A THEOREM ON POSITIVE HARMONIC FUNCTIONS

By S. VERBLUNSKY

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1. Let $z = re^{i\theta}$, and let $h(z)$ denote a (regular) positive harmonic function in the unit circle $r<1$. Then $h(r)(1-r)$ and $h(r)/(1-r)$ tend to limits as $r\to 1$. The first limit is finite; the second may be infinite. Such properties of h can be obtained in a straightforward way by using the fact that we can write

$$h(z) = \int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}\, d\alpha(\phi), \tag{1}$$

where $\alpha(\phi)$ is non-decreasing in the closed interval $(-\pi,\pi)$. Another method is to write

$$\eta(z) = h(z) + ih^*(z),$$

where h^* is a harmonic function conjugate to h. Then the function

$$f(z) = \frac{1-\eta}{1+\eta}$$

has the property $|f|<1$ in the unit circle. Such functions have been studied by Julia, Wolff, Carathéodory and others. One can deduce properties of h and h^* from known properties of f, and conversely. The main purpose of this note is to prove

THEOREM I. (i) $0<\lim_{r\to 1} h(r)/(1-r) = \lim_{z\to 1} h(z)/(1-r)$;

(ii) $\lim_{r\to 1} h(r)(1-r) = \overline{\lim_{z\to 1}}\, h(z)(1-r)$;

(iii) *if* $\lim h(r)/(1-r)$ *is finite, then* $\lim h^*(r)$ *is finite, where* $h^*(z)$ *is a harmonic function which is conjugate to* $h(z)$. *And if* h^* *is chosen so that* $\lim h^*(r) = 0$, *then* $\lim h^*(r)/(1-r) = 0$.

As an application, we prove a theorem of Carathéodory (§ 5). We conclude with a short proof of a theorem of Herzig (§ 6).

In what follows, integrals are extended over the interval $(-\pi,\pi)$ unless otherwise indicated.

2. *Proof of* (i). The function

$$g(r,\phi) = \frac{r}{1+r^2-2r\cos\phi}$$

is, for each ϕ, an increasing function of r in the interval $0<r<1$. Let

$$k(z) = \frac{h(z)}{1-r}\frac{r}{1+r}.$$

By (1),

$$k(z) = \int g(r,\theta-\phi)\, d\alpha(\phi),$$

and, for each θ, k is an increasing function of r. In particular, we can write

$$\lim_{r\to 1} k(r) = \lambda,$$

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where λ is positive if finite, and we have to prove that $\lambda \leqslant \mu$, where

$$\mu = \overline{\lim_{z \to 1}}\, k(z).$$

Let $z_n = r_n e^{i\theta_n}$, where $r_n \to 1$ and $n\theta_n \to 0$. Let

$$\rho_n = \min\left(r_n, \frac{1}{1+n\,|\,\theta_n\,|}\right), \quad \zeta_n = \rho_n e^{i\theta_n}.$$

Then

$$\rho_n \leqslant r_n, \quad \rho_n \to 1.$$

We shall prove that for all sufficiently large n,

$$k(\rho_n) < \left(1 + \frac{15}{n}\right) k(z_n), \tag{2}$$

and since z_n can be chosen so that the second member tends to μ, the desired result will follow.

Since $k(\zeta_n) \leqslant k(z_n)$, it suffices to prove (2) with z_n replaced by ζ_n. We have

$$k(\rho_n) - k(\zeta_n) = \int [g(\rho_n, \phi) - g(\rho_n, \theta_n - \phi)]\, d\alpha(\phi)$$

and

$$g(\rho_n, \phi) - g(\rho_n, \theta_n - \phi) = 2(\cos\phi - \cos(\theta_n - \phi))\, g(\rho_n, \phi)\, g(\rho_n, \theta_n - \phi).$$

The result will follow if we prove that

$$2(\cos\phi - \cos(\theta_n - \phi))\, g(\rho_n, \phi) < \frac{15}{n} \tag{3}$$

if $\rho_n > \frac{1}{2}$. We have

$$g(\rho_n, \phi) = \frac{\rho_n}{(1-\rho_n)^2 + 4\rho_n \sin^2 \frac{1}{2}\phi} < \frac{1}{(1-\rho_n)^2 + 2(\phi/\pi)^2}$$

$$< \frac{5}{(1-\rho_n)^2 + \phi^2}, \tag{4}$$

and

$$|\,\theta_n\,| \leqslant \frac{1-\rho_n}{n\rho_n} \leqslant \frac{2}{n}(1-\rho_n),$$

$$2(\cos\phi - \cos(\theta_n - \phi)) = 2\cos\phi(1 - \cos\theta_n) - 2\sin\theta_n \sin\phi < \theta_n^2 + 2\,|\,\theta_n\,|\,|\,\phi\,|$$

$$< \frac{2}{n}\left[\frac{2}{n}(1-\rho_n)^2 + 2\,|\,\phi\,|\,(1-\rho_n)\right] < \frac{3}{n}[(1-\rho_n)^2 + \phi^2] \tag{5}$$

if $n > 4$. Then (3) follows from (4) and (5).

We note that if $\lim h(r)/(1-r)$ is finite, then

$$\int \frac{d\alpha(\phi)}{(1-\cos\phi)} \tag{6}$$

is convergent and equals the limit.

3. *Proof of* (ii). The function

$$G(r, \phi) = \frac{(1-r)^2}{1 + r^2 - 2r\cos\phi}$$

is, for each ϕ, a decreasing function of r in the interval $0 < r < 1$. Let

$$K(z) = \frac{h(z)\,(1-r)}{1+r}.$$

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Then $$K(z) = \int G(r, \theta - \phi)\, d\alpha(\phi),$$

and, for each θ, K is a decreasing function of r. In particular, we can write

$$\lim_{r \to 1} K(r) = \Lambda,$$

and we have to prove that $\Lambda \geqslant M$, where

$$M = \overline{\lim_{z \to 1}}\, K(z).$$

Let z_n, ζ_n have the meanings of § 2. The result will follow if we prove that

$$K(\rho_n) > \left(1 - \frac{15}{n}\right) K(z_n) \tag{7}$$

for all sufficiently large n. Since $K(\zeta_n) \geqslant K(z_n)$, it suffices to prove (7) with z_n replaced by ζ_n. We have

$$K(\rho_n) - K(\zeta_n) = \int [G(\rho_n, \phi) - G(\rho_n, \theta_n - \phi)]\, d\alpha(\phi)$$

and $$G(\rho_n, \phi) - G(\rho_n, \theta_n - \phi) = G(\rho_n, \phi)\, G(\rho_n, \theta_n - \phi) \frac{2\rho_n}{(1-\rho_n)^2} (\cos\phi - \cos(\theta_n - \phi)).$$

Now $$\rho_n G(\rho_n, \phi) < \frac{5(1-\rho_n)^2}{(1-\rho_n)^2 + \phi^2}$$

and $$2(\cos\phi - \cos(\theta_n - \phi)) > -\frac{3}{n}((1-\rho_n)^2 + \phi^2)$$

for all sufficiently large n. Hence the result.

4. *Proof of* (iii). If $\lim h(r)/(1-r)$ is finite, then (6) is convergent. We have

$$h^*(z) = \int \frac{2r \sin(\theta - \phi)}{1 + r^2 - 2r\cos(\theta - \phi)}\, d\alpha(\phi) + c,$$

where c is a real constant; so that the convergence of (6) implies the existence of $\lim h^*(r)$ and the equation

$$\lim h^*(r) = -\int \frac{\sin\phi}{1 - \cos\phi}\, d\alpha(\phi) + c.$$

If c is chosen so that the first member is zero, then

$$h^*(r) = \int \sin\phi \left(\frac{1}{1-\cos\phi} - 2g(r, \varphi)\right) d\alpha(\phi),$$

and $$\frac{h^*(r)}{1-r} = \int \frac{(1-r)\sin\phi}{1 + r^2 - 2r\cos\phi} \frac{d\alpha(\phi)}{1 - \cos\phi}$$

$$= \int_{-\epsilon}^{\epsilon} + o(1),$$

if ϵ is an assigned positive number less than π. But

$$\left| \frac{(1-r)\sin\phi}{1 + r^2 - 2r\cos\phi} \right| < \frac{1}{2r}[(1-r)^2 + \phi^2]\, g < 5$$

if $r > \frac{1}{2}$. Hence the result.

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5. The following theorem is due to Carathéodory (*Conformal Representation* (Cambridge, 1932), 57): *If $f(z)$ is regular and $|f(z)|<1$ in the unit circle, and there is a sequence z_n such that*

$$z_n\to 1, \quad f(z_n)\to 1, \quad \lim\frac{1-|f(z_n)|}{1-|z_n|}<\infty,$$

then

$$\frac{1-|f(r)|}{1-r}, \quad \frac{|1-f(r)|}{1-r}, \quad \frac{1-f(r)}{1-r}$$

tend to the same positive limit as $r\to 1$.

Proof. Write

$$\eta(z)=\frac{1-f}{1+f}.$$

Then

$$h(z)=\Re\eta=\frac{1-|f|^2}{|1+f|^2}>0, \quad h^*(z)=\Im\eta=\frac{i(f-\bar f)}{|1+f|^2},$$

$$\frac{h}{1-r}=\frac{1-|f|}{1-r}\frac{1+|f|}{|1+f|^2},$$

and

$$\lim_{z\to 1}\frac{h(z)}{1-r}\leqslant\lim\frac{h(z_n)}{1-|z_n|}=\tfrac{1}{2}\lim\frac{1-|f(z_n)|}{1-|z_n|}<\infty.$$

By §2,

$$\lim\frac{h(r)}{1-r}=\int\frac{d\alpha(\phi)}{1-\cos\phi}<\infty. \tag{8}$$

In particular, $h(r)\to 0$. Again, $h^*(z_n)\to 0$. Let $z_n=r_n e^{i\theta_n}$ where $\theta_n\to 0$. Suppose that we can prove that

$$h^*(r_n)\to 0. \tag{9}$$

It will follow from (iii) Theorem I that

$$h^*(r)\to 0, \quad \lim\frac{h^*(r)}{1-r}=0.$$

Then $\eta(r)\to 0$ and $f(r)\to 1$. Hence

$$\frac{1-|f(r)|}{1-r}=\frac{h(r)}{1-r}\frac{|1+f(r)|^2}{1+|f(r)|}\to 2\lim\frac{h(r)}{1-r},$$

the limit being positive; and

$$\frac{1-f(r)}{1-r}=\frac{h(r)+ih^*(r)}{1-r}(1+f(r)),$$

and

$$\frac{|1-f(r)|}{1-r}$$

tend to the same limit.

It remains to prove (9), which will follow if we show that

$$\lambda_n=h^*(z_n)-h^*(r_n)\to 0.$$

We may suppose that $\theta_n\neq 0$ for relevant n, since otherwise there is nothing to prove.

Using the integral representation of $h^*(z)$ (see §4), we have

$$\lambda_n=I_1+I_2$$

where

$$I_1=2\sin\theta_n\int\frac{(1+r_n^2)\cos\phi-2r_n}{1+r_n^2-2r_n\cos\phi}g(r_n,\theta_n-\phi)\,d\alpha(\phi),$$

and

$$I_2=2(1-\cos\theta_n)(1+r_n^2)\int\frac{\sin\phi}{1+r_n^2-2r_n\cos\phi}g(r_n,\theta_n-\phi)\,d\alpha(\phi),$$

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By hypothesis, $$\frac{h(z_n)}{1-r_n} = \frac{1+r_n}{r_n}\int g(r_n, \theta_n - \phi)\, d\alpha(\phi) = O(1).$$

Since $$|(1+r_n^2)\cos\phi - 2r_n| \leqslant 1 + r_n^2 - 2r_n\cos\phi,$$

it follows that $I_1 = o(1)$.

Finally, if ϵ is a small positive number,

$$I_2 = O\left[\int_{-\epsilon}^{\epsilon} \frac{\theta_n^2 |\phi|}{1+r_n^2-2r_n\cos\phi} g(r_n, \theta_n-\phi)\, d\alpha(\phi)\right] + o(1).$$

For $|\phi| < \frac{1}{2}|\theta_n|$, $$g(r_n, \theta_n - \phi) < g(r_n, \tfrac{1}{2}\theta_n) = O(\theta_n^{-2}).$$

The contribution to I_2 is $$O\left[\int \frac{|\phi|}{1-\cos\phi} d\alpha(\phi)\right]$$

the integral being over the interval $|\phi| < \frac{1}{2}|\theta_n|$; and this is $o(1)$ by (8). For $|\phi| \geqslant \frac{1}{2}|\theta_n|$,

$$\frac{1}{1+r_n^2-2r_n\cos\phi} = O(\theta_n^{-2}),$$

and the contribution to I_2 is

$$O\left[\int_{-\epsilon}^{\epsilon} |\phi|\, g(r_n, \theta_n-\phi)\, d\alpha(\phi)\right] = O(\epsilon).$$

Since ϵ is arbitrary, the result follows.

Carathéodory deduced the preceding theorem from his generalization of Julia's theorem (op. cit. p. 54). This, as well as the theorem of Wolff and Carathéodory on the angular derivative (op. cit. p. 96) can easily be proved by examining the formula

$$\eta(z) = \int \frac{e^{i\phi}+z}{e^{i\phi}-z} d\alpha(\phi) + ih^*(0)$$

as was first done by R. Nevanlinna (*C.R. Acad. Sci., Paris*, 188 (1929), 1027–9), and subsequently by Herzig (*Math. Z.* 46 (1940), 129–56), whose paper contains a very detailed proof.

6. In the paper of Herzig (loc. cit. p. 149) the following theorem is proved: *If* $\alpha(\phi) = O(|\phi|^k)$ *for small* ϕ, *where* $0 < k < 1$, *then*

$$\int \frac{e^{i\phi}+z}{e^{i\phi}-z} d\alpha(\phi) = O(1-r)^{k-1},$$

as $z \to 1$ *in a sector whose sides are chords of the unit circle.* As Herzig's proof is lengthy, the following one is of interest. On integrating by parts, we obtain

$$\frac{1-z}{1+z}\{\alpha(\pi) - \alpha(-\pi)\} + 2iz\int \frac{e^{i\phi}}{(e^{i\phi}-z)^2}\alpha(\phi)\, d\phi.$$

The approach of z to 1 in the sector implies that

$$|1-z| = O(1-r), \quad |\theta| = O(1-r).$$

It is sufficient to estimate the last integral, whose modulus is less than a constant times I, where

$$I = \int \frac{|\phi|^k}{1+r^2-2r\cos(\theta-\phi)} d\phi.$$

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Observing that $|\phi|^k < |\phi-\theta|^k + |\theta|^k$,

we have $I < I_1 + I_2$, where

$$I_1 = \int \frac{|\phi-\theta|^k}{1+r^2-2r\cos(\phi-\theta)}\,d\phi = O\left[\int \frac{|\phi-\theta|^k}{(1-r)^2+(\phi-\theta)^2}\,d\phi\right]$$

$$= O\left[\int_{-\infty}^{\infty} \frac{|\phi|^k}{(1-r)^2+\phi^2}\,d\phi\right]$$

$$= O(1-r)^{k-1}$$

and

$$I_2 = \int \frac{|\theta|^k}{1+r^2-2r\cos(\phi-\theta)}\,d\phi = O\left[(1-r)^{k-1}\int \frac{1-r^2}{1+r^2-2r\cos(\phi-\theta)}\,d\phi\right]$$

$$= O(1-r)^{k-1}.$$

[*Added* 8 *June* 1948.] I add the following remarks in consequence of the referee's comments. (i) The monotonic character of $g(r,\phi)$ can be used in other connexions. See, for example, a paper by the author in *Proc. London Math. Soc.* 34 (1932), 441–56. (ii) The theorems of Carathéodory, mentioned in § 5, are proved by him by a geometrical method which requires an elaborate study of Möbius transformations. As we show, this is unnecessary if we use the Poisson-Stieltjes representation of positive harmonic functions.

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