = \dot{q}_1$ while ${}_0\omega_1^i = \dot{q}_i (i = 2, 3, \ldots, 6)$. Furthermore, taking into account that the screw ${}^3\$_1^4$ is reciprocal to all the screws, excluding the screw representing the prismatic joint of the first limb, then the application of the Klein form $\{*;*\}$, which is a bilinear symmetric form of the Lie algebra $se(3)$ of the Euclidean group $SE(3)$, of the screw ${}^3\$_1^4$ to both sides of the corresponding expression (10), with the reduction of terms we have

$$\left\{{}^3\$_1^4; \boldsymbol{V}_O\right\} = \dot{q}_1. \tag{11}$$

Similarly, owing to the fact that the screw ${}^4\$_i^5 (i = 2, 3, \ldots, 6)$ is reciprocal to all the screws in the same limb, excepting the screw ${}^0\$_i^1$, the application of the Klein form of ${}^4\$_i^5$ to both sides of the corresponding expression (10), with the cancelation of terms, one obtains

$$\left\{{}^4\$_i^5; \boldsymbol{V}_O\right\} = \dot{q}_i \left\{{}^4\$_i^5; {}^0\$_i^1\right\} \quad i = 2, 3, \ldots, 6. \tag{12}$$

Casting into a matrix-vector form Eqs. (11) and (12) the I/O equation of velocity of the parallel manipulator results in

$$\mathbf{A}\ \boldsymbol{V}_O = \mathbf{B}\dot{\mathbf{q}}, \tag{13}$$

therein $\mathbf{A} = \mathbf{J}^T\ \Delta$ where $\mathbf{J} = [{}^3\$_1^4\ {}^4\$_2^5\ {}^4\$_3^5\ {}^4\$_4^5\ {}^4\$_5^5\ {}^4\$_6^5]$ is the active screw-coordinate Jacobian matrix of the manipulator while $\Delta = \left[\begin{smallmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{I} & \mathbf{0}\end{smallmatrix}\right]$ is an operator of polarity given in accordance with the 3×3 identity matrix $\mathbf{I}$ and the 3×3 zero matrix $\mathbf{0}$. Furthermore, $\mathbf{B}$ is a diagonal matrix given by

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \left\{{}^0\$_2^1; {}^4\$_2^5\right\} & 0 & 0 & 0 & 0 \\ 0 & 0 & \left\{{}^0\$_3^1; {}^4\$_3^5\right\} & 0 & 0 & 0 \\ 0 & 0 & 0 & \left\{{}^0\$_4^1; {}^4\$_4^5\right\} & 0 & 0 \\ 0 & 0 & 0 & 0 & \left\{{}^0\$_5^1; {}^4\$_5^5\right\} & 0 \\ 0 & 0 & 0 & 0 & 0 & \left\{{}^0\$_6^1; {}^4\$_6^5\right\} \end{bmatrix}, \tag{14}$$

whereas $\dot{\mathbf{q}} = [\dot{q}_1\ \dot{q}_2\ \dot{q}_3\ \dot{q}_4\ \dot{q}_5\ \dot{q}_6]^T$ is the *first order driver matrix* of the parallel manipulator.

Following that trend, in order to find the I/O equation of acceleration of the parallel manipulator, consider that the reduced acceleration state, or accelerator for short, of the moving platform with respect to the fixed platform, the six-dimensional vector $\boldsymbol{A}_O = \left[\begin{smallmatrix}\dot{\boldsymbol{\omega}}\\ \boldsymbol{a}_O - \boldsymbol{\omega} \times \mathbf{v}_O\end{smallmatrix}\right]$ may be expressed in a screw form through any of the connector chains as

$${}_0\dot{\omega}_1^i\ {}^0\$_i^1 + {}_1\dot{\omega}_2^i\ {}^1\$_i^2 + \cdots + {}_5\dot{\omega}_6^i\ {}^5\$_i^6 + \boldsymbol{\mathcal{L}}_i = \boldsymbol{A}_O, \tag{15}$$

where $\dot{\boldsymbol{\omega}}$ is the angular acceleration vector of the moving platform with respect to the fixed platform while $\boldsymbol{a}_O$ is the linear (translational) acceleration vector of the point O. Furthermore, $\boldsymbol{\mathcal{L}}_i$ is the i-th Lie screw of acceleration which is computed as

$$\begin{aligned} \boldsymbol{\mathcal{L}}_i = &\left[{}_0\omega_1^i\ {}^0\$_i^1 \quad {}_1\omega_2^i\ {}^1\$_i^2 + \cdots + {}_5\omega_6^i\ {}^5\$_i^6\right] \\ &+ \left[{}_1\omega_2^i\ {}^1\$_i^2 \quad {}_2\omega_3^i\ {}^2\$_i^3 + \cdots + {}_5\omega_6^i\ {}^5\$_i^6\right] \\ &+ \left[{}_2\omega_3^i\ {}^2\$_i^3 \quad {}_3\omega_4^i\ {}^3\$_i^4 + {}_4\omega_5^i\ {}^4\$_i^5 + {}_5\omega_6^i\ {}^5\$_i^6\right] \\ &+ \left[{}_3\omega_4^i\ {}^3\$_i^4 \quad {}_4\omega_5^i\ {}^4\$_i^5 + {}_5\omega_6^i\ {}^5\$_i^6\right] \\ &+ \left[{}_4\omega_5^i\ {}^4\$_i^5 \quad {}_5\omega_6^i\ {}^5\$_i^6\right], \end{aligned} \tag{16}$$

where the brackets $[*\quad *]$ denote the Lie product of the Lie algebra $se(3)$ of the Euclidean group $SE(3)$. Naturally, the computation of the Lie screw of acceleration requires the solution of the velocity analysis of the robot. To this end, the joint-velocity rates of the parallel manipulator can be computed, see Eq. (10), by means of the following expression:

$$\Omega_i = \mathbf{J}_i^{-1}\ \boldsymbol{V}_O, \tag{17}$$

where $\Omega_i = [{}_0\omega_1^i\ {}_1\omega_2^i\ {}_2\omega_3^i\ {}_3\omega_4^i\ {}_4\omega_5^i\ {}_5\omega_6^i]^T$ is a matrix containing the joint-velocity rates of the i-th limb while $\mathbf{J}_i = [{}^0\$_i^1\ {}^1\$_i^2\ {}^2\$_i^3\ {}^3\$_i^4\ {}^4\$_i^5\ {}^5\$_i^6]$ is the screw-coordinate Jacobian matrix of the same limb.

Finally, it is straightforward to demonstrate that, similar to the velocity analysis, by resorting to reciprocal-screw theory the I/O equation of acceleration of the parallel manipulator is found as

$$\mathbf{A}\ \boldsymbol{A}_O = \mathbf{B}\ddot{\mathbf{q}} + \mathbf{C}, \tag{18}$$

where $\ddot{\mathbf{q}} = [\ddot{q}_1\ \ddot{q}_2\ \ddot{q}_3\ \ddot{q}_4\ \ddot{q}_5\ \ddot{q}_6]^T$ is the *second-order driver matrix* of the parallel manipulator and

$$\mathbf{C} = \begin{bmatrix} \left\{\boldsymbol{\mathcal{L}}_1; {}^3\$_1^4\right\} \\ \left\{\boldsymbol{\mathcal{L}}_2; {}^4\$_2^5\right\} \\ \left\{\boldsymbol{\mathcal{L}}_3; {}^4\$_3^5\right\} \\ \left\{\boldsymbol{\mathcal{L}}_4; {}^4\$_4^5\right\} \\ \left\{\boldsymbol{\mathcal{L}}_5; {}^4\$_5^5\right\} \\ \left\{\boldsymbol{\mathcal{L}}_6; {}^4\$_6^5\right\} \end{bmatrix} \tag{19}$$

is the complementary matrix of acceleration.

Finally, in agreement with Eq. (15), the joint-acceleration rates of the parallel manipulator can be computed via the following expression:

$$\dot{\Omega}_i = \mathbf{J}_i^{-1}\ (\boldsymbol{A}_O - \boldsymbol{\mathcal{L}}_i), \tag{20}$$

where $\dot{\Omega}_i = [{}_0\dot{\omega}_1^i\ {}_1\dot{\omega}_2^i\ {}_2\dot{\omega}_3^i\ {}_3\dot{\omega}_4^i\ {}_4\dot{\omega}_5^i\ {}_5\dot{\omega}_6^i]^T$ is a matrix containing the joint-acceleration rates of the i-th limb.

3.3. Singularity analysis

Singularities are configurations where the moving platform gains or loses degrees of freedom. In this subsection, relationship (13) is the fundamental expression to explain the so-called *illness regions* of the robot at hand.

It is interesting to note that since in each limb of the manipulator none of the axes of the two revolute joints passes through the common point of the spherical joint and the direction of the prismatic pair does not coincide

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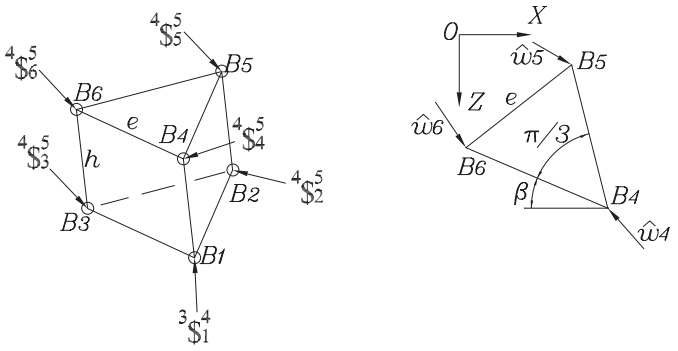


Fig. 5. The reciprocal screws of the parallel manipulator.

with the axes of the revolute joints, excluding singular configurations, then the serial connecting chain generates all the possible displacements of the Euclidean group. Thus, if the moving platform was connected to the fixed platform only by the serial chain RRPS, the moving platform could adopt any arbitrary pose, within the workspace of the serial connecting chain, with respect to the fixed platform.

In accordance with Eq. (13) a singularity occurs, see Gosselin and Angeles,[24] when: (i) matrix **A** is singular, namely singularity type 2, (ii) matrix **B** is singular, namely singular type 1, and (iii) both matrices **A** and **B** are singular, namely singular type 3.

Singularity type 1. Matrix **B**, see Eq. (14), is singular when $\det(\mathbf{B}) = 0$ which occurs when any of the elements of the diagonal of **B** vanishes. Clearly, this situation emerges if and only if a limb of the robot equipped with a rotary motor is folded-unfolded. In fact, if the i-th limb is folded-unfolded then the screws $^0\$_i^1$ and $^4\$_i^5$ are reciprocal to each other yielding $\{^0\$_i^1; ^4\$_i^5\} = 0$. This is a real possibility that must be taken into proper account in the design process of the parallel manipulator. It is interesting to note that the Lie product of the screw $^1\$_i^2$ with any of the remaining screws in the same limb does not belong to the set of elements of the Jacobian matrix $\mathbf{J}_i$. Then, actuating screw $^1\$_i^2$ the manipulator can escape from this singular configuration, if any.

Singularity type 2. Taking into account that $\mathbf{A} = \mathbf{J}^T \Delta$ then matrix **A** is singular when the active screw-coordinate Jacobian matrix **J** of the manipulator is singular. For clarity, matrix **J** is rewritten here

$$\mathbf{J} = [^3\$_1^4 \quad ^4\$_2^5 \quad ^4\$_3^5 \quad ^4\$_4^5 \quad ^4\$_5^5 \quad ^4\$_6^5], \tag{21}$$

where the screws are shown in Fig. 5. It is straightforward to demonstrate that some elements of **J** are linearly dependent when the plane $B_4B_5B_6$ is located on the XZ-plane, e.g., a linear combination of the primal parts of the screws $^4\$_4^5$, $^4\$_5^5$, and $^4\$_6^5$ may be expressed as

$$\lambda_4\hat{\boldsymbol{w}}_4 + \lambda_5\hat{\boldsymbol{w}}_5 + \lambda_6\hat{\boldsymbol{w}}_6 = \mathbf{0} \quad \forall\lambda_4, \lambda_5, \lambda_6 \in \Re. \tag{22}$$

However, since the component of each unit vector $\hat{\boldsymbol{w}}_i$ along the Y-axis is null then, excluding the trivial solution, it is evident that expression (22) admits arbitrary values for the scalars λ_4, λ_5, and λ_6 causing the singularity of **J**. Furthermore, in order to find the singularity loci of this situation, it is necessary to obtain a nearly closed-form solution for the direct displacement analysis. To this end, consider $B_4 = (X_4, Z_4)$, $B_5 = (X_5, Z_5)$, and $B_6 = (X_6, Z_6)$. Furthermore, by resorting to basic trigonometry, see Fig. 5, the coordinates of B_5 may be computed as

$$\begin{aligned} X_5 &= (X_4 + X_6 + \sqrt{3}Z_4 - \sqrt{3}Z_6)/2,\ Z_5 \\ &= (Z_4 + Z_6 - \sqrt{3}X_4 + \sqrt{3}X_6)/2. \end{aligned} \tag{23}$$

Hence, in accordance with Eq. (4) one obtains a linear system of three equations in the unknowns X_4, Z_4, X_6, and Z_6. Solving this system it follows that

$$X_4 = D_1 + E_1Z_4,\ X_6 = D_2 + E_2Z_4,\ Z_6 = D_3 + E_3Z_4, \tag{24}$$

where the coefficients are computed in accordance with the parameters and generalized coordinates of the parallel manipulator, e.g., $D_1 = a_{4X} + a_{4Z}\cot(q_4)$ and $E_1 = -\cot(q_4)$, notated $A_4 = (a_{4X}, a_{4Y}, a_{4Z})$. With the purpose to not distract the attention of the reader, in the rest of the contribution, coefficients containing long terms will not be listed.

After, in Plücker coordinates with the reduction of terms the screws $^4\$_4^5$, $^4\$_5^5$, and $^4\$_6^5$ are given by

$$^4\$_4^5 = \begin{bmatrix} \sin(q_4) \\ \cos(q_4) \\ F_1 + G_1Z_4 \end{bmatrix},\ ^4\$_5^5 = \begin{bmatrix} \sin(q_5) \\ \cos(q_5) \\ F_2 + G_2Z_4 \end{bmatrix},$$

$$^4\$_6^5 = \begin{bmatrix} \sin(q_6) \\ \cos(q_6) \\ F_3 + G_3Z_4 \end{bmatrix} \tag{25}$$

where $F_1 = -D_1\cos(q_4)$ and $G_1 = \sin(q_4) - E_1\cos(q_4)$.

Let $\mathbf{S} = [^4\$_4^5\ ^4\$_5^5\ ^4\$_6^5]$ be a subset of **J**. The manipulator is at singular posture type 2 when $\det(\mathbf{S}) = 0$, in other words when the screws of matrix **S** are coplanar. Hence, the singularity loci type 2 is governed by the following expression

$$\begin{aligned} &\big[G_1\sin(q_5 - q_6) + G_2\sin(q_6 - q_4) + G_3\sin(q_4 - q_5)\big]Z_4 \\ &\quad + F_1\sin(q_5 - q_6) + F_2\sin(q6 - q4) \\ &\quad + F_3\sin(q4 - q5) = 0, \end{aligned} \tag{26}$$

where Z_4 is computed, see Eq. (5), via the following quadratic equation

$$\begin{aligned} &(E_1^2 + E_2^2 + E_3^2 - 2E_2E_1 - 2E_3 + 1)Z_4^2 \\ &\quad + 2(-E_2D_1 + D_2E_2 - D_2E_1 \\ &\quad + D_3E_3 + D_1E_1 - D_3)Z_4 \\ &\quad + D_1^2 + D_2^2 + D_3^2 - e^2 - 2D_2D_1 = 0. \end{aligned} \tag{27}$$

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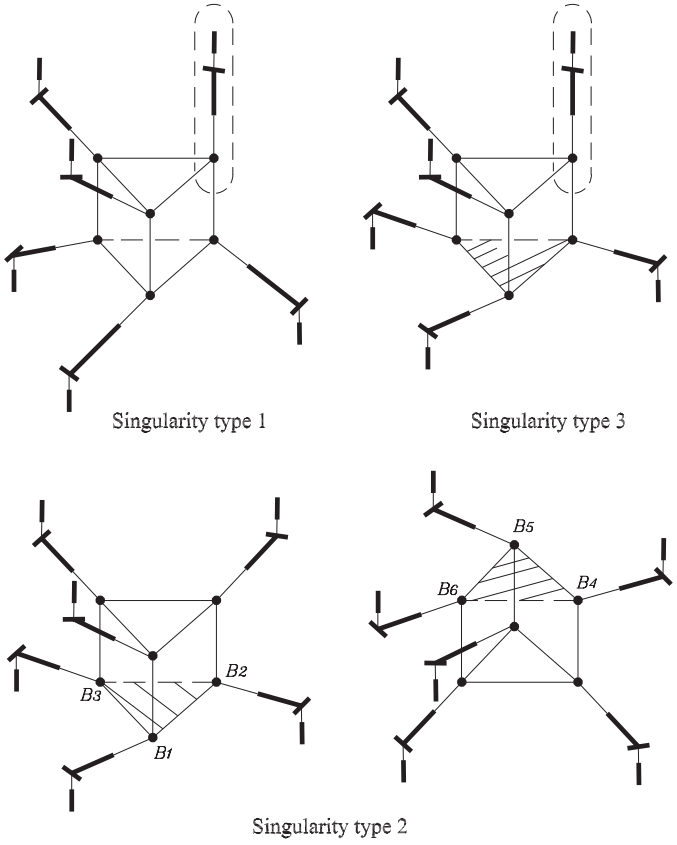


Fig. 6. Some singular postures of the parallel manipulator.

It is straightforward to demonstrate that a similar result is obtained when the primal parts of the screws ${}^3\$_1^4$, ${}^4\$_2^5$, and ${}^4\$_3^5$ are coplanar.

Singularity type 3. As previously noted, if one of the limbs that contain an active revolute joint is vertical then $\det(\mathbf{B}) = 0$. Furthermore, if in addition the primal parts of the screws ${}^4\$_4^5$, ${}^4\$_5^5$, and ${}^4\$_6^5$ are coplanar then $\det(\mathbf{A}) = 0$, which leads to a singular posture type 3.

Finally, some evident singular configurations of the parallel manipulator are depicted in Fig. 6.

3.4. Numerical example

In this subsection, a numerical exercise is provided. To this end, the dimensions, using hereafter SI units, of the moving platform are given by $e = 0.2165$ and $h = 0.08$, whereas the nominal positions of the revolute joints connecting the limbs to the fixed platform are chosen as $E_1 = (0.4, 0, 0)$, $E_2 = (-0.2, 0, -0.3464)$, $E_3 = (-0.2, 0, 0.3464)$, $E_4 = (0.25, 0.68, 0)$, $E_5 = (-0.125, 0.68, -0.2165)$, and $E_6 = (-0.125, 0.68, 0.2165)$. Moreover, the nominal positions of the passive revolute joints are $A_1 = (0.4, 0.11, 0)$, $A_2 = (-0.2, 0.11, -0.3464)$, $A_3 = (-0.2, 0.11, 0.3464)$, $A_4 = (0.25, 0.561, 0)$, $A_5 = (-0.125, 0.561, -0.2165)$, and $A_6 = (-0.125, 0.561, 0.2165)$.

The first part of the exercise consists of determining the singularity loci of the mechanism associated to the generalized coordinates q_4, q_5, and q_6. The resulting singular regions are provided in Fig. 7. It is noticeable that due to the selected scheme of actuation, the parallel manipulator is practically free of this kind of singularity.

On the other hand, consider that the generalized coordinates are governed by periodical functions given by $q_1 = 0.1\sin(t) + 0.4$, $q_2 = 0.15\sin(t) + 2\pi/3$, $q_3 = 0.25\sin(t) + \pi/6$, $q_4 = 0.5\sin(t)$, $q_5 = 0.25\sin(t) + 2\pi/3$, and $q_6 = 0.1\sin(t) + \pi/6$ where the time t is given in the interval $0 \leq t \leq 2\pi$.

At the time $t = 0$ there are eight solutions for the FDA, four real and four imaginary. The real solutions are listed in Table I.

In order to approach the forward infinitesimal kinematics of the numerical example, consider solution 1 of Table I as the reference configuration of the manipulator. After, the resulting temporal behavior of the angular and linear kinematic properties of the center C of the moving platform using the theory of screws is provided in Fig. 8.

Finally, the graphics obtained using the theory of screws are compared with simulations generated with the aid of special software like ADAMS©. These plots are added to Fig. 8. It is noticeable that the numerical results obtained via screw theory are in excellent agreement with those generated with ADAMS©.

4. Conclusions

The FDA of the general Gough–Stewart platform is a complicated open problem that does not allow for closed-form solutions due to its multiloop topology. As pointed out by Husty,[25] the FDAGS is governed by a set of seven quadratic equations which may be solved by applying continuation methods, implementing polynomial equation systems or using Gröbner bases. In reality the FDAGS is a rather complicated task, by any method. On the other hand, singular configurations are a recurrent problem of the Gough–Stewart platform.

With these considerations in mind, in this work a parallel manipulator inspired by the general Gough–Stewart platform is introduced. The proposed robot consists of two identical parallel manipulators with RRPS-type kinematic chains. The common moving platform is a regular triangular prism whose vertices are precisely the centers of the spherical joints. Compared with the Gough–Stewart platform, the

Table I. The four real solutions of the forward displacement analysis.

Sol.	B_1	B_2	B_3
1	(0.125, 0.400, 0)	(−0.062, 0.400, −0.108)	(−0.062, 0.400, 0.108)
2	(0.243, 0.478, 0)	(0.086, 0.478, 0.149)	(0.035, 0.478, −0.061)
3	(0.125, −0.180, 0)	(−0.0625, −0.180, −0.108)	(−0.0625, −0.180, 0.108)
4	(0.243, −0.258, 0)	(0.086, −0.258, 0.149)	(0.035, −0.258, −0.061)

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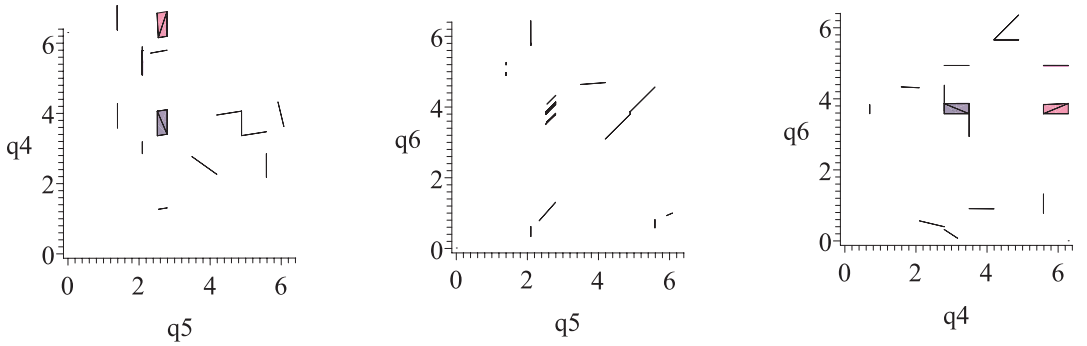


Fig. 7. (Colour online) Singular regions of the numerical example.

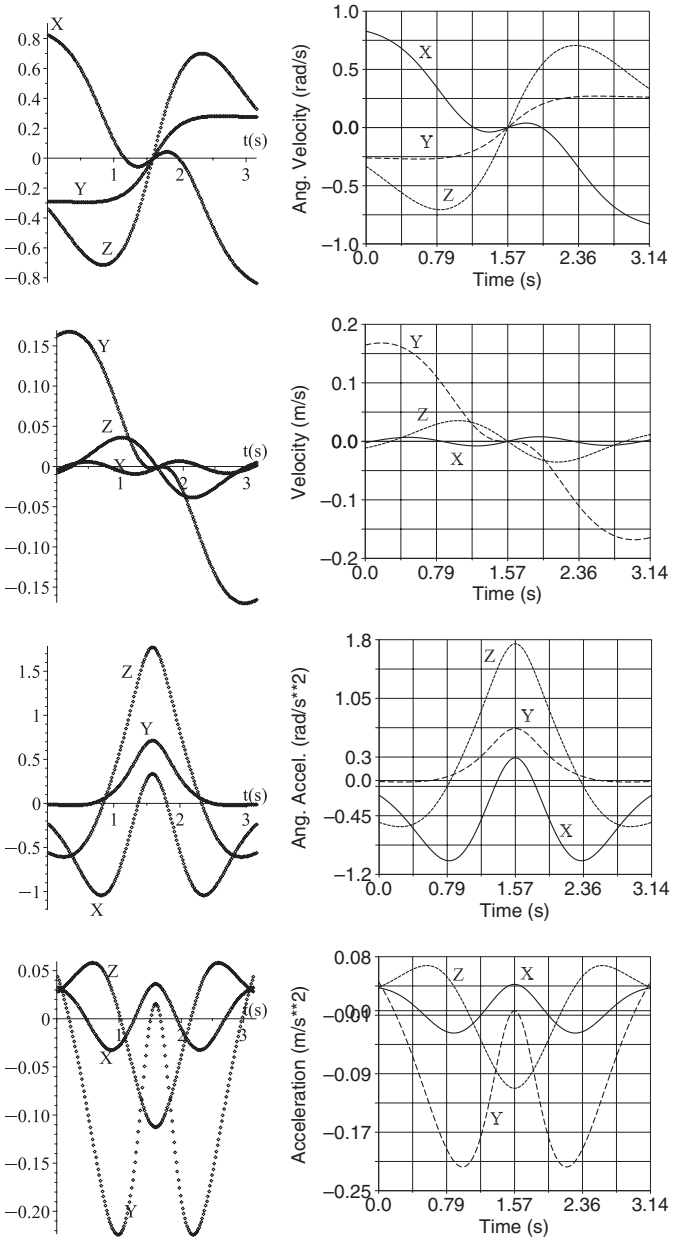


Fig. 8. Temporal behavior of the forward kinematics of the center of the moving platform.

proposed manipulator brings two interesting benefits: (i) a combination of rotary and linear actuators allows for nearly closed-form solutions for the direct position analysis, and (ii) singularities are ameliorated due to the geometry of the moving platform, which is a regular triangular prism. Furthermore, unlike classical serial-parallel manipulators, in other words mechanisms built with two serially connected parallel manipulators, the actuators can be conveniently installed near to the fixed platform. On the other hand, taking into account that all the limbs are connected to the fixed and moving platforms, it is straightforward to demonstrate that the PrismCQ robot is free of passive legs and keep the advantages of the Gough–Stewart platform. Hence, applications considered for the Gough–Stewart platform such as multiaxis machine tools, like-hexapod structures in surgical applications, flight simulators, telescopes, and so forth are also available for the proposed parallel manipulator.

Finally, a case study is provided in order to exemplify the effectiveness of the method of kinematic analysis here employed. Furthermore, the numerical results obtained via the theory of screws concerned with the velocity and acceleration analyses are successfully compared with simulations generated using commercially available software like ADAMS©.

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References

1. V. E. Gough, "Contribution to Discussion to Papers on Research in Automobile Stability and Control and in Type Performance," *Proceedings of the Automation Division Institution of Mechanical Engineers*, UK (1957) pp. 392–395.
2. V. E. Gough and S. G. Whitehall, "Universal Tyre Testing Machine," *Proceedings of the FISITA Ninth International Technical Congress, IMechE 1*, London, UK (1962) pp. 117–137.
3. D. Stewart, "A platform with six degrees of freedom," *Proc. Inst. Mech. Eng.* I **180**(15), 371–386 (1965).
4. M. Raghavan, "The Stewart platform of general geometry has 40 configurations," *ASME J. Mech. Des.* **115**(2), 277–282 (1993).
5. C. Innocenti, "Forward kinematics in polynomial form of the general Stewart platform," *ASME J. Mech. Des.* **123**(2), 254–260 (2001).
6. L. Rolland, "Certified solving of the forward kinematics problem with an exact algebraic method for the general parallel manipulator," *Adv. Robot.* **19**(9), 995–1025 (2005).
7. C. Innocenti and V. Parenti-Castelli, "Analytical form solution of the direct kinematics of a 4-4 fully in-parallel actuated six-degrees-of-freedom mechanism," *Informatica* **17**(1), 13–20 (1993).

https://doi.org/10.1017/S0263574712000392 Published online by Cambridge University Press

8. K. Wohlhart, "Displacement analysis of the general spherical Stewart platform," *Mech. Mach. Theory* **29**(4), 581–589 (1994).
9. C. Innocenti, "Direct kinematics in analytical form of the 6-4 fully-parallel mechanism," *ASME J. Mech. Des.* **117**(1), 89–95 (1995).
10. D.-M. Ku, "Forward kinematic analysis of a 6-3 type Stewart platform mechanism," *Proc. Instn. Mech. Eng. Part K: J. Multibody Dyn.* **214**(K4), 233–241 (2000).
11. R. Di Gregorio, "Analytic formulation of the 6-3 fully-parallel manipulator's singularity determination," *Robotica* **19**(6), 663–667 (2001).
12. J. Gallardo-Alvarado, "Jerk distribution of a 6-3 Gough-Stewart platform," *Proc. Instn. Mech. Eng. Part K: J. Multibody Dyn.* **217**(1), 77–84 (2003).
13. S.-L. Chen and I.-T. You, "Kinematic and singularity analyses of a six-DOF 6-3-3 parallel link machine tool," *Int. J. Adv. Manufact. Tech.* **16**(11), 835–842 (2000).
14. S.-L. Chen and Y.-C. Liu, "Post-processor development for a six degrees-of-freedom parallel-link machine tool," *Int. J. Adv. Manufact. Tech.* **18**(4), 254–265 (2001).
15. M. Zhang and B. Zhuo, "Workspace Analysis and Parameter Optimization of a Six-DOF 6-3-3 Parallel Link Machine Tool," **In**: *Intelligent Robotics and Computation, Lecture Notes in Computer Science* (R. Goebel, J. Siekmann and W. Wahlster, eds.) (Springer, Dec. 16–18, 2009), vol. 5928/2009, pp. 706–712.
16. H. Gao, P. Xiao and H. Zhang, "Configuration Optimization of Physical Prototype for 6-3-3 Parallel Mechanism," *Proceedings of 2011 International Conference on Electrical and Control Engineering (ICECE)* (Yichang, China, Sep. 16–18, 2011) pp. 2937–2939.
17. Y. Lu and B. Hu, "Solving driving forces of 2(3-SPR) serial-parallel manipulator by CAD variation geometry approach," *ASME J. Mech. Des.* **128**(6), 1349–1351 (2006).
18. J. Gallardo-Alvarado, C. R. Aguilar-Nájera, L. Casique-Rosas, J. M. Rico-Martínez and Md. Nazrul Islam, "Kinematics and dynamics of 2(3-RPS) manipulators by means of screw theory and the principle of virtual work," *Mech. Mach. Theory* **43**(10), 1281–1294 (2008).
19. J.-P. Merlet, "Manipulateurs paralleles, 4eme partie: mode d´assemblage et cinematique directe sous forme polynomiale," *INRIA Research Report no. 1135* (1989) Centre de Sophia Antipolis, Valbonne.
20. C. Innocenti and V. Parenti-Castelli, "Direct position analysis of the Stewart platform mechanism," *Mech. Mach. Theory* **25**(6), 611–621 (1990).
21. L.-W. Tsai, *Robot Analysis* (John Wiley & Sons, New York, 1999).
22. J. Gallardo-Alvarado, R. Rodríguez-Castro and Md Nazrul Islam, "Analytical solution of the forward position analysis of parallel manipulators that generate 3-RS structures," *Adv. Robot.* **22**(2–3), 215–234 (2008).
23. J. Gallardo-Alvarado, C. R. Aguilar-Nájera, L. Casique-Rosas, L. Pérez-González and J. M. Rico-Martínez, "Solving the kinematics and dynamics of amodular spatial hyper-redundant manipulator by means of screw theory," *Multibody Syst. Dyn.* **20**(4), 307–325 (2008).
24. C. Gosselin and J. Angeles, "Singularity analysis of closed-loop kinematic chains," *IEEE Trans. Rob. Autom.* **6**(32), 261–290 (1990).
25. M. L. Husty, "An algorithm for solving the direct kinematic of general Stewart-Gough platforms," *Mech. Mach. Theory* **31**(4), 365–380 (1996).

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