# BOUNDARY LAYER GROWTH

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[*Received* 17 February, *read* 26 October 1936]

## I. Introduction

When relative motion of a viscous incompressible fluid of constant density and an immersed solid body is started impulsively from rest, the initial motion of the fluid is irrotational, without circulation. This is shown by observation, and may be seen in many of the published photographs of fluid flow. The theoretical proof is exactly the same as that given, for inviscid fluids*, in treatises on hydrodynamics; for it may be assumed that the viscous stresses remain finite. The fluid in contact with the solid body is, however, at rest relative to the boundary, whilst the adjacent layer of fluid is slipping past the boundary with a velocity determined from the theory of the velocity potential. There is thus initially a surface of slip, or vortex sheet, in the fluid, coincident with the surface of the solid body. In other words, there is a "boundary layer" of zero thickness. The vorticity in the sheet diffuses from the boundary further into the fluid, and is convected by the stream. The boundary layer grows in thickness. (The same results follow from a consideration of the equations for the vorticity components in a viscous incompressible fluid†, or of the equation for the circulation in a circuit moving with the fluid‡.)

In any region along the boundary where the fluid is flowing against a pressure gradient, the forward stream will, after a time, leave the boundary (if the pressure gradient extends far enough), and the boundary layer will be continued, as a vortex-layer, in the main body of the fluid. Downstream of the position of separation, there will be a slow back flow along the boundary.

Up to the time when separation begins, the velocity and pressure outside the boundary layer may be taken to be the same as those at the surface in the irrotational motion without circulation, since this assumption provides a very close approximation to the facts. The pressure may also, as in boundary layer theory generally, be taken as constant throughout any section of the boundary layer. Separation begins when the velocity gradient normal to the boundary vanishes

* See, for example, Lamb, *Hydrodynamics* (1932 edition), p. 11. It is assumed that any extraneous impulsive body forces acting on the fluid are conservative.

† Lamb, *Hydrodynamics* (1932 edition), p. 578.

‡ Jeffreys, *Proc. Camb. Phil. Soc.* 24 (1928), 477–9.

https://doi.org/10.1017/S0305004100019101 Published online by Cambridge University Press

at the boundary. For two-dimensional motion, the time, $T$, that elapses before separation begins, and the distribution of velocity in the boundary layer, may be approximately calculated. The second approximation to the velocity, sufficient to give a first approximation to $T$, were calculated for an impulsive start by Blasius*, who applied his results to the motion of a circular cylinder. He also obtained the third approximation to the velocity distribution, and the second approximation to $T$, for a uniformly accelerated motion. For a circular cylinder, separation begins at the rear stagnation point, and Blasius appears to have thought that this would hold for a symmetrical cylinder of any section. Tollmien† pointed out that this is not the case: for an elliptic cylinder with its major axis across the stream, for example, it is untrue if the ratio of the squares of the axes exceeds 4/3.

In what follows, we have found the third approximation to the velocity distribution, and the second approximation to $T$, for an impulsive start. Since the formulae may be more briefly expressed than in the original, we have first summarized the results of Blasius for the second approximation.

For the first approximation to $T$, separation always begins where the velocity gradient is a maximum, but unless this point coincides with a stagnation point, this is not accurately true for the second approximation, in contradiction to certain published statements. It was partly to obtain some notion of the change in the position of initial separation in passing from the first to the second approximation that this work was undertaken.

Until recently, the formulae derived by Blasius had been applied to symmetrical cylinders only, but Mr Howarth has recently applied the first approximation for an impulsive start to motion past an elliptic cylinder, with axes in the ratio of 1 : 6, and with its major axis at an angle of 7° to the stream‡. The application of the results obtained here to the case of a circular cylinder is noted, and then second approximations, both for an impulsive start and for uniformly accelerated motion, are obtained for the case of the elliptic cylinder considered by Mr Howarth.

After separation has once begun, the position of separation moves upstream. The movement could be followed theoretically on the assumption that the velocity and pressure outside the boundary layer continue to be the same as in the irrotational motion without circulation; but this assumption would no longer be valid, and the results would have at best only a qualitative value—and then only for flow past a symmetrical cylinder, for example, since for an asymmetrical cylinder, the circulation around the cylinder begins to grow as soon as separation starts. Even for a symmetrical cylinder, the thickening of the boundary layer

* *Zeitschr. f. Math. u. Phys.* 56 (1908), 20–37.
† *Handbuch der exper. Phys.* 4, pt. 1 (1931), 272–9.
‡ *Proc. Camb. Phil. Soc.* 31 (1935), 582–4.

https://doi.org/10.1017/S0305004100019101 Published online by Cambridge University Press

beyond the position of separation—or, rather, its projection into the main body of the fluid—and the consequent formation of a wake, deprive results obtained on the above assumption of any quantitative value.

## II. PREVIOUS INVESTIGATIONS

We assume that at time $t=0$ a cylinder starts to move in a straight line with velocity $V$, and that this velocity remains constant thereafter. We take a frame of reference fixed relative to the cylinder. If $x$ is distance along a section of the cylinder from the forward stagnation point, and $y$ distance normal to the surface of the cylinder, the approximate equation of motion in the boundary layer is

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}=U\frac{\partial U}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}, \tag{1}$$

where $(u, v)$ are the velocity components in the direction of $x$ and $y$ increasing, respectively, $p$ is the pressure and $U$ the velocity just outside the boundary, and $\rho$ is the density and $\nu$ the kinematic viscosity of the fluid. The equation of continuity is approximately

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{2}$$

Initially the boundary layer has zero thickness, and at the beginning of the motion the diffusion far outweighs the convection and the influence of the pressure gradient, i.e. the convection terms in the acceleration on the right can be neglected compared with $\partial u/\partial t$, and the term $U\partial U/\partial x$ can be neglected compared with $\nu\partial^2 u/\partial y^2$. The equation for $u_1$, the first approximation to $u$, is

$$\frac{\partial u_1}{\partial t}=\nu\frac{\partial^2 u_1}{\partial y^2}, \tag{3}$$

and the solution required is

$$\left.\begin{aligned} &u_1=Uf_1(\eta), \\ \text{where}\quad &\eta=\frac{y}{2\sqrt{(\nu t)}},\quad f_1(\eta)=\operatorname{erf}\eta=\frac{2}{\sqrt{\pi}}\int_0^\eta e^{-z^2}dz. \end{aligned}\right\} \tag{4}$$

This solution makes $u_1=0$ when $\eta=0$, $u_1$ practically equal to $U$ when $\eta$ is large and theoretically equal to $U$ when $\eta=\infty$, and makes the thickness of the boundary layer zero when $t=0$.

The first approximation to $v$ must satisfy (2) and must vanish when $\eta=0$. It is therefore given by

$$v_1=-2\sqrt{(\nu t)}\,U'[\eta\operatorname{erf}\eta-\pi^{-\frac{1}{2}}(1-e^{-\eta^2})], \tag{5}$$

where the dash denotes differentiation with respect to $x$. When $\eta\to\infty$,

$$v_1\sim-2\sqrt{(\nu t)^{\frac{1}{2}}}\,\eta U'=-yU',$$

and becomes infinite. The solution therefore fails theoretically for infinite values of $\eta$. But for moderate values of $\eta$, at which $u_1$ is practically equal to $U$, $v_1$ is of order $(\nu t)^{\frac{1}{2}}U'$.

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To find a second approximation, put

$$u = u_1 + u_2, \quad v = v_1 + v_2 \tag{6}$$

in (1). $u_2/u_1$ and $v_2/v_1$ will be of order $tU'$, and to find $u_2$ it is sufficient to solve the equation

$$\nu \frac{\partial^2 u_2}{\partial y^2} - \frac{\partial u_2}{\partial t} = -UU' + u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y}$$

$$= UU' [\operatorname{erf}^2 \eta - 2\pi^{-\frac{1}{2}} \eta e^{-\eta^2} \operatorname{erf} \eta - 1 + 2\pi^{-\frac{1}{2}} (e^{-\eta^2} + e^{-2\eta^2})]. \tag{7}$$

Write

$$u_2 = tUU' f_2(\eta). \tag{8}$$

Then

$$f_2'' + 2\eta f_2' - 4 f_2 = 4 [\operatorname{erf}^2 \eta - 2\pi^{-\frac{1}{2}} \eta e^{-\eta^2} \operatorname{erf} \eta - 1 + 2\pi^{-1} (e^{-\eta^2} - e^{-2\eta^2})], \tag{9}$$

where dashes now denote differentiation with respect to $\eta$. The solution of this equation is

$$f_2 = l_2 + \alpha m_2 + \beta n_2, \tag{10}$$

where

$$\left.\begin{aligned} l_2 &= \tfrac{1}{2}(2\eta^2 - 1) \operatorname{erf}^2 \eta + 3\pi^{-\frac{1}{2}} \eta e^{-\eta^2} \operatorname{erf} \eta + 1 \\ &\quad - \tfrac{4}{3}\pi^{-1} e^{-\eta^2} + 2\pi^{-1} e^{-2\eta^2}, \\ m_2 &= 2\eta^2 + 1, \\ n_2 &= \tfrac{1}{2}\pi^{-\frac{1}{2}} (2\eta^2 + 1) \operatorname{erf} \eta + \eta e^{-\eta^2}, \end{aligned}\right\} \tag{11}$$

and $\alpha$ and $\beta$ are constants to be chosen so that $u_1 = 0$ at $\eta = 0$ and at $\eta = \infty$. These conditions require

$$\left.\begin{aligned} \alpha &= -\left(1 + \frac{2}{3\pi}\right) = -1{\cdot}21221, \\ \beta &= \frac{1}{\sqrt{\pi}} \left(1 + \frac{4}{3\pi}\right) = 0{\cdot}80364. \end{aligned}\right\} \tag{12}$$

The position of separation of forward flow from the wall is given by $\partial u/\partial y = 0$, or $\partial u/\partial \eta = 0$, at $\eta = 0$; and the time at which separation occurs at any particular place is therefore given by

$$1 + \left(1 + \frac{4}{3\pi}\right) U' t = 0. \tag{13}$$

Separation will occur where $U'$ is negative, and will occur first where $U'$ has its greatest negative value. The interval to separation is given by

$$T = 0{\cdot}70205/(-U')_{\max.} \tag{14}$$

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## III. THE THIRD APPROXIMATION TO THE VELOCITY DISTRIBUTION

The value of $u_2$ is given by (8), (10), (11) and (12). The value of $v_2$ must satisfy (2) and must vanish at $\eta = 0$; and is given by

$$v_2 = 2(\nu t^3)^{\frac{1}{2}}[UU'' + U'^2][\tfrac{1}{3}\beta - p_2 - \alpha q_2 - \beta r_2], \tag{15}$$

where
$$\left.\begin{aligned} p_2 &= \int l_2 d\eta = \tfrac{1}{6}(2\eta^3 - 3\eta)\operatorname{erf}^2\eta - \tfrac{1}{6}\pi^{-\frac{1}{2}}[4 - (4\eta^2 - 11)e^{-\eta^2}]\operatorname{erf}\eta \\ &\qquad + \eta + \tfrac{1}{3}\pi^{-1}\eta e^{-2\eta^2} + \frac{4\sqrt{2}}{3}\pi^{-\frac{1}{2}}\operatorname{erf}\sqrt{2}\eta, \\ q_2 &= \int m_2 d\eta = \tfrac{2}{3}\eta^3 + \eta, \\ r_2 &= \int n_2 d\eta = \tfrac{1}{6}\pi^{\frac{1}{2}}(2\eta^3 + 3\eta)\operatorname{erf}\eta + \tfrac{1}{3}(\eta^2 + 1)e^{-\eta^2}. \end{aligned}\right\} \tag{16}$$

[When $\eta \to \infty$
$$v_2 \to \frac{2}{3\sqrt{\pi}}\left(3 + \frac{4}{3\pi} - 4\sqrt{2}\right)(\nu t^3)^{\frac{1}{2}}(UU'' + U'^2).]$$

To find a third approximation, write

$$u = u_1 + u_2 + u_3, \quad v = v_1 + v_2 + v_3. \tag{17}$$

Then $u_3/u_2$, $v_3/v_2$ will be of order $tU'$, and to find $u_3$ it is sufficient to solve the equation

$$\begin{aligned} \nu\frac{\partial^2 u_3}{\partial y^2} - \frac{\partial u_3}{\partial t} &= u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_1}{\partial x} + v_1\frac{\partial u_2}{\partial y} + v_2\frac{\partial u_1}{\partial y} \\ &= tU^2U''\Theta(\eta) + tUU'^2\{\Theta(\eta) + \Phi(\eta)\}, \end{aligned} \tag{18}$$

where
$$\begin{aligned} \Theta(\eta) &= (l_2 + \alpha m_2 + \beta n_2)\operatorname{erf}\eta + 2\pi^{-\frac{1}{2}}(\tfrac{1}{3}\beta - p_2 - \alpha q_2 - \beta r_2)e^{-\eta^2} \\ &= \tfrac{1}{2}(2\eta^2 - 1)\operatorname{erf}^3\eta \\ &\quad + \{\tfrac{1}{2}\pi^{\frac{1}{2}}\beta(2\eta^2 + 1) - \tfrac{1}{3}\pi^{-\frac{1}{2}}(2\eta^3 - 12\eta)e^{-\eta^2}\}\operatorname{erf}^2\eta \\ &\quad + \{2\alpha\eta^2 + \alpha + 1 - \tfrac{2}{3}\beta\eta^3 e^{-\eta^2} - \tfrac{1}{3}\pi^{-1}(4\eta^2 - 17)e^{-2\eta^2}\}\operatorname{erf}\eta \\ &\quad - \frac{8\sqrt{2}}{3}\pi^{-1}e^{-\eta^2}\operatorname{erf}\sqrt{2}\eta - \tfrac{2}{3}\pi^{-\frac{1}{2}}(2\alpha\eta^3 - 2\pi^{-1}\eta - \beta)e^{-\eta^2} \\ &\quad - \tfrac{2}{3}\pi^{-\frac{1}{2}}\beta(\eta^2 + 1)e^{-2\eta^2} - \tfrac{2}{3}\pi^{-\frac{3}{2}}\eta e^{-3\eta^2} \end{aligned} \tag{19}$$

and

$$\begin{aligned} \Phi(\eta) &= (l_2 + \alpha m_2 + \beta n_2)\operatorname{erf}\eta - [\eta\operatorname{erf}\eta - \pi^{-\frac{1}{2}}(1 - e^{-\eta^2})]\frac{d}{d\eta}(l_2 + \alpha m_2 + \beta n_2) \\ &= -\tfrac{1}{2}(2\eta^2 + 1)\operatorname{erf}^3\eta - \tfrac{1}{2}\pi^{-\frac{1}{2}}\{2\pi\beta\eta^2 - 4\eta - \pi\beta - 4\eta^3 e^{-\eta^2}\}\operatorname{erf}^2\eta \\ &\quad - \{2\alpha\eta^2 - 2\beta\eta - \alpha - 1 + \tfrac{1}{3}\pi^{-1}(14\eta^2 + 9\pi\beta\eta + 1)e^{-\eta^2} - \pi^{-1}(4\eta^2 + 1)e^{-2\eta^2}\}\operatorname{erf}\eta \\ &\quad + 4\pi^{-\frac{1}{2}}\alpha\eta + 2\beta(2\eta + \pi^{-\frac{1}{2}})e^{-\eta^2} - \tfrac{2}{3}\pi^{-\frac{3}{2}}(7\eta + 3\pi\beta)e^{-2\eta^2} + 2\pi^{-\frac{3}{2}}\eta e^{-3\eta^2}. \end{aligned} \tag{20}$$

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The solution of (18) is $u_3 = t^2[U^2U''f_3(\eta) + UU'^2F_3(\eta)]$, (21)

where $f_3'' + 2\eta f_3' - 8f_3 = 4\Theta(\eta)$, (22)

$$F_3'' + 2\eta F_3' - 8F_3 = 4\Theta(\eta) + 4\Phi(\eta). \tag{23}$$

Complementary functions of either of these equations are given by

$$m_3 = 4\eta^4 + 12\eta^2 + 3, \tag{24}$$

$$n_3 = (4\eta^4 + 12\eta^2 + 3)\operatorname{erf}\eta + \pi^{-\frac{1}{2}}(4\eta^3 - 10\eta)e^{-\eta^2}. \tag{25}$$

From the nature of $\Theta(\eta)$ it appears that there will be a particular integral of (22) of the form

$$f_3 = X\operatorname{erf}^3\eta + Y\operatorname{erf}^2\eta + Z\operatorname{erf}\eta + T, \tag{26}$$

where $X$, $Y$, $Z$, $T$ do not involve $\operatorname{erf}\eta$ explicitly. Write

$$P \equiv \frac{d^2}{d\eta^2} + 2\eta\frac{d}{d\eta} - 8. \tag{27}$$

Then, if $f_3$ is given by (26),

$$\begin{aligned} P(f_3) = {} & \operatorname{erf}^3\eta\, P(X) + \operatorname{erf}^2\eta\,[P(Y) + 12\pi^{-\frac{1}{2}}e^{-\eta^2}X'] \\ & + \operatorname{erf}\eta\,[P(Z) + 8\pi^{-\frac{1}{2}}e^{-\eta^2}Y' + 24\pi^{-1}e^{-2\eta^2}X] \\ & + [P(T) + 4\pi^{-\frac{1}{2}}e^{-\eta^2}Z' + 8\pi^{-1}e^{-2\eta^2}Y]. \end{aligned} \tag{28}$$

If the right-hand side of (28) is equal to $4\Theta(\eta)$, we must have

$$P(X) = 4\eta^2 - 2, \tag{29}$$

$$P(Y) + 12\pi^{-\frac{1}{2}}e^{-\eta^2}X' = 2\pi^{\frac{1}{2}}\beta(2\eta^2 + 1) - \tfrac{4}{3}\pi^{-\frac{1}{2}}(2\eta^3 - 12\eta)e^{-\eta^2}, \tag{30}$$

$$\begin{aligned} P(Z) + 8\pi^{-\frac{1}{2}}e^{-\eta^2}Y' + 24\pi^{-1}e^{-2\eta^2}X \\ = 8\alpha\eta^2 + 4\alpha + 4 - \tfrac{8}{3}\beta\eta^3e^{-\eta^2} - \tfrac{4}{3}\pi^{-1}(4\eta^2 - 17)e^{-2\eta^2}, \end{aligned} \tag{31}$$

$$\begin{aligned} P(T) + 4\pi^{-\frac{1}{2}}e^{-\eta^2}Z' + 8\pi^{-1}e^{-2\eta^2}Y = -\frac{32\sqrt{2}}{3}\pi^{-1}e^{-\eta^2}\operatorname{erf}\sqrt{2}\eta \\ - \tfrac{8}{3}\pi^{-\frac{3}{2}}(2\alpha\pi\eta^3 - 2\eta - \beta\pi)e^{-\eta^2} - \tfrac{8}{3}\pi^{-\frac{1}{2}}\beta(\eta^2 + 1)e^{-2\eta^2} - \tfrac{8}{3}\pi^{-\frac{3}{2}}\eta e^{-3\eta^2}. \end{aligned} \tag{32}$$

A solution of (29), not involving $\operatorname{erf}\eta$ explicitly, is

$$X = -\eta^2 + A(4\eta^4 + 12\eta^2 + 3), \tag{33}$$

where $A$ is an arbitrary constant. With this value of $X$, (30) becomes

$$P(Y) = 2\beta\pi^{\frac{1}{2}}(2\eta^2 + 1) - \tfrac{4}{3}\pi^{-\frac{1}{2}}\{(2 + 144A)\eta^3 - (30 - 216A)\eta\}e^{-\eta^2}. \tag{34}$$

A solution of (34), not involving $\operatorname{erf}\eta$ explicitly, is

$$\begin{aligned} Y = -\tfrac{1}{2}\beta\pi^{\frac{1}{2}}(2\eta^2 + 1) + \tfrac{1}{12}\pi^{-\frac{1}{2}}\{(2 + 144A)\eta^3 - (39 - 360A)\eta\}e^{-\eta^2} \\ + B(4\eta^4 + 12\eta^2 + 3), \end{aligned} \tag{35}$$

where $B$ is also an arbitrary constant. With these values of $X$ and $Y$, (31) becomes

$$\begin{aligned} P(Z) = {} & 8\alpha\eta^2 + 4\alpha + 4 - 4\{(\tfrac{2}{3}\beta + 32B\pi^{-\frac{1}{2}})\eta^3 - (4\beta - 48B\pi^{-\frac{1}{2}})\eta\}e^{-\eta^2} \\ & + \tfrac{2}{3}\pi^{-1}\{(73 - 468A) - (56 + 144A)\eta^2 + (4 + 144A)\eta^4\}e^{-2\eta^2}. \end{aligned} \tag{36}$$

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A solution of (36) which is a sum of polynomials and products of polynomials and negative exponentials is possible only if $A=\frac{1}{12}$. With this value of $A$, a solution of (36) not involving $\operatorname{erf}\eta$ explicitly is

$$\begin{aligned} Z = &-\tfrac{1}{2}(4\alpha\eta^2+2\alpha+1)+C(4\eta^4+12\eta^2+3)\\ &+\tfrac{1}{12}\{(2\beta+96B\pi^{-\frac{1}{2}})\eta^3-(15\beta-240B\pi^{-\frac{1}{2}})\eta\}e^{-\eta^2}\\ &+\tfrac{1}{3}\pi^{-1}(4\eta^2-5)e^{-2\eta^2}. \end{aligned} \tag{37}$$

When the values of $X$, $Y$ and $Z$ are substituted into (32), we find that the resulting equation for $T$ has a solution not involving $\operatorname{erf}\eta$ explicitly only if $B=\frac{1}{8}\beta\pi^{\frac{1}{2}}$. The constant $C$ remains indeterminate, but the terms in the particular integral (26) for $f_3$ which involve $C$ are, in fact, $Cn_3$, where $n_3$ is the complementary function given in (25), so that it is sufficient to put $C=0$. The solution for $T$ with these values of $B$ and $C$ is given below (eq. 39).

Collecting our results, we find that the solution for $f_3$ is

$$f_3 = X\operatorname{erf}^3\eta + Y\operatorname{erf}^2\eta + Z\operatorname{erf}\eta + T + \gamma m_3 + \delta n_3, \tag{38}$$

where $m_3$, $n_3$ are given in (24) and (25), and

$$\left.\begin{aligned} X &= \tfrac{1}{12}(4\eta^4+3),\\ Y &= \tfrac{1}{8}\pi^{\frac{1}{2}}\beta(4\eta^4+4\eta^2-1)+\tfrac{1}{12}\pi^{-\frac{1}{2}}(14\eta^3-9\eta)e^{-\eta^2},\\ Z &= -\tfrac{1}{2}(4\alpha\eta^2+2\alpha+1)+\tfrac{1}{12}\beta(14\eta^3+15\eta)e^{-\eta^2}\\ &\quad+\tfrac{1}{3}\pi^{-1}(4\eta^2-5)e^{-2\eta^2},\\ T &= \tfrac{1}{30}\pi^{-\frac{1}{2}}[10\eta^3+(20-15\alpha)\eta-8\beta]e^{-\eta^2}\\ &\quad+\tfrac{2}{3}\pi^{-\frac{1}{2}}\beta(\eta^2+1)e^{-2\eta^2}-\tfrac{1}{40}\pi^{-\frac{3}{2}}(18\eta^3+31\eta)e^{-3\eta^2}\\ &\quad+\frac{16\sqrt{2}}{15}\pi^{-1}e^{-\eta^2}\operatorname{erf}\sqrt{2}\eta-\frac{9\sqrt{3}}{80}\pi^{-1}(4\eta^4+12\eta^2+3)\operatorname{erf}\sqrt{3}\eta. \end{aligned}\right\} \tag{39}$$

When $\eta=0$,
$$f_3=\tfrac{2}{5}\pi^{-\frac{1}{2}}\beta+3\gamma. \tag{40}$$

When $\eta\to\infty$
$$f_3\sim\left(\tfrac{1}{12}+\tfrac{1}{8}\beta\pi^{\frac{1}{2}}-\frac{9\sqrt{3}}{80\pi}+\gamma+\delta\right)(4\eta^4+12\eta^2+3). \tag{41}$$

Since $f_3$ must vanish at $\eta=0$ and $\eta=\infty$, we must have

$$\left.\begin{aligned} \gamma &= -\left(\frac{8}{45\pi^2}+\frac{2}{15\pi}\right)=-0{\cdot}06045,\\ \delta &= \frac{8}{45\pi^2}+\frac{9\sqrt{3}}{80\pi}-\frac{1}{30\pi}-\frac{5}{24}=-0{\cdot}13891. \end{aligned}\right\} \tag{42}$$

In exactly the same way, we find that

$$F_3-f_3 = X^*\operatorname{erf}^3\eta + Y^*\operatorname{erf}^2\eta + Z^*\operatorname{erf}\eta + T^* + \Gamma m_3 + \Delta n_3, \tag{43}$$

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where

$$\left.\begin{aligned}
X^* &= -\tfrac{1}{3}(2\eta^4+3\eta^2),\\
Y^* &= -\tfrac{1}{12}\pi^{\frac{1}{2}}\beta(4\eta^4+3)-\tfrac{4}{3}\pi^{-\frac{1}{2}}\eta-\tfrac{1}{4}\pi^{-\frac{1}{2}}(10\eta^3+13\eta)\,e^{-\eta^2},\\
Z^* &= \tfrac{1}{6}(12\alpha\eta^2-8\beta\eta-3)-\tfrac{2}{3}\pi^{-1}\{\pi\beta(\eta^3-\eta)-2\eta^2+1\}e^{-\eta^2}\\
&\quad -3\pi^{-1}(\eta^2+1)\,e^{-2\eta^2},\\
T^* &= -\tfrac{8}{3}\pi^{-\frac{1}{2}}\alpha\eta-\tfrac{4}{3}\pi^{-\frac{1}{2}}\{(\beta\pi^{\frac{1}{2}}-\alpha)\,\eta+\beta\}e^{-\eta^2}-\tfrac{1}{3}\pi^{-\frac{3}{2}}\{\pi\beta(\eta^2-2)-4\eta\}e^{-2\eta^2}\\
&\quad +\tfrac{1}{24}\pi^{-\frac{3}{2}}(54\eta^3+125\eta)\,e^{-3\eta^2}+\frac{9\sqrt{3}}{16}\pi^{-1}(4\eta^4+12\eta^2+3)\operatorname{erf}\sqrt{3}\eta,\\
\Gamma &= \frac{8}{27\pi^2}+\frac{2}{9\pi}=0{\cdot}10076,\\
\Delta &= \frac{8}{27\pi^2}-\frac{1}{9\pi}-\frac{9\sqrt{3}}{16\pi}+\frac{1}{4}=-0{\cdot}12551.
\end{aligned}\right\} \quad (44)$$

Correct to the third order of approximation the velocity in the boundary layer is therefore

$$\frac{u}{U}=f_1(\eta)+tU'f_2(\eta)+t^2[UU''f_3(\eta)+U'^2F_3(\eta)]. \quad (45)$$

Some numerical values of the various functions are given in the accompanying table, and the results are shown diagrammatically in Fig. 1. For all practical purposes the functions may be assumed to have their asymptotic values by about $\eta=2$.

| $\eta$ | $f_1(\eta)$ | $f_2(\eta)$ | $f_3(\eta)$ | $F_3(\eta)$ |
|---|---|---|---|---|
| 0·0 | 0·000 | 0·000 | 0·000 | 0·000 |
| 0·5 | 0·520 | 0·381 | −0·031 | −0·127 |
| 1·0 | 0·843 | 0·269 | −0·243 | −0·392 |
| 1·5 | 0·966 | 0·098 | −0·021 | −0·122 |
| 2·0 | 0·995 | 0·020 | −0·016 | −0·047 |

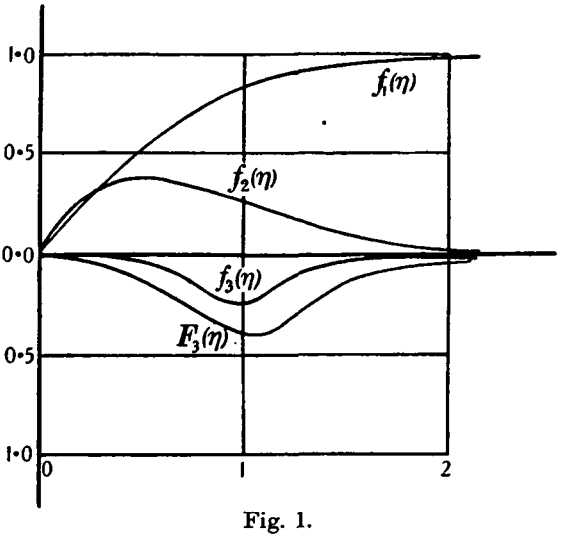



Fig. 1.

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Separation commences where $(\partial u/\partial \eta)_{\eta=0}=0$, that is, where

$$0=1+\left(1+\frac{4}{3\pi}\right)U't+\left\{UU''\left[\frac{64}{45\pi^2}+\frac{27\sqrt{3}-11}{30\pi}-\frac{7}{12}\right]\right.$$
$$\left.+U'^2\left[-\frac{128}{135\pi^2}+\frac{89-108\sqrt{3}}{30\pi}+\frac{11}{12}\right]\right\}t^2, \qquad (46)$$

i.e.
$$1+1{\cdot}4244U't-\{0{\cdot}05975UU''+0{\cdot}21987U'^2\}t^2=0. \qquad (47)$$

## IV. Applications to a circular cylinder and a special case of an elliptic cylinder

The solution of (47) is

$$\frac{1}{t}=-0{\cdot}7122U'+\sqrt{\{0{\cdot}7271U'^2+0{\cdot}05975UU''\}}. \qquad (48)$$

$t$ has its least value where $-U'$ is greatest if, and only if, $U$ is zero there. As pointed out in the introduction, we may take $U$ to be the velocity at the surface in an irrotational motion without circulation. Then for a circular cylinder, for example, $-U'$ is greatest at the rear stagnation point, and separation begins there both on the first and second approximations. If $r$ is the radius of the cylinder,

$$U=2V\sin\frac{x}{r}, \qquad (49)$$

and the time that elapses from the beginning of the motion until separation first begins is given by
$$VT_1=0{\cdot}35r, \qquad (50)$$

for the first approximation, and
$$VT_2=0{\cdot}32r \qquad (51)$$

for the second approximation. The left-hand sides give the distance travelled by the cylinder from the commencement of the motion, and the second approximation is about 9 per cent. less than the first.

For uniformly accelerated motion, we must write

$$V=ft, \quad U=Wt, \qquad (52)$$

and Blasius found that the time to separation is given by

$$1+0{\cdot}427W't^2-\{0{\cdot}010WW''+0{\cdot}026W'^2\}t^4=0. \qquad (53)$$

For the circular cylinder, separation begins again at the rear stagnation point for either the first or second approximation, and we find that

$$\tfrac{1}{2}fT_1^2=0{\cdot}585r, \qquad (54)$$
$$\tfrac{1}{2}fT_2^2=0{\cdot}52r. \qquad (55)$$

The left-hand sides again give the distance travelled by the cylinder from the commencement of the motion, and the second approximation is about 11 per cent. less than the first.

https://doi.org/10.1017/S0305004100019101 Published online by Cambridge University Press

Mr Howarth has lately considered the case of an ellipse with axes in the ratio 1 : 6, and with its major axis at an angle of 7° to the stream. For the irrotational motion without circulation the rear stagnation point is at a distance of $0{\cdot}0224c$ from the end of the major axis, towards the upper side of the ellipse, where $2c$ is the distance between the foci. For the first approximation, for an impulsive start, he informs us that he found that separation begins at a distance of $0{\cdot}0175c$ from the rear stagnation point towards the lower side of the ellipse, after a time given by

$$VT_1 = 0{\cdot}0160c. \tag{56}$$

Mr Howarth kindly placed values of $U$, $U'$, $U''$ at our disposal, from which it appears that for the second approximation for an impulsive start, separation begins at $0{\cdot}0172c$ from the rear stagnation point after a time given by

$$VT_2 = 0{\cdot}0146c. \tag{57}$$

The position of initial separation is not much altered. The interval is again reduced by about 9 per cent.

Similarly, for uniformly accelerated motion, we find that for the first approximation, separation begins at $0{\cdot}0175c$ from the rear stagnation point after a time interval given by

$$\tfrac{1}{2}fT_1^2 = 0{\cdot}02675c, \tag{58}$$

and for the second approximation at $0{\cdot}0171c$ from the rear stagnation point after a time interval given by

$$\tfrac{1}{2}fT_2^2 = 0{\cdot}0237c. \tag{59}$$

The position of initial separation is again not much altered; and the distance traversed before separation begins is reduced by about 11 per cent., as for the circular cylinder. Since the position of initial separation is not much altered in either case, the terms in $UU''$ or $WW''$ in (48) or (53), in fact, make very little difference, and the same percentage reduction would always be found.



https://doi.org/10.1017/S0305004100019101 Published online by Cambridge University Press