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# Symplectic periodic flows with exactly three equilibrium points

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*Abstract*. Let the circle act symplectically on a compact, connected symplectic manifold $M$. If there are exactly three fixed points, $M$ is equivariantly symplectomorphic to $\mathbb{CP}^2$.

## 1. *Introduction*

The study of fixed points of flows or maps is a classical and important topic studied in geometry and dynamical systems. In this paper we focus on the case where manifolds admit symplectic structures and circle actions on the manifolds preserve the symplectic structures. Circle actions in symplectic geometry correspond to periodic flows in mechanical systems. Fixed points by the actions correspond to equilibrium points by the flows. If a circle action has fixed points, a lot of information is encoded by the fixed point data of the action.

Hamiltonian actions are symplectic but symplectic actions need not be Hamiltonian. Hence it is a natural question to ask if there is a non-Hamiltonian symplectic action on a compact symplectic manifold when given a certain number of fixed points. It is a classical fact that a Hamiltonian circle action on a compact symplectic manifold $(M, \omega)$ has at least $\frac{1}{2}\dim M + 1$ fixed points.

Frankel proved that a symplectic $S^1$-action on a Kähler manifold is Hamiltonian if and only if is has a fixed point [**F59**]. The property that a symplectic $S^1$-action is Hamiltonian if and only if it has a fixed point holds on four-dimensional symplectic manifolds [**MD88**], and it also holds for semi-free actions on symplectic manifolds with discrete fixed points [**TW00**]. On the other hand, in the same paper McDuff constructed an example of a non-Hamiltonian symplectic circle action with fixed points. However, in this case the fixed point set are tori. Therefore, the question still remains whether there is a non-Hamiltonian symplectic circle action with isolated fixed points.

Let the circle act symplectically on a compact, connected symplectic manifold $M$. First of all, there cannot be exactly one fixed point. Also, due to Kosniowski, if there are exactly

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two fixed points, then either $M$ is the 2-sphere or $\dim M = 6$ [**Ko74**]. This is re-proved by Pelayo and Tolman using another method [**PT11**]. This result itself does not rule out the possibility that there is a six-dimensional compact symplectic manifold $M$ with exactly two fixed points by the symplectic circle action. In this paper, we follow the method that Pelayo and Tolman use.

In this paper we study symplectic circle actions on compact, connected symplectic manifolds with exactly three fixed points. The conclusion is that in this case the manifold must be four-dimensional and it is equivariantly symplectomorphic to $\mathbb{CP}^2$ with the standard action. In particular, if there are exactly three fixed points, symplectic circle actions are Hamiltonian.

A 2-form $\omega \in \Omega^2(M)$ on a manifold $M$ is called a *symplectic form* if it is closed and non-degenerate. If a circle action on $M$ preserves $\omega$, the action is called *symplectic*. Let $X_M$ be the vector field on $M$ induced by the circle action. The action is called *Hamiltonian*, if there exists a map $\mu : M \to \mathbb{R}$ such that

$$-d\mu = \iota_{X_M}\omega.$$

This implies that every symplectic action is Hamiltonian if $H_1(M; \mathbb{R}) = 0$, since $\iota_{X_M}\omega$ is closed. We now state the main result of this paper.

THEOREM 1.1. *Let the circle act symplectically on a compact, connected symplectic manifold $M$. If there are exactly three fixed points, $M$ is equivariantly symplectomorphic to $\mathbb{CP}^2$.*

We now give a brief overview of the proof. The proof is based on induction on the dimension of $M$. For each fixed point, there are well-defined non-zero integers, called *weights*. The main idea of the proof is to get restrictions on the weights at the three fixed points and show that if $\dim M > 4$, the weights cannot satisfy all the restrictions.

One of the key facts is that, for any non-zero integer $l$, the subgroup $\mathbb{Z}_l \subset S^1$ also acts on $M$. Moreover, the set of points $M^{\mathbb{Z}_l}$ fixed by the $\mathbb{Z}_l$-action is a union of symplectic submanifolds, and the isotropy weights in $M^{\mathbb{Z}_l}$ are multiples of $l$.

Two important isotropy weights are the largest weight and the isotropy weight two. First, let $d$ be the biggest weight among all the weights that occurs at the three fixed points. We show that $M^{\mathbb{Z}_d}$ is a union of a 2-sphere and a point. Moreover, we show that this gives significant restrictions on other weights. Second, we show that $M^{\mathbb{Z}_2}$ is $\mathbb{CP}^2$, provided that $\dim M > 4$. This implies that for manifolds of dimension greater than four, exactly two weights at each fixed point are even. If the largest weight $d$ is even, it is itself one of the weights in the isotropy submanifold fixed by the $\mathbb{Z}_2$-action. Because of this, we divide the theorem into two cases, depending on whether the largest weight is odd or even. The former case is much easier and requires less work. Finally, if $\dim M > 4$, the fact that there are exactly three fixed points implies that the sum of the weights at any fixed point is zero. When combined with the restrictions described above, this determines the number of negative weights at each fixed point.

Finally, the ABBV localization formula describes the push-forward map from the equivariant cohomology of $M$ to the equivariant cohomology of a point in terms of a formula in the weights at the fixed points [**AB84**]; see Theorem 2.1 below. Since the

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push-forward of 1 is 0, this gives additional restrictions on the weights, and we use these to complete the proof.

*Remark 1.2.* Let us now consider a symplectic circle action on a compact, connected symplectic manifold $M$ with exactly four fixed points. Unfortunately, the techniques for the case of three fixed points are not sufficient to analyze the case of four fixed points. One of difficulties in this case is that there are more cases to consider for the isotropy submanifolds $M^{\mathbb{Z}_2}$ and $M^{\mathbb{Z}_d}$, where $d$ is the biggest weight that occurs. Moreover, in this case the sum of the weights at each fixed point need not be identically zero. Therefore, there are many possibilities for the number of negative weights at each fixed point as well.

This paper is organized as follows. In §2 we introduce the background and notation needed. In §3 we prove preliminary lemmas. In §4 we prove the base case, that is, when $\dim M < 8$. In §5 we consider the case where the largest weight is odd. Sections 6 and 7 are preliminaries for §8, in which we consider the case where the largest weight is even. Once this is completed, a one-paragraph proof is given in §9.

## 2. *Background and notation*

Consider a circle action on a manifold $M$. *The equivariant cohomology of* $M$ is defined by $H^*_{S^1}(M) = H^*(M \times_{S^1} S^\infty)$. If $M$ is oriented and compact, then from the projection map $\pi : M \times_{S^1} S^\infty \to \mathbb{CP}^\infty$ we obtain a natural push-forward map

$$\pi_* : H^i_{S^1}(M; \mathbb{Z}) \to H^{i-\dim M}_{S^1}(\mathbb{CP}^\infty; \mathbb{Z})$$

for $i \in \mathbb{Z}$. This map is given by 'integration over the fiber' and denoted by $\int_M$.

THEOREM 2.1. (ABBV localization) [**AB84**] *Let the circle act on a compact oriented manifold $M$. Fix $\alpha \in H^*_{S^1}(M; \mathbb{Q})$. As elements of $\mathbb{Q}(t)$,*

$$\int_M \alpha = \sum_{F \subset M^{S^1}} \int_F \frac{\alpha|_F}{e_{S^1}(N_F)},$$

*where the sum is over all fixed components, and $e_{S^1}(N_F)$ denotes the equivariant Euler class of the normal bundle to $F$.*

Let the circle act symplectically on a symplectic manifold $(M, \omega)$. To each isolated fixed point $p \in M^{S^1}$ we can assign well-defined non-zero (integer) weights $\xi^1_p, \ldots, \xi^n_p$ in the isotropy representation $T_pM$ (repeated with multiplicity). Denote by $\sigma_i$ the $i$th elementary symmetric polynomial. Then the $i$th equivariant Chern class at the fixed point $p$ is given by

$$c_i(M)|_p = \sigma_i(\xi^1_p, \ldots, \xi^n_p)t^i,$$

where $t$ is the generator of $H^2_{S^1}(p; \mathbb{Z})$. For instance, the first Chern class map at $p$ is given by $c_1(M)|p = \Sigma \xi^i_p t$, and the equivariant Euler class of the tangent bundle at $p$ is given by $e_{S^1}(N_F) = c_n(M)|_p = (\prod \xi^i_p)t^n$. Hence,

$$\int_p \frac{c_i(M)|_p}{e_{S^1}(N_p)} = \frac{\sigma_i(\xi^1_p, \ldots, \xi^n_p)}{\prod \xi^j_p} t^{i-n}.$$

Denote by $\lambda_p$ twice of the number of negative weights at $p$ for all $p \in M^{S^1}$. This is called *the index* at $p$, and this notion agrees with the index of a fixed point of a Hamiltonian

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circle action. Weights in the isotropy representation $T_pM$ also satisfy the following lemma.

LEMMA 2.2. [**PT11**] *Let the circle act symplectically on a compact symplectic* $2n$*-manifold with isolated fixed points. Then*

$$|\{p \in M^{S^1} \mid \lambda_p = 2i\}| = |\{p \in M^{S^1} \mid \lambda_p = 2n - 2i\}| \quad \text{for all } i,$$

COROLLARY 2.3. [**PT11**] *Let the circle act symplectically on a compact symplectic* $2n$*-manifold with* $k$ *isolated fixed points. If* $k$ *is odd, then* $n$ *is even.*

LEMMA 2.4. [**PT11**] *Let the circle act symplectically on a compact symplectic manifold* $M$ *with isolated fixed points. Then*

$$\sum_{p \in M^{S^1}} N_p(l) = \sum_{p \in M^{S^1}} N_p(-l) \quad \text{for all } l \in \mathbb{Z}.$$

*Here,* $N_p(l)$ *is the multiplicity of* $l$ *in the isotropy representation* $T_pM$ *for all weights* $l \in \mathbb{Z}$ *and all* $p \in M^{S^1}$.

LEMMA 2.5. [**T10**] *Let the circle act on a compact symplectic manifold* $(M, \omega)$. *Let* $p$ *and* $p'$ *be fixed points which lie in the same component* $N$ *of* $M^{\mathbb{Z}_k}$, *for some* $k > 1$. *Then the* $S^1$*-weights at* $p$ *and at* $p'$ *are equal modulo* $k$.

Let the circle act on a compact symplectic manifold $M$. Let $p$ and $p'$ be fixed points which lie in the same component $N$ of $M^{\mathbb{Z}_k}$, for some $k > 1$. Denote by $\Sigma_p$ and $\Sigma_{p'}$ the multisets of weights at $p$ and $p'$, respectively. Lemma 2.5 states that there exists a bijection between $\Sigma_p$ and $\Sigma_{p'}$ that takes each weight $\alpha$ at $p$ to a weight $\beta$ at $p'$ such that $\alpha \equiv \beta \bmod k$.

THEOREM 2.6. [**PT11**] *Let the circle act symplectically on a compact, connected symplectic manifold* $M$ *with exactly two fixed points. Then either* $M$ *is the* 2*-sphere or* $\dim M = 6$ *and there exist natural numbers* $a$ *and* $b$ *so that the weights at the two fixed points are* $\{a, b, -a - b\}$ *and* $\{a + b, -a, -b\}$.

COROLLARY 2.7. [**PT11**] *Let the circle act symplectically on a compact symplectic manifold* $M$ *with non-empty fixed point set. Then there are at least two fixed points, and if* $\dim M \geq 8$, *then there are at least three fixed points. Moreover, if the Chern class map is not identically zero and* $\dim M \geq 6$, *then there are at least four fixed points.*

## 3. *Preliminaries*

One of the main ideas to prove Theorem 1.1 is to look at the biggest weight among all the weights of fixed points.

Consider a symplectic circle action on a $2n$-dimensional compact, connected symplectic manifold $M$ with exactly three fixed points. The key fact is that the largest weight occurs only once. From this it follows that if without loss of generality we assume that $\lambda_p \leq \lambda_q \leq \lambda_r$ where $p$, $q$, and $r$ are the three fixed points, then $\lambda_p = n - 2$, $\lambda_q = n$, and $\lambda_r = n + 2$.

*Definition 3.1.* A weight $d$ is *the largest weight* if it is the biggest weight such that $\sum_{u \in M^{S^1}} N_u(d) > 0$.

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PROPOSITION 3.2. *Let the circle act symplectically on a compact, connected symplectic manifold $M$ and suppose that there are exactly three fixed points. Let $d$ be the largest weight. Then the isotropy submanifold $M^{\mathbb{Z}_d}$ contains exactly two components that have fixed points: one isolated fixed point and one 2-sphere that contains two fixed points.*

*Proof.* Consider the isotropy submanifold $M^{\mathbb{Z}_d}$. By Theorem 2.6, there are only three possible cases.

(1) The isotropy submanifold $M^{\mathbb{Z}_d}$ contains a 2-sphere with two fixed points. The third fixed point is another component of $M^{\mathbb{Z}_d}$.
(2) The isotropy submanifold $M^{\mathbb{Z}_e}$ contains a six-dimensional component with two fixed points. The third fixed point is another component of $M^{\mathbb{Z}_d}$.
(3) The isotropy submanifold $M^{\mathbb{Z}_d}$ contains a component with the three fixed points.

The subset inclusions may not be equalities since $M^{\mathbb{Z}_d}$ may contain other components with no fixed points.

Suppose that the second case holds. By Theorem 2.6, the weights in the isotropy submanifold $M^{\mathbb{Z}_d}$ at two fixed points that lie in the six-dimensional component are $\{a, b, -a-b\}$ and $\{-a, -b, a+b\}$ for some natural numbers $a$ and $b$. Moreover, $a$, $b$, and $a+b$ are multiples of $d$, which is impossible since $d$ is the largest weight.

Suppose that the third case holds. Let $Z$ be the component. Let $\dim Z = 2m$. Since all the weights in the isotropy submanifold $M^{\mathbb{Z}_d}$ are either $d$ or $-d$, by Theorem 2.1,

$$0 = \int_Z 1 = \frac{1}{\prod_{i=1}^m \pm d} + \frac{1}{\prod_{i=1}^m \pm d} + \frac{1}{\prod_{i=1}^m \pm d} = \pm\frac{1}{d^m} \mp \frac{1}{d^m} \pm \frac{1}{d^m} \neq 0,$$

which is a contradiction.

Hence the first case is the case and the weights in the isotropy submanifold $M^{\mathbb{Z}_d}$ at the two fixed points in the 2-sphere are $\{-d\}$ and $\{d\}$. □

Considering $S^1$ as a subset of $\mathbb{C}$, denote an $S^1$-action on a manifold $M$ by $\lambda \cdot p$ for $\lambda \in S^1$, $p \in M$. For technical reasons, throughout this paper we sometimes reverse the $S^1$-action. By reversing the action, we mean an $S^1$-action on $M$ by $\lambda^{-1} \cdot p$.

LEMMA 3.3. *Let the circle act symplectically on a $2n$-dimensional compact, connected symplectic manifold $M$ and suppose that there are exactly three fixed points. Then we can label the fixed points $p$, $q$, and $r$ so that $\lambda_p = n-2$, $\lambda_q = n$, and $\lambda_r = n+2$. Moreover, if $\dim M \neq 4$, then after possibly reversing the circle action, we may assume that $-d \in \Sigma_p$ and $d \in \Sigma_q$, where $d$ is the largest weight.*

*Proof.* Let $p$, $q$, and $r$ be the fixed points. Without loss of generality, assume that $\lambda_p \leq \lambda_q \leq \lambda_r$. By Corollary 2.3, $n$ is even. Also, since the number of fixed points is odd, $\lambda_q = n$ by Lemma 2.2. Moreover, since $M$ is connected, $\dim M \neq 0$.

First, assume that $\dim M = 4$. Then by Lemma 2.2, either $\lambda_p = \lambda_q = \lambda_r = 2$, or $\lambda_p = 0$, $\lambda_q = 2$, and $\lambda_r = 4$. Suppose that $\lambda_p = \lambda_q = \lambda_r = 2$. Then by Theorem 2.1,

$$0 = \int_M 1 = \frac{1}{\prod_{i=1}^2 \xi_p^i} + \frac{1}{\prod_{i=1}^2 \xi_q^i} + \frac{1}{\prod_{i=1}^2 \xi_r^i} < 0,$$

which is a contradiction. Hence $\lambda_p = 0$, $\lambda_q = 2$, and $\lambda_r = 4$.

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Next, assume that $\dim M \geq 8$. By Lemma 3.2, we can label the fixed points $\alpha$, $\beta$, and $\gamma$ such that $\alpha$ and $\beta$ lie in the same two-dimensional connected component of $M^{\mathbb{Z}_d}$ such that $-d \in \Sigma_\alpha$, $d \in \Sigma_\beta$, and $N_\gamma(d) = N_\gamma(-d) = 0$. By reversing the circle action if necessary, we may assume that either $\lambda_\alpha \leq \lambda_\gamma$ or $\lambda_\beta \leq \lambda_\gamma$. Moreover, by Corollary 2.7, the first Chern class map is identically zero. By Lemma 3.4 below, $\lambda_\alpha + 2 = \lambda_\beta$. Together with Lemma 2.2, the above statements imply that $\lambda_p = n-2$, $\lambda_q = n$, $\lambda_r = n+2$, $-d \in \Sigma_p$, and $d \in \Sigma_q$. □

To prove Lemma 3.3, we need the following technical lemma.

LEMMA 3.4. *Let the circle act on a $2n$-dimensional compact symplectic manifold $(M, \omega)$. Let $v$ and $w$ be fixed points in the same two-dimensional component $Z$ of $M^{\mathbb{Z}_d}$, where $d$ is the largest weight. Also suppose that $-d \in \Sigma_v$, $d \in \Sigma_w$, and $c_1(M)|_v = c_1(M)|_w$. Then $\lambda_v + 2 = \lambda_w$.*

*Proof.* By Lemma 2.5, $\Sigma_v \equiv \Sigma_w \bmod d$. Let $\xi_v^i$, $1 \leq i \leq n$, and $\xi_w^i$, $1 \leq i \leq n$, be the weights at $v$ and $w$, respectively, where $\xi_v^i, \xi_w^i \in \mathbb{Z} \setminus \{0\}$. By permuting if necessary, we can assume that $\xi_v^i \equiv \xi_w^i \bmod d$, for all $i < n$, $\xi_v^n = -d$, and $\xi_w^n = d$. By Lemma 2.4, $d > |\xi_v^i|$ and $d > |\xi_w^i|$, for $i < n$. Then for all $i < n$, the following statements hold:

(1) if $\xi_v^i > 0$ and $\xi_w^i > 0$, or if $\xi_v^i < 0$ and $\xi_w^i < 0$, then $\xi_v^i \equiv \xi_w^i \bmod d$ implies $\xi_v^i - \xi_w^i = 0$;
(2) if $\xi_v^i > 0$ and $\xi_w^i < 0$, then $\xi_v^i \equiv \xi_w^i \bmod d$ implies $\xi_v^i - \xi_w^i = d$;
(3) if $\xi_v^i < 0$ and $\xi_w^i > 0$, then $\xi_v^i \equiv \xi_w^i \bmod d$ implies $\xi_v^i - \xi_w^i = -d$.

Moreover, there are $(\lambda_v/2) - 1$ negative weights in $\Sigma_v$ excluding $-d$, and $\lambda_w/2$ negative weights in $\Sigma_w$. Hence,

$$\begin{aligned} 0 = c_1(M)|_v - c_1(M)|_w &= (\xi_v^1 + \cdots + \xi_v^{n-1} - d) - (\xi_w^1 + \cdots + \xi_w^{n-1} + d) \\ &= (\xi_v^1 - \xi_w^1) + \cdots + (\xi_v^{n-1} - \xi_w^{n-1}) - 2d = d\left(\frac{\lambda_w}{2} - \frac{\lambda_v}{2} + 1\right) - 2d \\ &= d\left(\frac{\lambda_w}{2} - \frac{\lambda_v}{2} - 1\right). \end{aligned}$$

Therefore, $\lambda_v + 2 = \lambda_w$. □

*Remark 3.5.* We can generalize Lemma 3.4 in the following way. Let the circle act on a $2n$-dimensional compact symplectic manifold $(M, \omega)$. Let $v$ and $w$ be fixed points in the same component $Z$ of $M^{\mathbb{Z}_d}$, where $d$ is the largest weight. Then

$$\lambda_v(M) - \lambda_w(M) + \lambda_v(Z) - \lambda_w(Z) = -\frac{2}{d}(c_1(M)|_v - c_1(M)|_w).$$

The proof goes similarly to that of Lemma 3.4; by Lemma 2.5, their weights are equal modulo $d$. Then as a bijection between $\Sigma_v$ and $\Sigma_w$ modulo $d$, $\pm d$ at $v$ is paired with $\pm d$ at $w$. Consider other weights; a positive weight $\xi$ at $v$ is paired with either $\xi$ or $\xi - d$ at $w$, etc. Finally, we consider the difference of the first Chern class at $v$ and $w$ together with indices at $v$ and $w$.

Finally, when the largest weight is odd, we will need the following closely related technical lemma.

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LEMMA 3.6. *Let the circle act on a $2n$-dimensional compact symplectic manifold $(M, \omega)$. Suppose that fixed points $v$ and $w$ satisfy the conditions in Lemma 3.4. Let $d$ be the largest weight and assume that $d$ is odd. Suppose that $\Sigma_v$ and $\Sigma_w$ have $E_v^+$ and $E_w^+$ positive even weights and $E_v^-$ and $E_w^-$ negative even weights, respectively. Then $E_v^+ - E_v^- - E_w^+ + E_w^- = 2$.*

*Proof.* By Lemma 2.5, $\Sigma_v \equiv \Sigma_w \bmod d$. Define $\xi_v^i$ and $\xi_w^i$ as in Lemma 3.4 and recall that the following statements hold:

(a) if $\xi_v^i > 0$ and $\xi_w^i > 0$, or if $\xi_v^i < 0$ and $\xi_w^i < 0$, then $\xi_v^i - \xi_w^i = 0$;
(b) if $\xi_v^i > 0$ and $\xi_w^i < 0$, then $\xi_v^i - \xi_w^i = d$;
(c) if $\xi_v^i < 0$ and $\xi_w^i > 0$, then $\xi_v^i - \xi_w^i = -d$.

Let $e_v^+$ be a positive even weight at $v$, $e_v^-$ a negative even weight at $v$, $o_v^+$ a positive odd weight at $v$, and $o_v^-$ a negative odd weight at $v$, and similarly for $w$. Then since the largest weight $d$ is odd, we have the following.

(1) $e_v^+ \equiv e_w^+ \bmod d$ implies that $e_v^+ = e_w^+$. Hence in $c_1(M)|_v - c_1(M)|_w$, this pair contributes 0. Suppose that there are $k_1$ such pairs.
(2) $e_v^+ \equiv o_w^- \bmod d$ implies that $e_v^+ - o_w^- = d$. Hence in $c_1(M)|_v - c_1(M)|_w$, this pair contributes $d$. There are $E_v^+ - k_1$ such pairs.
(3) $e_v^- \equiv e_w^- \bmod d$ implies that $e_v^- = e_w^-$. Hence in $c_1(M)|_v - c_1(M)|_w$, this pair contributes 0. Suppose that there are $k_2$ such pairs.
(4) $e_v^- \equiv o_w^+ \bmod d$ implies that $e_v^- - o_w^+ = -d$. Hence in $c_1(M)|_v - c_1(M)|_w$, this pair contributes $-d$. There are $E_v^- - k_2$ such pairs.
(5) $o_v^+ \equiv o_w^+ \bmod d$ implies that $o_v^+ = o_w^+$. Hence in $c_1(M)|_v - c_1(M)|_w$, this pair contributes 0. Suppose that there are $k_3$ such pairs.
(6) $o_v^+ \equiv e_w^- \bmod d$ implies that $o_v^+ - e_w^- = d$. Hence in $c_1(M)|_v - c_1(M)|_w$, this pair contributes $d$. There are $E_w^- - k_2$ such pairs.
(7) $o_v^- \equiv e_w^+ \bmod d$ implies that $o_v^- - e_w^+ = -d$. Hence in $c_1(M)|_v - c_1(M)|_w$, this pair contributes $-d$. There are $E_w^+ - k_1$ such pairs.
(8) $o_v^- \equiv o_w^- \bmod d$ implies that $o_v^- = o_w^-$. Hence in $c_1(M)|_v - c_1(M)|_w$, this pair contributes 0. Suppose that there are $k_4$ such pairs.

Then

$$\begin{aligned} 0 &= c_1(M)|_v - c_1(M)|_w \\ &= d(E_v^+ - k_1) - d(E_v^- - k_2) + d(E_w^- - k_2) - d(E_w^+ - k_1) - 2d \\ &= d(E_v^+ - E_v^- - E_w^+ + E_w^- - 2). \end{aligned}$$

□

*Remark 3.7.* We can also generalize Lemma 3.6 in the following way. Let the circle act on a $2n$-dimensional compact symplectic manifold $(M, \omega)$. Let $v$ and $w$ be fixed points in the same component $Z$ of $M^{\mathbb{Z}_d}$, where $d$ is the largest weight. Assume that the largest weight $d$ is odd. Then

$$d(E_v^+ - E_v^- - E_w^+ + E_w^-) = c_1(M)|_v - c_1(M)|_w - c_1(Z)|_v + c_1(Z)|_w.$$

The proof goes similarly to that of Lemma 3.6.

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4. *Proof of Theorem 1.1: the base case*

The proof of Theorem 1.1 is based on induction. The key fact to prove Theorem 1.1 is that an isotropy submanifold of a symplectic manifold is itself a smaller symplectic manifold. To prove the base case, we need several theorems.

PROPOSITION 4.1. [**MD88**] *An effective symplectic circle action on a four-dimensional compact, connected symplectic manifold is Hamiltonian if and only if the fixed point set is non-empty.*

Let the circle act symplectically on a four-dimensional compact, connected symplectic manifold $M$ with isolated fixed points. Assume that the action is effective. Then we can associate a graph to $M$ in the following way. We assign a vertex to each fixed point. Label each fixed point by its moment image. Additionally, given two fixed points $p$ and $q$, we say that $(p, q)$ is an edge if there exists $k > 1$ such that $p$ and $q$ are contained in the same component of the isotropy submanifold $M^{\mathbb{Z}_k}$, where $k$ is the largest such. We label the edge by $k$. We say that two graphs $\Pi$ and $\Pi'$ are isomorphic, if there is a one-to-one correspondence between vertices in $\Pi$ and vertices in $\Pi'$, and a one-to-one correspondence between edges in $\Pi$ and edges in $\Pi'$ such that if $\sigma : \Pi \longrightarrow \Pi'$ is such a map and if $(p, q)$ is a $k$-edge, then $\sigma((p, q))$ is a $k$-edge.

THEOREM 4.2. (Uniqueness theorem) [**K99**] *Let* $(M, \omega, \Pi)$ *and* $(M', \omega', \Pi')$ *be two compact four-dimensional Hamiltonian* $S^1$ *spaces. Then any isomorphism between their corresponding graphs is induced by an equivariant symplectomorphism.*

We now prove the base case.

PROPOSITION 4.3. *Let the circle act symplectically on a compact, connected symplectic manifold* $M$ *and suppose that there are exactly three fixed points. If* $\dim M < 8$*, then* $M$ *is equivariantly symplectomorphic to* $\mathbb{CP}^2$.

*Proof.* Suppose that $\dim M < 8$. By quotienting out by the subgroup which acts trivially, we may assume that the action is effective. Since the manifold $M$ is connected, $\dim M \neq 0$. Hence by Corollary 2.3, $\dim M = 4$. Let $p$, $q$, and $r$ denote the three fixed points and without loss of generality assume that $\lambda_p \leq \lambda_q \leq \lambda_r$. Then by Lemma 3.4, $\lambda_p = 0$, $\lambda_q = 2$, and $\lambda_r = 4$.

By a standard action on $\mathbb{CP}^2$ we mean that for each $\lambda \in S^1$, $\lambda$ acts on $\mathbb{CP}^2$ by

$$\lambda \cdot [z_0 : z_1 : z_2] = [\lambda^{a+b} z_0 : \lambda^a z_1 : z_2]$$

for some positive integers $a$ and $b$. This action has three fixed points $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$. And the weights at these points are $\{-b, -a - b\}$, $\{b, -a\}$, and $\{a, a + b\}$.

Since $\dim M = 4$, by Proposition 4.1, the action is Hamiltonian. Furthermore, by Lemma 2.4, there exist natural numbers $a$, $b$, and $c$ such that the weights are $\Sigma_p = \{a, c\}$, $\Sigma_q = \{-a, b\}$, and $\Sigma_r = \{-b, -c\}$. By Theorem 2.1,

$$0 = \int_M 1 = \frac{1}{ac} - \frac{1}{ab} + \frac{1}{bc} = \frac{b - c + a}{abc}.$$

Thus $c = a + b$. It is straightforward to check that the corresponding graph is isomorphic to a graph corresponding to some standard action on $\mathbb{CP}^2$ where the action is given

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by $\lambda \cdot [z_0 : z_1 : z_2] = [\lambda^{a+b} z_0 : \lambda^a z_1 : z_2]$, and hence this induces an equivariant symplectomorphism on manifolds by Theorem 4.2. □

Hence from now on we assume that $\dim M \geq 8$. Then note that, by Corollary 2.7, the Chern class map is identically zero.

LEMMA 4.4. *Fix a natural number $n$ such that $n \geq 4$. Assume that Theorem 1.1 holds for all manifolds $M$ such that $\dim M < 2n$. Let the circle act symplectically on a $2n$-dimensional compact, connected symplectic manifold $M$ and suppose that there are exactly three fixed points. Assume that the action is effective. Then there exist even natural numbers $a$ and $b$ such that the weights at the three fixed points in the isotropy submanifold $M^{\mathbb{Z}_2}$ are $\{a, c\}$, $\{-a, b\}$, and $\{-b, -c\}$, where $c = a + b$.*

*Proof.* Since the action is effective, the isotropy submanifold $\mathbb{Z}_2$ is a smaller manifold, i.e., for any component $Z$ of $M^{\mathbb{Z}_2}$, we have that $\dim Z < \dim M$. Then by the inductive hypothesis and Theorem 2.6, there are only four possible cases.

(1) Each fixed point is a component of the isotropy submanifold $M^{\mathbb{Z}_2}$.
(2) The isotropy submanifold $M^{\mathbb{Z}_2}$ contains a 2-sphere with two fixed points. The third fixed point is another component of $M^{\mathbb{Z}_2}$.
(3) The isotropy submanifold $M^{\mathbb{Z}_2}$ contains a four-dimensional component with the three fixed points.
(4) The isotropy submanifold $M^{\mathbb{Z}_2}$ contains a six-dimensional component with two fixed points. The third fixed point is another component of $M^{\mathbb{Z}_2}$.

Assume that the first case holds. Let $p$, $q$, and $r$ be the fixed points. The first case means that all the weights at $p$, $q$, and $r$ are odd. Let $A$, $B$, and $C$ be the products of the weights at $p$, $q$, and $r$, respectively. Then by Theorem 2.1,

$$\int_M 1 = \sum_{F \subset M^{S^1}} \int_F \frac{1}{e_{S^1}(N_F)} = \frac{1}{\prod \xi_p^i} t^{-n} + \frac{1}{\prod \xi_q^i} t^{-n} + \frac{1}{\prod \xi_r^i} t^{-n}$$
$$= \left( \frac{1}{A} + \frac{1}{B} + \frac{1}{C} \right) t^{-n} = 0.$$

Hence

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} = 0.$$

Multiplying both sides by $ABC$ yields

$$BC + AC + AB = 0.$$

However, since $A$, $B$, and $C$ are odd,

$$BC + AC + AB \equiv 1 \mod 2,$$

which is a contradiction.

Assume that the second case holds. Then the two fixed points in the 2-sphere have one even weight and $n - 1$ odd weights. By Corollary 2.3, $n$ is even. Then sums of the weights at these points are congruent to 1 mod 2, which contradicts Corollary 2.7 that the first Chern class map (the sum of the weights at a fixed point) is zero for all fixed points if $\dim M \geq 8$ and there are exactly three fixed points.

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The fourth case is similar. The two fixed points in the 2-sphere have three even weights and $n-3$ odd weights, hence the sums of weights at these points are congruent to 1 mod 2.

Hence the third case is the case. Thus as in the proof of Proposition 4.3, there are even natural numbers $a$ and $b$ such that the fixed points of the isotropy submanifold $M^{\mathbb{Z}_2}$ have weights $\{a+b, a\}$, $\{-a, b\}$, and $\{-b, -a-b\}$. □

LEMMA 4.5. *Fix a natural number $n$. Assume that Theorem 1.1 holds for all manifolds $M$ such that* $\dim M < 2n$. *Let the circle act symplectically on a $2n$-dimensional compact, connected symplectic manifold $M$ and suppose that there are exactly three fixed points. Assume that the action is effective. Given an integer $e \in \mathbb{Z} \setminus \{-1, 0, 1\}$, exactly one of the following statements holds.*

(1) *Each fixed point is a component of the isotropy submanifold $M^{\mathbb{Z}_e}$.*
(2) *The isotropy submanifold $M^{\mathbb{Z}_e}$ contains a 2-sphere with two fixed points, and the weights in the 2-sphere at these points are $\{a\}$ and $\{-a\}$ for some natural number $a$ that is a multiple of $e$. The third fixed point is another component of $M^{\mathbb{Z}_e}$.*
(3) *The isotropy submanifold $M^{\mathbb{Z}_e}$ contains a four-dimensional component with the three fixed points, and the weights in the isotropy submanifold at these points are $\{a+b, a\}$, $\{-a, b\}$, and $\{-b, -a-b\}$ for some natural numbers $a$ and $b$ that are multiples of $e$.*
(4) *The isotropy submanifold $M^{\mathbb{Z}_e}$ contains a six-dimensional component with two fixed points, and the weights in the isotropy submanifold at these points are $\{a, b, -a-b\}$ and $\{a+b, -a, -b\}$ for some natural numbers $a$ and $b$ that are multiples of $e$. The third fixed point is another component of $M^{\mathbb{Z}_e}$.*

*Proof.* Fix an integer $e \in \mathbb{Z} \setminus \{-1, 0, 1\}$. Since the action on $M$ is effective, for any component $Z$ of the isotropy submanifold $M^{\mathbb{Z}_e}$, we have that $\dim Z < \dim M$.

By ABBV localization (Theorem 2.1), if any component of the isotropy submanifold $M^{\mathbb{Z}_e}$ only contains one fixed point, then the fixed point itself is the component.

If every fixed point is itself a component of the isotropy submanifold $M^{\mathbb{Z}_e}$, this is the first case of the lemma.

Suppose instead that there exists a component $Z$ of the isotropy submanifold $M^{\mathbb{Z}_e}$ that contains exactly two fixed points. Then by Theorem 2.6, one of the following statements holds.

(a) The component is 2-sphere and the weights in the isotropy submanifold $M^{\mathbb{Z}_e}$ at these points are $\{a\}$ and $\{-a\}$ for some natural number $a$ that is a multiple of $e$. By the previous argument, the third fixed point is another component of $M^{\mathbb{Z}_e}$. This is the second case of the lemma.

(b) The component is six-dimensional and the weights in the isotropy submanifold $M^{\mathbb{Z}_e}$ at these points are $\{a, b, -a-b\}$ and $\{a+b, -a, -b\}$ for some natural numbers $a$ and $b$ that are multiples of $e$. The third fixed point is another component of $M^{\mathbb{Z}_e}$. This is the fourth case of the lemma.

Finally, suppose that a component of the isotropy submanifold $M^{\mathbb{Z}_e}$ contains the three fixed points. Then by the inductive hypothesis, the component is four-dimensional and the

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weights in the isotropy submanifold are $\{a+b, a\}$, $\{-a, b\}$, and $\{-b, -a-b\}$ for some natural numbers $a$ and $b$ that are multiples of $e$. This is the third case of the lemma. □

As particular cases of Lemma 4.5, we need the following lemma.

LEMMA 4.6. *Fix a natural number $n$. Assume that Theorem 1.1 holds for all manifolds $M$ such that* $\dim M < 2n$. *Let the circle act symplectically on a $2n$-dimensional compact, connected symplectic manifold $M$ and suppose that there are exactly three fixed points. Assume that the action is effective. Fix an integer $e \in \mathbb{Z} \setminus \{-1, 0, 1\}$.*

(1) *Suppose that there exist distinct fixed points $\alpha$ and $\beta$ such that $N_\alpha(e) > 0$ and $N_\beta(-e) > 0$ such that $|e| > d/2$ where $d$ is the largest weight. Then $N_\alpha(e) = 1$, $N_\beta(-e) = 1$, $\Sigma_\alpha \equiv \Sigma_\beta \bmod e$, and no additional multiples of $e$ appear as weights.*

(2) *If there exist two distinct fixed points $\alpha$ and $\beta$ such that $N_\alpha(e) > 0$ and $N_\beta(e) > 0$, then after possibly switching $\alpha$ and $\beta$,*

$$\{2e, e\} \subset \Sigma_\alpha, \quad \{-e, e\} \subset \Sigma_\beta, \quad \text{and} \quad \{-2e, -e\} \subset \Sigma_\gamma$$

*where $\gamma$ is the remaining fixed point. Moreover, no additional multiples of $e$ appear as weights.*

(3) *If there exists a fixed point $\alpha$ such that $N_\alpha(e) > 1$, $\{-2e, e, e\} \subset \Sigma_\alpha$ and $\{2e, -e, -e\} \subset \Sigma_\beta$ for some fixed point $\beta \neq \alpha$. Moreover, no additional multiples of $e$ appear as weights.*

(4) *Suppose that there exists a fixed point $\beta$ such that $N_\beta(e) > 0$ and $N_\beta(-e) > 0$. Then*

$$\{2e, e\} \subset \Sigma_\alpha, \quad \{-e, e\} \subset \Sigma_\beta, \quad \text{and} \quad \{-2e, -e\} \subset \Sigma_\gamma$$

*where $\alpha$ and $\gamma$ are the remaining two fixed points. Moreover, no additional multiples of $e$ appear as weights.*

*Proof.* Fix an integer $e \in \mathbb{Z} \setminus \{-1, 0, 1\}$ and consider the isotropy submanifold $M^{\mathbb{Z}_e}$.

(1) By looking at the weights in the isotropy submanifold $M^{\mathbb{Z}_e}$, the second, third, and fourth cases are possible. In the third or fourth case, $a \geq |e|$ and $b \geq |e|$ hence $a + b \geq 2|e| > d$, which is a contradiction. Hence this must be the second case of Lemma 4.5 with $a = |e|$. Moreover, $\alpha$ and $\beta$ lie in the same 2-sphere of $M^{\mathbb{Z}_e}$. Hence by Lemma 2.5, $\Sigma_\alpha \equiv \Sigma_\beta \bmod e$.

(2) By looking at the weights in the isotropy submanifold $M^{\mathbb{Z}_e}$, this must be the third case of Lemma 4.5 with $a = b = |e|$.

(3) By looking at the weights in the isotropy submanifold $M^{\mathbb{Z}_e}$, this must be the fourth case of Lemma 4.5 with $a = b = |e|$.

(4) By looking at the weights in the isotropy submanifold $M^{\mathbb{Z}_e}$, this must be the third case of Lemma 4.5 with $a = b = |e|$. □

## 5. *The case where the largest weight is odd*

Let the circle act symplectically on a compact, connected symplectic manifold $M$ with exactly three fixed points. In this section, we show that if $\dim M \geq 8$, the largest weight cannot be odd.

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PROPOSITION 5.1. *Fix a natural number $n$ such that $n \geq 4$. Assume that Theorem 1.1 holds for all manifolds $M$ such that* $\dim M < 2n$. *Let the circle act symplectically on a $2n$-dimensional compact, connected symplectic manifold $M$ and suppose that there are exactly three fixed points. Assume that the action is effective. Then the largest weight is even.*

*Proof.* Assume to the contrary that the largest weight is odd. Then this is an immediate consequence of Lemmas 5.2, 5.6, and 5.7 below. □

LEMMA 5.2. *Fix a natural number $n$ such that $n \geq 4$. Assume that Theorem 1.1 holds for all manifolds $M$ such that* $\dim M < 2n$. *Let the circle act symplectically on a $2n$-dimensional compact, connected symplectic manifold $M$ and suppose that there are exactly three fixed points $p$, $q$, and $r$, with $\lambda_p \leq \lambda_q \leq \lambda_r$. Assume that the action is effective and the largest weight $d$ is odd. Then after possibly reversing the circle action we may assume that $-d \in \Sigma_p$ and $d \in \Sigma_q$, and there exist even natural numbers $a$ and $b$ such that either*
(1) $\{a, c\} \subset \Sigma_p$, $\{-a, b\} \subset \Sigma_q$, *and* $\{-b, -c\} \subset \Sigma_r$; *or*
(2) $\{-a, b\} \subset \Sigma_p$, $\{-b, -c\} \subset \Sigma_q$, *and* $\{a, c\} \subset \Sigma_r$,
*where $c = a + b$. Moreover, these are the only even weights.*

*Proof.* By Lemma 3.2, $N_p(d) + N_q(d) + N_r(d) = 1$ and $N_p(-d) + N_q(-d) + N_r(-d) = 1$. By Lemma 3.3, $\lambda_p = n - 2$, $\lambda_q = n$, and $\lambda_r = n + 2$. Moreover, after possibly reversing the circle action, we may assume that $-d \in \Sigma_p$ and $d \in \Sigma_q$.

By Lemma 4.4, there exist even natural numbers $a$ and $b$ such that the weights at the three fixed points in the isotropy submanifold $M^{\mathbb{Z}_2}$ are $\{a, c\}$, $\{-a, b\}$, and $\{-b, -c\}$, where $c = a + b$. In the lemma, the order is not specified. We have six possible cases. The other four cases are:
(3) $\{a, c\} \subset \Sigma_p$, $\{-b, -c\} \subset \Sigma_q$, and $\{-a, b\} \subset \Sigma_r$;
(4) $\{-a, b\} \subset \Sigma_p$, $\{a, c\} \subset \Sigma_q$, and $\{-b, -c\} \subset \Sigma_r$;
(5) $\{-b, -c\} \subset \Sigma_p$, $\{-a, b\} \subset \Sigma_q$, and $\{a, c\} \subset \Sigma_r$;
(6) $\{-b, -c\} \subset \Sigma_p$, $\{a, c\} \subset \Sigma_q$, and $\{-a, b\} \subset \Sigma_r$.
The fixed points $p$ and $q$ satisfy the conditions in Lemma 3.6. Therefore, $E_v^+ - E_v^- - E_w^+ + E_w^- = 2$ and the other cases are ruled out. □

LEMMA 5.3. *Under the assumption of Lemma 5.2, $a \neq b$.*

*Proof.* Assume to the contrary that $a = b$. Since $-d \in \Sigma_p$ and $d \in \Sigma_q$ where $d$ is the largest weight, by Lemma 4.6(1) for $d$, $\Sigma_p \equiv \Sigma_q \bmod d$. As a result, we can find a bijection between the weights at $p$ and the weights at $q$ that takes each weight $\alpha$ at $p$ to a weight $\beta$ at $q$ such that $\alpha \equiv \beta \bmod d$. Moreover, since $a = b$, we can take this bijection to take $a$ at $p$ to $b$ at $q$ in the first case, and $-a$ at $p$ to $-b$ at $q$ in the second case.

Assume that the first case in Lemma 5.2 holds, i.e., $\{a, c\} \subset \Sigma_p$, $\{-a, b\} \subset \Sigma_q$, and $\{-b, -c\} \subset \Sigma_r$. Moreover, these are the only even weights. First, $-d$ at $p$ has to go to $d$ at $q$ since all the other weights are non-zero and have absolute values less than $d$. Next, $c$ at $p$ must go to $c - d$ at $q$ and $-a$ at $q$ must go to $d - a$ at $p$. If $l$ is any remaining positive odd weight at $p$, then it has to go to $l$ at $q$ since the largest weight $d$ is odd. Similarly, any negative odd weight $-k$ at $p$ must go to $-k$ at $q$.

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By Corollary 2.3, $\frac{1}{2}\dim M$ is even. Since $\lambda_p = \frac{1}{2}\dim M - 2$ and $\lambda_q = \frac{1}{2}\dim M$ by Lemma 3.3, this implies that the weights at $p$ and $q$ are

$$\Sigma_p = \{-d, a, c, d-a\} \cup \{x_i\}_{i=1}^t \cup \{-y_i\}_{i=1}^t,$$
$$\Sigma_q = \{d, b, c-d, -a\} \cup \{x_i\}_{i=1}^t \cup \{-y_i\}_{i=1}^t,$$

for some odd natural numbers $x_i$ and $y_i$ where $\dim M = 8 + 4t$, for some $t \geq 0$.

Suppose that $x_i > 1$ for some $i$. Then by Lemma 4.6(2), $\{2x_i, x_i\} \subset \Sigma_p$, $\{-x_i, x_i\} \subset \Sigma_q$, and $\{-2x_i, -x_i\} \subset \Sigma_r$, or $\{-x_i, x_i\} \subset \Sigma_p$, $\{2x_i, x_i\} \subset \Sigma_q$, and $\{-2x_i, -x_i\} \subset \Sigma_r$. Moreover, no more multiples of $x_i$ should appear as weights. This implies that $-x_i \neq -y_j$ for all $j$. Since $-y_j$ are the only negative odd weights at $p$, this implies that the second case is impossible. Assume that the first case holds. Then we must have $c = 2x_i$. Also, since $\{-x_i, x_i\} \subset \Sigma_q$ but $-x_i \neq -y_j$ for all $j$, $-x_i = c - d$. However, this means that $2x_i - d = c - d = -x_i$ hence $d = 3x_i$, which is a contradiction since no more multiples of $x_i$ should appear.

Hence $x_i = 1$ for all $i$. Similarly, one can show that $y_i = 1$ for all $i$. Then $c_1(M)|_p = -d + a + c + d - a = c > 0$, which by Corollary 2.7 contradicts that the first Chern class map is identically zero.

Similarly, we get a contradiction of the second case of Lemma 5.2 with $a = b$ by a slight variation of this argument. □

LEMMA 5.4. *Assume that the first case in Lemma 5.2 holds. Then the weights are*

$$\Sigma_p = \{-d, a, c, d-a, b-d, 1\} \cup \{-1, 1\}_{i=1}^t,$$
$$\Sigma_q = \{d, a-d, c-d, -a, b, 1\} \cup \{-1, 1\}_{i=1}^t,$$
$$\Sigma_r = \{-b, -c, \ldots\},$$

*where the largest weight $d$ is odd, $a$, $b$, and $c$ are even natural numbers such that $c = a + b$, and $\dim M = 12 + 4t$ for some $t \geq 0$. Moreover, $a \neq b$ and the remaining weights at $r$ are odd.*

*Proof.* Assume that the first case in Lemma 5.2 holds, i.e., there exist even natural numbers $a$, $b$, and $c$ such that $\{a, c\} \subset \Sigma_p$, $\{-a, b\} \subset \Sigma_q$, and $\{-b, -c\} \subset \Sigma_r$ where $c = a + b$. Moreover, these are the only even weights.

As in the proof of Lemma 5.3, combining the fact that $\Sigma_p \equiv \Sigma_q \bmod d$ by Lemma 4.6(1) for $d$, $\frac{1}{2}\dim M$ is even by Corollary 2.3, $\lambda_p = \frac{1}{2}\dim M - 2$ and $\lambda_q = \frac{1}{2}\dim M$ by Lemma 3.3, one can show that

$$\Sigma_p = \{-d, a, c, d-a, b-d\} \cup \{x_i\}_{i=1}^{t+1} \cup \{-y_i\}_{i=1}^t,$$
$$\Sigma_q = \{d, a-d, c-d, -a, b\} \cup \{x_i\}_{i=1}^{t+1} \cup \{-y_i\}_{i=1}^t,$$

for some odd natural numbers $x_i$ and $y_i$ where $\dim M = 12 + 4t$, for some $t \geq 0$. Moreover, with a slight variation of the argument as in the proof of Lemma 5.3, one can show that $x_i = y_i = 1$ for all $i$ by using Lemma 4.6(2) for $x_i$ and $y_i$, respectively. □

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LEMMA 5.5. *Assume that the second case in Lemma 5.2 holds. Then the weights are*

$$\begin{aligned}\Sigma_p &= \{-d, -a, b, d-b, d-c, 1\} \cup \{-1, 1\}_{i=1}^{t},\\ \Sigma_q &= \{d, -b, -c, d-a, b-d, 1\} \cup \{-1, 1\}_{i=1}^{t},\\ \Sigma_r &= \{a, c, \ldots\},\end{aligned}$$

*where the largest weight $d$ is odd, $a$, $b$, and $c$ are even natural numbers such that $c = a + b$, and* $\dim M = 12 + 4t$ *for some $t \geq 0$. Moreover, $a \neq b$ and the remaining weights at $r$ are odd.*

*Proof.* The proof is analogous to that of Lemma 5.4. □

LEMMA 5.6. *The first case in Lemma 5.2 is not possible.*

*Proof.* By Lemma 5.4, the weights are

$$\begin{aligned}\Sigma_p &= \{-d, a, c, d-a, b-d, 1\} \cup \{-1, 1\}_{i=1}^{t},\\ \Sigma_q &= \{d, a-d, c-d, -a, b, 1\} \cup \{-1, 1\}_{i=1}^{t},\\ \Sigma_r &= \{-b, -c, \ldots\},\end{aligned}$$

where the largest weight $d$ is odd, $a$, $b$, and $c$ are even natural numbers such that $c = a + b$, and $\dim M = 12 + 4t$ for some $t \geq 0$. Moreover, $a \neq b$ and the remaining weights at $r$ are odd.

We consider Lemma 2.4 for each integer. Lemma 2.4 holds for $d$, $a$, $b$, and $c$. Since $d > c > b$, $b - d < -1$. Since $a \neq b$, by Lemma 2.4 for $b - d$, it is straightforward to show that $d - b \in \Sigma_r$.

First, suppose that $c - d \neq -1$. Then $N_p(1) = N_p(-1) + 1$ and $N_q(1) = N_q(-1) + 1$. Hence by Lemma 2.4 for 1, $N_r(1) + 2 = N_r(-1)$. Considering Lemma 2.4 for each integer, one can show that

$$\Sigma_r = \{-b, -c, d-b, -1, d-c, -1\} \cup \{-e_i, e_i\}_{i=1}^{t}$$

for some odd natural numbers $e_i$. By Lemma 4.6(4) for $e_i$, one can show that $e_i = 1$ for all $i$.

Second, suppose that $c - d = -1$. Then $N_p(1) = N_p(-1) + 1$ and $N_q(1) = N_q(-1)$. Hence by Lemma 2.4 for 1, $N_r(1) + 1 = N_r(-1)$. Considering Lemma 2.4 for each integer, one can show that

$$\Sigma_r = \{-b, -c, d-b, -1\} \cup \{-e_i, e_i\}_{i=1}^{t+1}$$

for some odd natural numbers $e_i$. Again, $e_i = 1$ for all $i$.

Hence in either case the weights are

$$\begin{aligned}\Sigma_p &= \{-d, a, c, d-a, b-d, 1\} \cup \{-1, 1\}^{t},\\ \Sigma_q &= \{d, -a, b, a-d, c-d, 1\} \cup \{-1, 1\}^{t},\\ \Sigma_r &= \{-b, -c, d-b, -1, d-c, -1\} \cup \{-1, 1\}^{t}.\end{aligned}$$

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Moreover, since $c_1(M)|_p = 0$ by Corollary 2.7, we have that $-d + a + c + d - a + b - d + 1 = 0$. Therefore, $d = c + b + 1$. Let $A = c_n(M)|_p = \prod \xi_p^j$, $B = c_n(M)|_q = \prod \xi_q^j$, and $C = c_n(M)|_r = \prod \xi_r^j$. Then

$$\begin{aligned}(-1)^{t+1}(B + C) &= dab(d - a)(d - c) - bc(d - b)(d - c)\\ &= b(d - c)\{da(d - a) - c(d - b)\}\\ &= b(d - c)\{(c + b + 1)a(c + b + 1 - a) - c(c + b + 1 - b)\}\\ &= b(d - c)\{(c + b + 1)a(2b + 1) - c(c + 1)\}\\ &= b(d - c)\{(a + 2b + 1)a(2b + 1) - (a + b)(a + b + 1)\}\\ &= b(d - c)\{(a^2 + 2ab + a)(2b + 1) - (a^2 + 2ab + b^2 + a + b)\}\\ &= b(d - c)\{2a^2b + 4ab^2 + 2ab + a^2 + 2ab\\ &\qquad + a - (a^2 + 2ab + b^2 + a + b)\}\\ &= b(d - c)\{(2a^2b - a^2) + (4ab^2 - b^2)\\ &\qquad + (2ab - 2ab) + (a^2 - a) + (2ab - b) + a\} > 0.\end{aligned}$$

Hence $(-1)^{t+1}B > -(-1)^{t+1}C > 0$, i.e., $(-1)^{t+1}((1/B) + (1/C)) < 0$. We also have that $(-1)^{t+1}1/A < 0$. Then, by Theorem 2.1,

$$0 = (-1)^{t+1}\int_M 1 = (-1)^{t+1}\left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C}\right) < 0,$$

which is a contradiction. □

LEMMA 5.7. *The second case in Lemma 5.2 is not possible.*

*Proof.* By Lemma 5.5, the weights in this case are

$$\begin{aligned}\Sigma_p &= \{-d, -a, b, d - b, d - c, 1\} \cup \{-1, 1\}_{i=1}^t,\\ \Sigma_q &= \{d, -b, -c, d - a, b - d, 1\} \cup \{-1, 1\}_{i=1}^t,\\ \Sigma_r &= \{a, c, \ldots\},\end{aligned}$$

where the largest weight $d$ is odd, $a$, $b$, and $c$ are even natural numbers such that $c = a + b$, and $\dim M = 12 + 4t$ for some $t \geq 0$.

Let $A = c_n(M)|_p = \prod \xi_p^j$, $B = c_n(M)|_q = \prod \xi_q^j$, and $C = c_n(M)|_r = \prod \xi_r^j$. Then

$$\begin{aligned}(-1)^{t+1}(B + A) &= dbc(d - a)(d - b) - dab(d - b)(d - c)\\ &= db(d - b)\{c(d - a) - a(d - c)\} > 0,\end{aligned}$$

since $c > a$ and $d - a > d - c$. Hence it follows that $(-1)^{t+1}((1/A) + (1/B)) < 0$. Since $\lambda_r = \frac{1}{2}\dim M + 2$, we also have that $(-1)^{t+1}1/C < 0$. Then, by Theorem 2.1,

$$0 = (-1)^{t+1}\int_M 1 = (-1)^{t+1}\left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C}\right) < 0,$$

which is a contradiction. □

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6. *Preliminaries for the case where the largest weight is even, part I*

Let the circle act symplectically on a compact, connected symplectic manifold $M$ with exactly three fixed points. Also assume that $\dim M \geq 8$ and the largest weight is even. The main idea to prove Theorem 1.1 is to rule out manifolds such that $\dim M \geq 8$. In this section, we investigate properties that the manifold $M$ should satisfy, if it exists.

PROPOSITION 6.1. *Fix a natural number $n$ such that $n \geq 4$. Assume that Theorem 1.1 holds for all manifolds $M$ such that $\dim M < 2n$. Let the circle act symplectically on a $2n$-dimensional compact, connected symplectic manifold $M$ and suppose that there are exactly three fixed points. Assume that the action is effective. Then the largest weight is odd.*

*Proof.* Assume to the contrary that the largest weight is even. Then this is an immediate consequence of Lemmas 6.2, 7.1, and 8.1–8.11 below. □

LEMMA 6.2. *Fix a natural number $n$ such that $n \geq 4$. Assume that Theorem 1.1 holds for all manifolds $M$ such that $\dim M < 2n$. Let the circle act symplectically on a $2n$-dimensional compact, connected symplectic manifold $M$ and suppose that there are exactly three fixed points $p$, $q$, and $r$, with $\lambda_p \leq \lambda_q \leq \lambda_r$. Assume that the action is effective and the largest weight $c$ is even. Then after possibly reversing the circle action we may assume that the weights are*

$$\begin{aligned}
\Sigma_p &= \{-c, -b\} \cup \{x_i\}_{i=1}^{t+3} \cup \{-y_i\}_{i=1}^{t} \cup \{1\} \cup \{-1, 1\}_{i=1}^{s}, \\
\Sigma_q &= \{c, a\} \cup \{x_i - c\}_{i=1}^{t+3} \cup \{c - y_i\}_{i=1}^{t} \cup \{1\} \cup \{-1, 1\}_{i=1}^{s}, \\
\Sigma_r &= \{-a, b, \ldots\},
\end{aligned}$$

*for some $s \geq 0$ and $t \geq 0$ such that $\dim M = 2n = 12 + 4t + 4s$, where $a$ and $b$ are even natural numbers such that $c = a + b$, and $x_i$ and $y_i$ are odd natural numbers for all $i$. Moreover, the remaining weights at $r$ are odd.*

*Proof.* Let $c$ be the largest weight. By Lemma 3.2, $N_p(c) + N_q(c) + N_r(c) = 1$ and $N_p(-c) + N_q(-c) + N_r(-c) = 1$. By Lemma 3.3, $\lambda_p = n - 2$, $\lambda_q = n$, and $\lambda_r = n + 2$. Moreover, after possibly reversing the circle action, we may assume that $-c \in \Sigma_p$ and $c \in \Sigma_q$.

By Lemma 4.4, there exist even natural numbers $a$ and $b$ such that the weights at the three fixed points in the isotropy submanifold $M^{\mathbb{Z}_2}$ are $\{a, d\}$, $\{-a, b\}$, and $\{-b, -d\}$, where $d = a + b$. In Lemma 4.4, the order is not specified. However, since we can assume without loss of generality that $-c \in \Sigma_p$ and $c \in \Sigma_q$, we can assume that $d = c$, hence $\{-c, -b\} \subset \Sigma_p$, $\{c, a\} \subset \Sigma_q$, and $\{-a, b\} \subset \Sigma_r$. Moreover, these are the only even weights.

Next, by Lemma 4.6(1) for $c$, $\Sigma_p \equiv \Sigma_q \bmod c$. As a result, we can find a bijection between weights at $p$ and weights at $q$ that takes each weight $\alpha$ at $p$ to a weight $\beta$ at $q$ such that $\alpha \equiv \beta \bmod c$.

First, $-c$ at $p$ has to go to $c$ at $q$ since all the other weights are non-zero and have absolute values less than $c$. Second, $-b$ at $p$ must go to $a$ at $q$. Next, if $l$ is any positive odd weight at $p$, then it goes to either $l$ or $l - c$ at $q$. Suppose that there are $t_0$ positive

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odd weights at $p$ that go to negative odd weights at $q$. Then since $\lambda_p = n-2$ and there is no positive even weight at $p$, there are $n/2+1-t_0$ positive odd weights $p$ that go to positive odd weights at $q$. Similarly, if $-k$ is any negative odd weight at $p$, it has to go to either $-k$ or $c-k$ at $q$. Suppose that there are $t_1$ negative odd weights at $p$ that go to positive odd weights at $q$. On the other hand, since $\lambda_q = n$ and $q$ has two positive even weights, the number of positive odd weights at $q$ that go to positive odd weights at $p$ is equal to $n/2-2-t_1$. Hence $n/2+1-t_0 = n/2-2-t_1$, i.e., $t_0 = t_1+3$. Let $t = t_1$ and $s = n/2-t-3$. By Corollary 2.3, $\frac{1}{2}\dim M$ is even. This implies that the weights are

$$\begin{aligned}
\Sigma_p &= \{-c, -b\} \cup \{x_i\}_{i=1}^{t+3} \cup \{-y_i\}_{i=1}^{t} \cup \{e_i\}_{i=1}^{s+1} \cup \{-f_i\}_{i=1}^{s},\\
\Sigma_q &= \{c, a\} \cup \{x_i - c\}_{i=1}^{t+3} \cup \{c-y_i\}_{i=1}^{t} \cup \{e_i\}_{i=1}^{s+1} \cup \{-f_i\}_{i=1}^{s},\\
\Sigma_r &= \{-a, b, \ldots\},
\end{aligned}$$

for some odd natural numbers $x_i$, $y_i$, $e_i$, and $f_i$, where $\dim M = 2n = 12+4t+4s$, for some $t \geq 0$ and $s \geq 0$.

Next, we show that $e_i = f_i = 1$ for all $i$.

First, assume to the contrary that $e_i > 1$ for some $i$. Denote $e = e_i$. Then by Lemma 4.6(3) for $e$, either $\{2e, e\} \subset \Sigma_p$, $\{-e, e\} \subset \Sigma_q$, and $\{-2e, -e\} \subset \Sigma_r$, or $\{-e, e\} \subset \Sigma_p$, $\{2e, e\} \subset \Sigma_q$, and $\{-2e, -e\} \subset \Sigma_r$. Moreover, no additional multiples of $e$ should appear as weights. Since $\{-2e, -e\} \subset \Sigma_r$ in either case, the only possibility is that $a = 2e$. Therefore, the latter is the case. Thus, we have that $\{-e, e\} \subset \Sigma_p$. Therefore, $-e = -y_i$ for some $i$ or $-e = -f_i$ for some $i$. Since no additional multiples of $e$ should appear as weights at $q$, $-f_i \neq -e$ for all $i$. Hence, $-y_i = -e$ for some $i$. Without loss of generality, let $y_1 = e$. Moreover, since no additional multiples of $e$ should appear as weights, $b \neq 2e$. In particular, $a = 2e \neq b$. Since $2e = a < a+b = c$, $e < c/2$. Thus $c-e > c/2$.

Next, we show that $e-c \in \Sigma_r$. We have that $c-e = c-y_1 \in \Sigma_q$. By Lemma 2.4 for $c-e$, either $e-c \in \Sigma_p$, $e-c \in \Sigma_q$, or $e-c \in \Sigma_r$.

(a) $e-c \notin \Sigma_p$. Suppose that $e-c \in \Sigma_p$. Since $e-c$ is odd, either $e-c = -y_i$ for some $i$ or $e-c = -f_i$ for some $i$.

First, assume that $e-c = -y_i$ for some $i$. If $e-c = -y_1$, this implies that $e-c = -e$ hence $c = 2e$, which is a contradiction. Hence if $e-c = -y_i$ for some $i$, $i \neq 1$. Without loss of generality, let $e-c = -y_2$. Then we have that $\{-y_1, -y_2\} = \{e-c, e-c\} \subset \Sigma_p$. Then by Lemma 4.6(3) for $e-c$, $2(c-e) \in \Sigma_p$, which is a contradiction since $2(c-e) > c$.

Second, assume that $e-c = -f_i$ for some $i$. Then we have that $e-c = -f_i \in \Sigma_p$ and $e-c = -f_i \in \Sigma_q$. Then by Lemma 4.6(2) for $c-e$, $2(c-e) \in \Sigma_r$, which is a contradiction since $2(c-e) > c$. Hence $e-c \neq \Sigma_p$.

(b) $e-c \notin \Sigma_q$. Suppose that $e-c \in \Sigma_q$. Then we have that $\{c-e, e-c\} = \{c-y_1, e-c\} \subset \Sigma_q$. Hence by Lemma 4.6(4) for $c-e$, either $2(c-e) \in \Sigma_p$ or $2(c-e) \in \Sigma_r$. However, $2(c-e) > c$, which is a contradiction.

Therefore $e-c \in \Sigma_r$. Then by Lemma 4.6(1) for $c-e$, $\Sigma_q \equiv \Sigma_r \bmod c-e$. Consider $\{c, e\} \subset \Sigma_q$. We have that $c \notin \Sigma_r$ and $e \notin \Sigma_r$. Also, $e-(c-e) = 2e-c = a-a-b = -b$, but $-b \notin \Sigma_r$ since $-b$ is a negative even integer and $-a$ is the only

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negative even weight in $\Sigma_r$, but $a \neq b$. Since $|e + k(c-e)| > c$ for $k < -2$ or $k > 1$, $\Sigma_q \equiv \Sigma_r \bmod c - e$ and $\{c, e\} \subset \Sigma_q$ imply that $N_r(e - 2(c-e)) = N_r(3e - 2c) = 2$. Then by Lemma 4.6(3) for $3e - 2c$, $2(2c - 3e) \in \Sigma_r$, which is a contradiction since $2(2c - 3e) = 4c - 6e = c + 3c - 6e = c + 3a + 3b - 6e = c + 6e + 3b - 6e = c + 3b > c$, where $c$ is the largest weight. Therefore, $e_i = 1$ for all $i$.

Second, assume to the contrary that $f_i > 1$ for some $i$. Denote $f = f_i$. Then by Lemma 4.6(3) for $f$, either $\{-2f, -f\} \subset \Sigma_p$, $\{-f, f\} \subset \Sigma_q$, and $\{2f, f\} \subset \Sigma_r$, or $\{-f, f\} \subset \Sigma_p$, $\{-2f, -f\} \subset \Sigma_q$, and $\{2f, f\} \subset \Sigma_r$. Moreover, no additional multiples of $f$ should appear as weights. Since $\{2f, f\} \subset \Sigma_r$ in either case, the only possibility is that $b = 2f$. Therefore, the former is the case. Thus, we have that $\{-f, f\} \subset \Sigma_q$. Therefore, $f = c - y_i$ for some $i$. Without loss of generality, let $c - y_1 = f$. Moreover, since no additional multiples of $f$ should appear as weights, $a \neq 2f$. In particular, $b = 2f \neq a$. Since $2f = b < a + b = c$, $f < c/2$. Thus $c - f > c/2$.

Next, we show that $c - f \in \Sigma_r$. We have that $f - c = -y_1 \in \Sigma_p$. By Lemma 2.4 for $c - f$, either $c - f \in \Sigma_p$, $c - f \in \Sigma_q$, or $c - f \in \Sigma_r$.

(a) $c - f \notin \Sigma_p$. Suppose that $c - f \in \Sigma_p$. Then we have that $\{c - f, f - c\} = \{c - f, -y_1\} \subset \Sigma_p$. Then by Lemma 2.4(4) for $c - f$, either $2(c - f) \in \Sigma_q$ or $2(c - f) \in \Sigma_r$, which is a contradiction since $2(c - f) > c$.

(b) $c - f \notin \Sigma_q$. Suppose that $c - f \in \Sigma_q$. Then $c - f = c - y_i$ for some $i$. If $c - f = c - y_1$, then $c - f = c - y_1 = f$ hence $c = 2f < c$, which is a contradiction. Next, suppose that $c - f = c - y_i$ for some $i \neq 1$. Then we have that $\{f - c, f - c\} = \{-y_1, -y_i\} \subset \Sigma_p$. Hence, by Lemma 4.6(3) for $f - c$, $2(c - f) \in \Sigma_p$, which is a contradiction since $2(c - f) > c$.

Therefore $c - f \in \Sigma_r$. We also have that $f - c = -y_1 \in \Sigma_p$. Then by Lemma 4.6(1) for $c - f$, $\Sigma_p \equiv \Sigma_r \bmod c - f$. Consider $\{-c, -f\} \subset \Sigma_p$. We have that $-c \notin \Sigma_r$ and $-f \notin \Sigma_r$. Also, $-f + (c - f) = c - 2f = a + b - 2f = a + 2f - 2f = a$, but $a \notin \Sigma_r$ since $a$ is a positive even integer and $b$ is the only positive even weight in $\Sigma_r$, but $a \neq b$. Since $|-f + k(c-f)| > c$ for $k < -1$ or $k > 2$, $\Sigma_p \equiv \Sigma_r \bmod c - f$ and $\{-c, -f\} \subset \Sigma_p$ imply that $N_r(-f + 2(c-f)) = N_r(2c - 3f) = 2$. Then by Lemma 4.6(3) for $2c - 3f$, $-2(2c - 3f) \in \Sigma_r$, which is a contradiction since $-2(2c - 3f) = -4c + 6f = -c - 3c + 6f = -c - 3a - 3b + 6f = -c - 3a - 6f + 6f = -c - 3a < -c$, where $-c$ is the smallest weight. Therefore, $f_i = 1$ for all $i$. □

LEMMA 6.3. *In Lemma 6.2, $x_i \neq c - y_j$, for all $i$ and $j$.*

*Proof.* This follows from Lemma 4.6. □

LEMMA 6.4. *In Lemma 6.2, if $x_i = c - x_j$ for $i \neq j$, then $2x_i = 2x_j = c$. Also, if $x_i = c - x_i$ for some $i$, then $2x_i = 2x_j = c$ for some $j \neq i$. Moreover, there could be at most one such pair $(x_i, x_j)$ for $i \neq j$ such that $x_i = c - x_j$.*

*Proof.* This follows from Lemma 4.6. □

LEMMA 6.5. *In Lemma 6.2, $y_i \neq c - y_j$, if $i \neq j$.*

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*Proof.* Suppose not. First, assume that $y_i = c - y_j$ and $y_i = y_j$ for some $i \neq j$ , i.e., $2y_i = 2y_j = c$. Then $\{-2y_i, -y_i, -y_i\} = \{-c, -y_i, -y_j\} \subset \Sigma_p$, which contradicts Lemma 4.6(3) for $y_i$.

Second, assume that $y_i = c - y_j$ and $y_i \neq y_j$ for some $i \neq j$. Without loss of generality assume that $y_1 = c - y_2$ and $y_1 \neq y_2$. We can also assume that $y_1 > c/2 > y_2$. Then we have that $-y_1 \in \Sigma_p$ and $c - y_2 = y_1 \in \Sigma_q$. Hence by Lemma 4.6(1) for $y_1$, $\Sigma_p \equiv \Sigma_q \bmod y_1$.

First, we can choose a bijection between $\Sigma_p$ and $\Sigma_q$ so that

$$\Sigma_p \supset \{-y_1, 1\} \cup \{-1, 1\}^s \equiv \{c - y_2 = y_1, 1\} \cup \{-1, 1\}^s \subset \Sigma_q \mod y_1.$$

Then we are left with

$$\begin{aligned}\Sigma_p &\supset \{-c, -b\} \cup \{x_i\}_{i=1}^{t+3} \cup \{-y_i\}_{i=2}^{t} \\ &\equiv \{c, a\} \cup \{x_i - c\}_{i=1}^{t+3} \cup \{c - y_1 = y_2\} \cup \{c - y_i\}_{i=3}^{t} \subset \Sigma_q \mod y_1.\end{aligned}$$

Without loss of generality, let $x_1 \leq x_2 \leq \cdots \leq x_{t+3}$ and $-y_3 \leq -y_4 \leq \cdots \leq -y_t$, i.e.,

$$\begin{aligned}&\{-c, -b, -y_2, -y_3 \leq -y_4 \leq \cdots \leq -y_t < 0 < x_1 \leq x_2 \leq \cdots \leq x_{t+3}\} \\ &\quad \equiv \{x_1 - c \leq \cdots \leq x_{t+3} - c < 0 < c - y_3 \leq \cdots \leq c - y_t, c, a, c - y_2\} \mod x_1.\end{aligned}$$

Recall that $y_1$ is odd and $2y_1 > c$ where $c$ is the largest weight.

Consider $x_1 \in \Sigma_p$. If $x_1 \equiv c - y_i \bmod y_1$ for some $i \neq 1$ and 2, then $x_1 = c - y_i$, which contradicts Lemma 6.3. If $x_1 \equiv x_i - c$ for some $i$, $x_1 - 2y_1 = x_i - c$, which is a contradiction since $x_1 - 2y_1 < x_i - c$ for all $i$. If $x_1 \equiv c \bmod y_1$, then $x_1 + y_1 = c$, which contradicts Lemma 6.3. Therefore, $x_1 \equiv a \bmod y_1$.

Next, consider $-y_t \in \Sigma_p$. If $-y_t \equiv c - y_i$ for some $i \neq 1$ and 2, then $2y_1 - y_t = c - y_i$, which is a contradiction since $2y_1 - y_t > c - y_i$ for all $i$. If $-y_t \equiv x_i - c \bmod y_1$ for some $i$, then $-y_t = x_i - c$, which contradicts Lemma 6.3. Therefore, we have that either $-y_t \equiv c \bmod y_1$ or $-y_t \equiv c - y_2 \bmod y_1$.

Suppose that $-y_t \equiv c \bmod y_1$. This means that $-y_t + 3y_1 = c$. Then we have that $-y_t = c - 3y_1 = y_1 + y_2 - 3y_1 = y_2 - 2y_1 < y_1 - 2y_1 = -y_1 < -y_2$, hence $-y_t < -y_2$. Next, we consider $-y_2 \in \Sigma_p$. Using the same argument for $-y_t$, we have that $-y_2 \in \Sigma_p$ is congruent to no element in $\Sigma_q$ modulo $y_1$, which is a contradiction.

Next, suppose that $-y_t \equiv c - y_2 \bmod y_1$. This means that $2y_1 - y_t = c - y_2$. Then we have that $2y_1 - y_t = c - y_2 = y_1 + y_2 - y_2 = y_1$, hence $y_1 = y_t$. Hence, we have $\{-y_1, -y_1\} = \{-y_1, -y_t\} \subset \Sigma_p$. Then by Lemma 4.6(3) for $-y_1$, $2y_1 \in \Sigma_p$, which is a contradiction since $c < 2y_1$ where $c$ is the largest weight. □

LEMMA 6.6. *Fix a natural number $e$ such that $e \neq c/2$. In Lemma 6.2, at most one of $x_i$, $y_i$, $c - x_i$, and $c - y_i$ can be $e$.*

*Proof.* This follows easily from Lemmas 4.6, 6.3, and 6.4. □

LEMMA 6.7. *In Lemma 6.2, $x_i \neq y_j$ for all $i$ and $j$.*

*Proof.* Assume to the contrary that $x_i = y_j$ for some $i$ and $j$. Then by Lemma 6.6, $x_i = y_j = c/2$. Hence, we have $\{-2x_i, x_i, -x_i\} = \{-c, x_i, -y_j\} \subset \Sigma_p$, which contradicts Lemma 4.6(4) for $x_i$. □

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LEMMA 6.8. *In Lemma 6.2, assume that* $\{-f, f\} \subset \Sigma_r$ *for some natural number* $f$. *If* $f > 1$, *then* $c = 2f$, $\{-c = -2f, -f\} \subset \Sigma_p$, $\{2f = c, f\} \subset \Sigma_q$, *and* $\{-f, f\} \subset \Sigma_r$. *Moreover, no additional multiples of* $f$ *should appear as weights.*

*Proof.* This follows easily from Lemma 4.6(4) for $f$. □

LEMMA 6.9. *In Lemma 6.2, if* $N_p(1) > N_r(1)$ *and* $N_q(1) > N_r(1)$, *then* $N_p(1) < N_r(1) + 3$ *or* $N_q(1) < N_r(1) + 3$. *Similarly, if* $N_p(-1) > N_r(-1)$ *and* $N_q(-1) > N_r(-1)$, *then* $N_p(-1) < N_r(-1) + 3$ *or* $N_q(-1) < N_r(-1) + 3$.

*Proof.* First we prove the former. To do so, suppose that $N_p(1) \geq N_r(1) + 3$ and $N_q(1) \geq N_r(1) + 3$. There are three cases:

First, assume that $a > c/2$. Since $a \in \Sigma_q$ and $-a \in \Sigma_r$, by Lemma 4.6(1) for $a$, $\Sigma_q \equiv \Sigma_r \bmod a$. With $N_q(1) \geq N_r(1) + 3$, this implies that $N_r(1 + a) \geq 2$ or $N_r(1 - a) \geq 2$, since $|1 + ka| > c$ for $|k| \geq 2$. If $N_r(1 + a) \geq 2$, $-2(1 + a) \in \Sigma_r$ by Lemma 4.6(3) for $1 + a$, but $2(1 + a) > c$, which is a contradiction. If $N_r(1 - a) \geq 2$, $2(a - 1) \in \Sigma_r$ by Lemma 4.6(3) for $1 - a$. However, $2(a - 1) \geq c$ but $c \notin \Sigma_r$.

Second, assume that $a < c/2$. Then $b = c - a > c/2$. Since $-b \in \Sigma_p$ and $b \in \Sigma_r$, by Lemma 4.6(1) for $b$, $\Sigma_p \equiv \Sigma_r \bmod b$. With $N_p(1) \geq N_r(1) + 3$, this implies that $N_r(1 + b) \geq 2$ or $N_r(1 - b) \geq 2$, since $|1 + kb| > c$ for $|k| \geq 2$. If $N_r(1 + b) \geq 2$, $-2(1 + b) \in \Sigma_r$ by Lemma 4.6(3) for $1 + b$. However, $2(1 + b) > c$, which is a contradiction. If $N_r(1 - b) \geq 2$, $2(b - 1) \in \Sigma_r$ by Lemma 4.6(3) for $1 - b$. However, $2(b - 1) \geq c$ but $c \notin \Sigma_r$.

Finally, suppose that $a = c/2$. We have $b = c - a = c/2$. Then the isotropy submanifold $M^{\mathbb{Z}_a}$ must be the third case of Lemma 4.5. This means that the three fixed points lie in the same component of $M^{\mathbb{Z}_a}$, hence $\Sigma_p \equiv \Sigma_q \equiv \Sigma_r \bmod a$ by Lemma 2.5. With $N_q(1) \geq N_r(1) + 3$, $\Sigma_q \equiv \Sigma_r \bmod a$ implies that $N_r(1 + a) \geq 2$, $N_r(1 - a) \geq 2$, $N_r(1 - 2a) \geq 2$, or $N_r(1 + a) = N_r(1 - a) = N_r(1 - 2a) = 1$, since $|1 + ka| > c$ for $k \neq -2, -1, 0$, and 1.

(a) $N_r(1 + a) \geq 2$. Since $N_r(1 + a) \geq 2$, $-2(1 + a) \in \Sigma_r$ by Lemma 4.6(3) for $1 + a$. However, $-2(1 + a) < -c$ where $-c$ is the smallest weight, which is a contradiction.

(b) $N_r(1 - 2a) \geq 2$. By Lemma 4.6(3) for $1 - 2a$, $2(2a - 1) \in \Sigma_r$. However, $2(2a - 1) = 2(c - 1) > c$ where $c$ is the largest weight, which is a contradiction.

(c) $N_r(1 - a) \geq 2$. By Lemma 4.6(3) for $1 - a$, $N_r(1 - a) = 2$ and $2(a - 1) \in \Sigma_r$. Since $b$ is the only positive even weight at $r$, this means that $2(a - 1) = b$. Hence $a = b = 2$ and $c = a + b = 4$. Then the weights at $p$ and $q$ are

$$\Sigma_p = \{-4, -2\} \cup \{x_i\}_{i=1}^{t+3} \cup \{-y_i\}_{i=1}^{t} \cup \{1\} \cup \{-1, 1\}^s,$$
$$\Sigma_q = \{4, 2\} \cup \{x_i - 4\}_{i=1}^{t+3} \cup \{4 - y_i\}_{i=1}^{t} \cup \{1\} \cup \{-1, 1\}^s.$$

Since $c = 4$ is the largest weight, all of $x_i$ and $y_i$ are either 1 or 3. If at least two of $x_i$ are 3, $N_p(3) \geq 2$ hence $-6 \in \Sigma_p$ by Lemma 4.6(3) for 3, which is a contradiction since $-4$ is the smallest weight. If at most one of $x_i$ is 1, then at least two $x_i - c$ are $-3$. This means that $N_q(-3) \geq 2$ and hence $6 \in \Sigma_q$ by Lemma 4.6(3) for $-3$, which is a contradiction since 4 is the largest weight.

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(d) $N_r(1+a) = N_r(1-a) = N_r(1-2a) = 1$. Since $1-2a \in \Sigma_r$, by Lemma 2.4 for $1-2a$, there must be a weight of $2a-1$ for some fixed point. If it is $r$, then $\{2a-1, 1-2a\} \subset \Sigma_r$. Hence $2(2a-1) \in \Sigma_p$ or $2(2a-1) \in \Sigma_q$ by Lemma 4.6(4) for $2a-1$, which is a contradiction since $2(2a-1) = 2(c-1) > c$ where $c$ is the largest weight. Hence either $2a-1 \in \Sigma_p$ or $2a-1 \in \Sigma_q$. Suppose that $2a-1 \in \Sigma_p$. Then by Lemma 4.6(1) for $2a-1$, $\Sigma_p \equiv \Sigma_r \bmod 2a-1$. That $N_p(1) \geq N_r(1)+3$ implies that $N_r(2-2a) \geq 3$ since $|1+k(2a-1)| \geq c$ for $k \neq 0$ and $-1$, and the fixed point $r$ does not have a weight of $c$. However, $r$ has only one negative even weight $-a$, which is a contradiction. Similarly, $2a-1 \in \Sigma_q$ is also impossible.

With a slight variation of this argument, one can prove the latter statement of the lemma. □

## 7. *Preliminaries for the case where the largest weight is even, part II*

Let the circle act symplectically on a compact, connected symplectic manifold $M$ with exactly three fixed points. Also, assume that $\dim M \geq 8$ and the largest weight is even. In this section, for technical reasons, we consider $w = \min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\}$ and rewrite the weights in terms of $w$. And then we further investigate properties that the manifold $M$ should satisfy in terms of $w$, if such a manifold exists.

LEMMA 7.1. *Let $w = \min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\}$. In Lemma 6.2, the weights are*

$$\begin{aligned}
\Sigma_p &= \{-c, -b\} \cup \{x_i\}_{i=1}^{t+3} \cup \{-y_i\}_{i=1}^{t} \cup \{1\} \cup \{-1, 1\}^{v+w},\\
\Sigma_q &= \{c, a\} \cup \{x_i - c\}_{i=1}^{t+3} \cup \{c - y_i\}_{i=1}^{t} \cup \{1\} \cup \{-1, 1\}^{v+w},\\
\Sigma_r &= \{-a, b, \ldots\} \cup \{-1, 1\}^{w},
\end{aligned}$$

*for some $t \geq 0$ and $v \geq 0$, where $a$, $b$, and $c$ are even natural numbers such that $c = a+b$ is the largest weight, and $x_i$ and $y_i$ are odd natural numbers for all $i$. Moreover, the remaining weights at $r$ are odd.*

*Proof.* In Lemma 6.2, the weights are

$$\begin{aligned}
\Sigma_p &= \{-c, -b\} \cup \{x_i\}_{i=1}^{t+3} \cup \{-y_i\}_{i=1}^{t} \cup \{1\} \cup \{-1, 1\}^{s},\\
\Sigma_q &= \{c, a\} \cup \{x_i - c\}_{i=1}^{t+3} \cup \{c - y_i\}_{i=1}^{t} \cup \{1\} \cup \{-1, 1\}^{s},\\
\Sigma_r &= \{-a, b, \ldots\},
\end{aligned}$$

for some $s \geq 0$ and $t \geq 0$ such that $\dim M = 2n = 12 + 4t + 4s$, where $a$, $b$, and $c$ are even natural numbers such that $c = a+b$ is the largest weight, and $x_i$ and $y_i$ are odd natural numbers for all $i$. Moreover, the remaining weights at $r$ are odd.

Let $w = \min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\}$. We rewrite the weights in terms of $w$. We show that $\{-c, -b\} \cup \{x_i\}_{i=1}^{t+3} \cup \{-y_i\}_{i=1}^{t} \cup \{1\}$ in $\Sigma_p$ and $\{c, a\} \cup \{x_i - c\}_{i=1}^{t+3} \cup \{c - y_i\}_{i=1}^{t} \cup \{1\}$ in $\Sigma_q$ do not contribute to $w$, i.e.,

$$\begin{aligned}
\{-1, 1\} \nsubseteq (&\{-c, -b\} \cup \{x_i\}_{i=1}^{t+3} \cup \{-y_i\}_{i=1}^{t} \cup \{1\})\\
&\cap (\{c, a\} \cup \{x_i - c\}_{i=1}^{t+3} \cup \{c - y_i\}_{i=1}^{t} \cup \{1\}).
\end{aligned}$$

First, $a$, $b$, and $c$ are even natural numbers. Second, by Lemma 6.6, at most one of $c - x_i$ or $y_i$ can be 1.

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Suppose that $y_i = 1$ for some $i$. Then $c - x_j \neq 1$ for all $j$ by Lemma 6.6. Hence in $\Sigma_q$, $\{-1, 1\} \not\subseteq (\{c, a\} \cup \{x_i - c\}_{i=1}^{t+3} \cup \{c - y_i\}_{i=1}^{t} \cup \{1\})$. The case $c - x_i = 1$ for some $i$ is analogous.

Finally, if $y_i \neq 1$ and $c - x_i \neq 1$ for all $i$, then $\{-1, 1\} \not\subseteq (\{-c, -b\} \cup \{x_i\}_{i=1}^{t+3} \cup \{-y_i\}_{i=1}^{t} \cup \{1\})$ in $\Sigma_p$ and $\{-1, 1\} \not\subseteq (\{c, a\} \cup \{x_i - c\}_{i=1}^{t+3} \cup \{c - y_i\}_{i=1}^{t} \cup \{1\})$ in $\Sigma_q$. □

LEMMA 7.2. *In Lemma 7.1, for each $x_i$, either $x_i = c - x_j$ for some $j$ or $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$.*

*Proof.* By Lemma 2.4, for each $x_i$, either $-x_i \in \Sigma_p$, $-x_i \in \Sigma_q$, or $-x_i \in \Sigma_r$.

First, assume that $x_i > 1$. By Lemma 6.6, $x_i \neq y_j$ for all $j$. Hence $-x_i \notin \Sigma_p$. Next, if $-x_i \in \Sigma_q$, then $-x_i = x_j - c$ for some $j$. If $-x_i \in \Sigma_r$, $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$.

Second, assume that $x_i = 1$. By Lemma 6.6, at most one of $c - x_i$ and $y_i$ can be 1. Hence, either $N_p(1) > N_p(-1)$ and $N_q(1) \geq N_q(-1)$, or $N_p(1) \geq N_p(-1)$ and $N_q(1) > N_q(-1)$. By Lemma 2.4, this implies that $N_r(1) < N_r(-1)$. Therefore, $-x_i = -1 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. □

LEMMA 7.3. *In Lemma 7.1, for each $c - y_i$, $y_i - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$.*

*Proof.* By Lemma 2.4, for each $c - y_i$, either $y_i - c \in \Sigma_p$, $y_i - c \in \Sigma_q$, or $y_i - c \in \Sigma_r$.

First, assume that $c - y_i > 1$. Suppose that $c - y_i \in \Sigma_q$. Then $c - y_i = c - x_j$ for some $j$, which is a contradiction since $c - y_i \neq c - x_j$ for all $j$ by Lemma 6.7. Next, suppose that $y_i - c \in \Sigma_p$. Then $y_i - c = -y_j$ for some $j$. By Lemma 6.5, $i = j$, i.e., $c = 2y_i$. Hence, $\{-2y_i, -y_i\} = \{-c, -y_i\} \subset \Sigma_p$ and $\{2y_i, y_i\} = \{c, c - y_i\} \subset \Sigma_q$. The isotropy submanifold $M^{\mathbb{Z}_{y_i}}$ must be the third case of Lemma 4.5. Therefore, we have that $\{-y_i, y_i\} \subset \Sigma_r$. In particular, $y_i - c = -y_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$, since $c - y_i = y_i = c/2 \geq 2$ and $c - y_i$ is odd.

Second, assume that $c - y_i = 1$. By Lemma 6.6, at most one of $c - x_i$ or $y_i$ can be 1. Hence, either $N_p(1) > N_p(-1)$ and $N_q(1) \geq N_q(-1)$, or $N_p(1) \geq N_p(-1)$ and $N_q(1) > N_q(-1)$. By Lemma 2.4, this implies that $N_r(1) < N_r(-1)$. Therefore, $y_i - c = -1 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. □

LEMMA 7.4. *In Lemma 7.1, suppose that $c - x_i \neq 1$. Then either $c - x_i = x_j$ for some $j$ or $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$.*

*Proof.* This follows easily from Lemma 2.4 for $c - x_i$ and Lemma 6.7. □

LEMMA 7.5. *In Lemma 7.1, suppose that $y_i \neq 1$. Then $y_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$.*

*Proof.* Suppose that $y_i \neq 1$. By Lemma 2.4, for each $y_i$, either $y_i \in \Sigma_p$, $y_i \in \Sigma_q$, or $y_i \in \Sigma_r$. From Lemma 6.7, it follows that $y_i \notin \Sigma_p$. Next, assume that $y_i \in \Sigma_q$. Then $y_i = c - y_j$ for some $j$. By Lemma 6.5, $i = j$, i.e., $c = 2y_i$. Hence, $\{-2y_i, -y_i\} = \{-c, -y_i\} \subset \Sigma_p$ and $\{2y_i, y_i\} = \{c, c - y_i\} \subset \Sigma_q$. The isotropy submanifold $M^{\mathbb{Z}_{y_i}}$ must be the third case of Lemma 4.5. Therefore, we have that $\{-y_i, y_i\} \subset \Sigma_r$. In particular, $y_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$, since $y_i = c/2 \geq 2$ and $y_i$ is odd. □

LEMMA 7.6. *In Lemma 7.1, if $x_i \neq c - x_j$ for all $i$ and $j$, then $t < v$.*

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*Proof.* Assume to the contrary that $x_i \neq c - x_j$ for all $i$ and $j$, and $t \geq v$. By Lemma 7.2, $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$. Also, by Lemma 7.3, $y_i - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$. By Lemmas 6.5 and 6.6, it follows that $x_i \neq x_j$ and $y_i \neq y_j$ for $i \neq j$.

First, suppose that $c - x_i = 1$ for some $i$. Without loss of generality, let $c - x_1 = 1$. Then by Lemma 6.6, $c - x_i \neq 1$ for $i \neq 1$ and $y_j \neq 1$ for all $j$. Hence $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i \neq 1$ by Lemma 7.4. Also, $y_j \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $j$ by Lemma 7.5. Also, by Lemma 6.3, $x_i \neq c - y_j$ for all $i$ and $j$. Therefore, we have that $\{-x_i\}_{i=1}^{t+3} \cup \{c - x_i\}_{i=2}^{t+3} \cup \{y_i\}_{i=1}^{t} \cup \{c - y_i\}_{i=1}^{t} \subset \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$, which is a contradiction since $|\{-x_i\}_{i=1}^{t+3} \cup \{c - x_i\}_{i=2}^{t+3} \cup \{y_i\}_{i=1}^{t} \cup \{c - y_i\}_{i=1}^{t}| = 4t + 5$ and $|\Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)| = 2t + 4 + 2u + 2v$, but $4t + 5 = 2t + 2t + 5 > 2t + 4 + 2v$. The cases where $y_i = 1$ for some $i$ and where $c - x_i \neq 1$ and $y_i \neq 1$ for all $i$ are similar. □

## 8. *The case where the largest weight is even*

Let the circle act symplectically on a compact, connected symplectic manifold $M$ with exactly three fixed points. In this section, we show that if $\dim M \geq 8$, the largest weight cannot be even. We proceed by elimination. In Lemma 7.1, we have the following cases:

(1) $t = 0$ and $v = 0$;
(2) $t = 0$ and $v = 1$;
(3) $t = 0$ and $v = 2$;
(4) $t = 1$ and $v = 0$;
(5) $t = 1$ and $v = 1$;
(6) $t = 1$ and $v = 2$;
(7) $t = 2$ and $v = 1$;
(8) $t = 2$ and $v = 2$;
(9) $t = 3$ and $v = 2$;
(10) $t \geq 2 + v$;
(11) $v \geq 3$.

LEMMA 8.1. *In Lemma 7.1, $t = 0$ and $v = 0$ are impossible.*

*Proof.* The weights in this case are

$$\Sigma_p = \{-c, -b, x_1, x_2, x_3, 1\} \cup \{-1, 1\}^w,$$
$$\Sigma_q = \{c, a, x_1 - c, x_2 - c, x_3 - c, 1\} \cup \{-1, 1\}^w,$$
$$\Sigma_r = \{-a, b, \ldots\} \cup \{-1, 1\}^w,$$

where $w = \min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\}$, $a$, $b$, and $c$ are even natural numbers such that $c = a + b$ is the largest weight, and $x_i$ are odd natural numbers for all $i$. Moreover, the remaining weights at $r$ are odd.

By Lemma 7.6, $x_i = c - x_j$ for some $i$ and $j$. Then by Lemma 6.4, there exist $x_i$ and $x_j$ where $i \neq j$ such that $2x_i = 2x_j = c$. Without loss of generality, let $2x_1 = 2x_2 = c$. Lemma 6.4 also implies that $x_3 \neq c - x_i$ for all $i$. Therefore, $-x_3 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. Note that $x_1 = c - x_1 = c/2 \geq 2$.

First, suppose that $c - x_3 = 1$. Then we have that $x_3 = c - 1 > 1$. Hence, $N_p(1) = N_p(-1) + 1 = w + 1$ and $N_q(1) = N_q(-1) = w + 1$. Therefore, $N_r(1) + 1 = N_r(-1)$ by

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Lemma 2.4 for 1. Considering Lemma 2.4 for each integer, one can show that the weights are

$$\begin{aligned}\Sigma_p &= \{-c, -b, x_1, x_1, x_3, 1\} \cup \{-1, 1\}^w,\\ \Sigma_q &= \{c, a, -x_1, -x_1, -1, 1\} \cup \{-1, 1\}^w,\\ \Sigma_r &= \{-a, b, -x_3, f, -f, -1\} \cup \{-1, 1\}^w,\end{aligned}$$

for some odd natural number $f$. If $f > 1$, by Lemma 6.8, we have that $c = 2x_1 = 2f$, which is a contradiction since no additional multiples of $x_1$ should appear by Lemma 4.6(3) for $x_1$. Hence $f = 1$.

Second, suppose that $c - x_3 \neq 1$. Then by Lemma 7.4, $c - x_3 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. Also, $N_p(1) \geq N_p(-1) + 1 = w + 1$ and $N_q(1) = N_q(-1) + 1 = w + 1$. Therefore, $N_r(1) + 2 \leq N_r(-1)$ by Lemma 2.4 for 1. Considering Lemma 2.4 for each integer, one can show that the weights are

$$\begin{aligned}\Sigma_p &= \{-c, -b, x_1, x_1, x_3, 1\} \cup \{-1, 1\}^w,\\ \Sigma_q &= \{c, a, -x_1, -x_1, x_3 - c, 1\} \cup \{-1, 1\}^w,\\ \Sigma_r &= \{-a, b, -x_3, c - x_3, -1, -1\} \cup \{-1, 1\}^w.\end{aligned}$$

Therefore, in either case the weights are

$$\begin{aligned}\Sigma_p &= \{-c, -b, x_1, x_1, x_3, 1\} \cup \{-1, 1\}^w,\\ \Sigma_q &= \{c, a, -x_1, -x_1, x_3 - c, 1\} \cup \{-1, 1\}^w,\\ \Sigma_r &= \{-a, b, -x_3, c - x_3, -1, -1\} \cup \{-1, 1\}^w.\end{aligned}$$

Let $A = c_n(M)|_q = \prod \xi_p^j$, $B = c_n(M)|_q = \prod \xi_q^j$, and $C = c_n(M)|_r = \prod \xi_r^j$. First, $2x_1 = c = a + b > b$. Since $c = a + b \geq 2 + 2 = 4$, $x_1 = c/2 \geq 2$. Therefore, $c = 2x_1 > x_3$ implies that $x_1^2 > x_3$. Then

$$(-1)^w \frac{-B - C}{a(c - x_3)} = cx_1^2 - bx_3 > 0,$$

and this implies that $(-1)^{w+1}B > (-1)^w C$. Also, we have that $(-1)^w A > 0$. Then, by Theorem 2.1,

$$0 = (-1)^w \int_M 1 = (-1)^w \left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C}\right) > 0,$$

which is a contradiction. □

LEMMA 8.2. *In Lemma 7.1, $t = 0$ and $v = 1$ are impossible.*

*Proof.* The weights in this case are

$$\begin{aligned}\Sigma_p &= \{-c, -b, x_1, x_2, x_3, 1\} \cup \{-1, 1\}^{w+1},\\ \Sigma_q &= \{c, a, x_1 - c, x_2 - c, x_3 - c, 1\} \cup \{-1, 1\}^{w+1},\\ \Sigma_r &= \{-a, b, \ldots\} \cup \{-1, 1\}^w,\end{aligned}$$

where $a$, $b$, $c$, $w$, $x_i$ are as in Lemma 7.1, and the remaining weights at $r$ are odd.

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(1) $x_i \neq c - x_j$ for all $i$ and $j$. By Lemma 7.2, $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$. From Lemma 6.6 it follows that $x_i \neq x_j$ for $i \neq j$.

First, assume that $c - x_i \neq 1$ for all $i$. Then by Lemma 7.4, $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$. Then

$$\Sigma_r = \{-a, b, -x_1, -x_2, -x_3, c - x_1, c - x_2, c - x_3\} \cup \{-1, 1\}^w$$

and hence $\lambda_r = \frac{1}{2} \dim M$, which contradicts $\lambda_r = \frac{1}{2} \dim M + 2$ in Lemma 3.3.

Next, assume that $c - x_i = 1$ for some $i$. Without loss of generality, let $c - x_3 = 1$. Then by Lemma 6.6, $c - x_i \neq 1$ for $i \neq 3$. By Lemma 7.4, $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i \neq 3$. Also, by the assumption, $x_i \neq 1$ for $i \neq 3$. Then $N_p(1) = N_p(-1) + 1 = w + 2$ and $N_q(1) = N_q(-1) = w + 2$. Therefore, $N_r(1) + 1 = N_r(-1)$ by Lemma 2.4 for 1. Then

$$\Sigma_r = \{-a, b, -x_1, -x_2, 1 - c, c - x_1, c - x_2, -1\} \cup \{-1, 1\}^w.$$

By Lemma 4.6(1) for $x_3 = c - 1$, $\Sigma_p \equiv \Sigma_r \bmod c - 1$. $N_r(1) = w$. Then $N_p(1) \geq w + 2$, $N_r(1) = w$, and $\Sigma_p \equiv \Sigma_r \bmod c - 1$ imply that $N_r(2 - c) \geq 2$ since $|1 + k(c - 1)| > c$ for $|k| \geq 2$ and $c \notin \Sigma_r$. However, $r$ has only one negative even weight, which is a contradiction.

(2) $x_i = c - x_j$ for some $i$ and $j$. By Lemma 6.4, there exist $x_i$ and $x_j$ where $i \neq j$ such that $2x_i = 2x_j = c$. Without loss of generality, let $2x_1 = 2x_2 = c$. Lemma 6.4 also implies that $x_3 \neq c - x_i$ for all $i$. Therefore, by Lemma 7.2, $-x_3 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. Note that $x_1 = c - x_1 = c/2 \geq 2$.

First, assume that $c - x_3 = 1$. Then $N_p(1) = N_p(-1) + 1 = w + 2$ and $N_q(1) = N_q(-1) = w + 2$. Therefore, $N_r(1) + 1 = N_r(-1)$ by Lemma 2.4 for 1. Considering Lemma 2.4 for each integer, one can show that

$$\Sigma_r = \{-a, b, 1 - c, -1, -f, f, -g, g\} \cup \{-1, 1\}^w,$$

for some odd natural numbers $f$ and $g$. From Lemma 6.8 it follows that $f = g = 1$, which contradicts the definition of $w$.

Second, assume that $c - x_3 \neq 1$. By Lemma 7.4, $c - x_3 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. Then $N_p(1) \geq N_p(-1) + 1 = w + 2$ and $N_q(1) = N_q(-1) + 1 = w + 2$. Therefore, $N_r(1) + 2 \leq N_r(-1)$ by Lemma 2.4 for 1. Similarly, we consider Lemma 2.4 for each integer and have

$$\Sigma_r = \{-a, b, -x_3, c - x_3, -1, -1, -f, f\} \cup \{-1, 1\}^w.$$

As above, $f = 1$ and we get a contradiction. □

LEMMA 8.3. *In Lemma 7.1, $t = 0$ and $v = 2$ are impossible.*

*Proof.* The weights in this case are

$$\begin{aligned}
\Sigma_p &= \{-c, -b, x_1, x_2, x_3, 1\} \cup \{-1, 1\}^{w+2}, \\
\Sigma_q &= \{c, a, x_1 - c, x_2 - c, x_3 - c, 1\} \cup \{-1, 1\}^{w+2}, \\
\Sigma_r &= \{-a, b, \ldots\} \cup \{-1, 1\}^w,
\end{aligned}$$

where $a$, $b$, $c$, $w$, $x_i$ are as in Lemma 7.1, and the remaining weights at $r$ are odd.

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(1) $x_i \neq c - x_j$ for all $i$ and $j$. By Lemma 7.2, $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$. By Lemma 6.6, it follows that $x_i \neq x_j$ for $i \neq j$.

First, assume that $c - x_i \neq 1$ for all $i$. Then by Lemma 7.4, $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$. Also, $N_p(1) \geq N_p(-1) + 1 = w + 3$ and $N_q(1) = N_q(-1) + 1 = w + 3$. Therefore, $N_r(1) + 2 \leq N_r(-1)$ by Lemma 2.4 for 1. Considering Lemma 2.4 for each integer, one can show that

$$\Sigma_r = \{-a, b, -x_1, -x_2, -x_3, c - x_1, c - x_2, c - x_3, -1, -1\} \cup \{-1, 1\}^w.$$

Then we have that $N_p(1) \geq w + 3$, $N_q(1) \geq w + 3$, and $N_r(1) = w$, which contradict Lemma 6.9.

Next, assume that $c - x_i = 1$ for some $i$. Without loss of generality, let $c - x_3 = 1$. Then by Lemma 6.6, $c - x_i \neq 1$ for $i \neq 3$. Hence, by Lemma 7.4, $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i \neq 3$. Also, $N_p(1) \geq N_p(-1) + 1 = w + 3$ and $N_q(1) = N_q(-1) = w + 3$. Therefore, $N_r(1) + 1 \leq N_r(-1)$ by Lemma 2.4 for 1. Considering Lemma 2.4 for each integer, one can show that

$$\Sigma_r = \{-a, b, -x_1, -x_2, 1 - c, c - x_1, c - x_2, -1, -f, f\} \cup \{-1, 1\}^w,$$

for some odd natural number $f$. From Lemma 6.8 it follows that $f = 1$, which contradicts the definition of $w$.

(2) $x_i = c - x_j$ for some $i$ and $j$. By Lemma 6.4, there exist $x_i$ and $x_j$ where $i \neq j$ such that $2x_i = 2x_j = c$. Without loss of generality, let $2x_1 = 2x_2 = c$. Lemma 6.4 also implies that $x_3 \neq c - x_i$ for all $i$. Therefore, $-x_3 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. Note that $x_1 = c - x_1 = c/2 \geq 2$.

First, assume that $c - x_3 = 1$. Then we have that $N_p(1) = N_p(-1) + 1 = w + 3$ and $N_q(1) = N_q(-1) = w + 3$. Therefore, $N_r(1) + 1 = N_r(-1)$ by Lemma 2.4 for 1. Considering Lemma 2.4 for each integer, one can show that

$$\Sigma_r = \{-a, b, 1 - c, -1, -f, f, -h, h, -k, k\} \cup \{-1, 1\}^w,$$

for some odd natural numbers $f$, $h$, and $k$. As above, $f = 1$ and we get a contradiction.

Second, assume that $c - x_3 \neq 1$. By Lemma 7.4, $c - x_3 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. Also, $N_p(1) \geq N_p(-1) + 1 = w + 3$ and $N_q(1) = N_q(-1) + 1 = w + 3$. Therefore, $N_r(1) + 2 \leq N_r(-1)$ by Lemma 2.4 for 1. Considering Lemma 2.4 for each integer, one can show that

$$\Sigma_r = \{-a, b, -x_3, c - x_3, -1, -1, -f, f, -h, h\} \cup \{-1, 1\}^w,$$

for some odd natural numbers $f$ and $h$. As above, $f = 1$ and we get a contradiction. □

LEMMA 8.4. *In Lemma 7.1, $t = 1$ and $v = 0$ are impossible.*

*Proof.* The weights in this case are

$$\Sigma_p = \{-c, -b, x_1, x_2, x_3, x_4, -y, 1\} \cup \{-1, 1\}^w,$$
$$\Sigma_q = \{c, a, x_1 - c, x_2 - c, x_3 - c, x_4 - c, c - y, 1\} \cup \{-1, 1\}^w,$$
$$\Sigma_r = \{-a, b, \ldots\} \cup \{-1, 1\}^w,$$

where $a$, $b$, $c$, $w$, $x_i$, $y$ are as in Lemma 7.1, and the remaining weights at $r$ are odd.

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By Lemma 7.6, $x_i = c - x_j$ for some $i$ and $j$. By Lemma 6.4, there exist $x_i$ and $x_j$ where $i \neq j$ such that $2x_i = 2x_j = c$. Without loss of generality, let $2x_1 = 2x_2 = c$. Lemma 6.4 also implies that $x_i \neq c - x_j$ for $i \neq 1$ and 2, and for all $j$. Therefore, $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i = 3$ and 4 by Lemma 7.2. Also, by Lemma 7.3, $y - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. Moreover, by Lemma 4.6(3) for $x$, none of $x_i$, $y$, $c - x_i$ and $c - y$ can be $x$ for $i \neq 1$ and 2. Hence, by Lemma 6.6, all of $x_i$, $y$, $c - x_i$ and $c - y$ are different for $i \neq 1$ and 2.

First, suppose that $c - x_i \neq 1$ for all $i$ and $y \neq 1$. By Lemma 7.4, $x_i - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i = 3$ and 4. Also, by Lemma 7.5, $y \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. Then

$$\Sigma_r = \{-a, b, -x_3, -x_4, y, c - x_3, c - x_4, y - c\} \cup \{-1, 1\}^w$$

and hence $\lambda_r = \frac{1}{2} \dim M$, which contradicts $\lambda_r = \frac{1}{2} \dim M + 2$ in Lemma 3.3.

Second, suppose that $y = 1$. By Lemma 6.6, none of $x_i$, $c - x_i$, and $c - y$ is 1 for all $i$. Hence, by Lemma 7.4, $x_i - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i = 3$ and 4. Moreover, $N_p(1) = N_p(-1) = w + 1$ and $N_q(1) - 1 = N_q(-1) = w$. By Lemma 2.4 for 1, this implies that $N_r(1) = N_r(-1) - 1$. Then the weights are

$$\begin{aligned}
\Sigma_p &= \{-c, -b, x_1, x_1, x_3, x_4, -1, 1\} \cup \{-1, 1\}^w, \\
\Sigma_q &= \{c, a, -x_1, -x_1, x_3 - c, x_4 - c, c - 1, 1\} \cup \{-1, 1\}^w, \\
\Sigma_r &= \{-a, b, -x_3, -x_4, c - x_3, c - x_4, 1 - c, -1\} \cup \{-1, 1\}^w.
\end{aligned}$$

By Corollary 2.7, $c_1(M)|_p = 0$ and this implies that $x_3 + x_4 = b$. Since $x_3 + x_4 = b < a + b = c = 2x_1$ and $x_1 = c/2 \geq 2$, we have that $x_1^2 > x_3 x_4$. Therefore,

$$(-1)^w \frac{B + C}{a(c - x_3)(c - x_4)(c - 1)} = cx_1^2 - bx_3x_4 > 0.$$

Also, $(-1)^w A < 0$. Then, by Theorem 2.1,

$$0 = \int_M 1 = (-1)^w \left( \frac{1}{A} + \frac{1}{B} + \frac{1}{C} \right) < 0,$$

which is a contradiction.

Finally, suppose that $c - x_i = 1$ for some $i$. Since $c \geq 4$, $c - x_1 = x_1 = c/2 \geq 2$. Hence, $c - x_i = x_i \neq 1$ for $i = 1$ and 2. Therefore, without loss of generality, let $c - x_4 = 1$. By Lemma 6.6, none of $x_i$, $c - x_j$, $y$, and $c - y$ is 1 for all $i$ and for $j \neq 4$. Hence, by Lemma 7.4, $c - x_3 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. Also, by Lemma 7.5, $y \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. Moreover, $N_p(1) = N_p(-1) + 1 = w + 1$ and $N_q(1) = N_q(-1) = w + 1$. By Lemma 2.4 for 1, this implies that $N_r(1) = N_r(-1) - 1$. Then the weights are

$$\begin{aligned}
\Sigma_p &= \{-c, -b, x_1, x_1, x_3, c - 1, -y, 1\} \cup \{-1, 1\}^w, \\
\Sigma_q &= \{c, a, -x_1, -x_1, x_3 - c, -1, c - y, 1\} \cup \{-1, 1\}^w, \\
\Sigma_r &= \{-a, b, -x_3, 1 - c, c - x_3, y - c, y, -1\} \cup \{-1, 1\}^w.
\end{aligned}$$

By Lemma 4.6(1) for $c - 1 = x_4$, $\Sigma_p \equiv \Sigma_r \bmod c - 1$. First, we can choose a bijection between $\Sigma_p$ and $\Sigma_q$ so that $\Sigma_p \supset \{c - 1, -c\} \cup \{-1, 1\}^w \equiv \{1 - c, -1\} \cup \{-1, 1\}^w \subset \Sigma_r \bmod c - 1$. Since $N_p(1) = w + 1$ and $N_r(1) = w$, $\Sigma_p \equiv \Sigma_r \bmod c - 1$ implies that

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$2 - c \in \Sigma_r$ since $|1 + k(c-1)| > c$ for $|k| \geq 2$ and $c \notin \Sigma_r$. Since $-a$ is the only negative even weight at $r$, we have that $-a = 2 - c$, i.e., $a + 2 = c = a + b$. Hence, $b = 2$. Then we are left with

$$\{-2 = -b, x_1, x_1, x_3, -y\} \equiv \{2 = b, -x_3, c - x_3, y - c, y\} \mod c - 1.$$

Since for $2 \in \Sigma_r$, $2 \neq -2, x_1$, and $x_3 \bmod c - 1$, the only possibility is that $2 \equiv -y \bmod c - 1$, i.e., $2 - c + 1 = -y$. Thus $y = c - 3$. By Corollary 2.7, $c_1(M)|_p = -c - 2 + x_1 + x_1 + x_3 + c - 1 - y + 1 = 0$. Hence, we have that $x_3 + c = y + 2$. However, $x_3 + c = y + 2 = c - 3 + 2 = c - 1$ and so $0 < x_3 = -1$, which is a contradiction. □

LEMMA 8.5. *In Lemma 7.1, $t = 1$ and $v = 1$ are impossible.*

*Proof.* The weights in this case are

$$\begin{aligned}
\Sigma_p &= \{-c, -b, x_1, x_2, x_3, x_4, -y, 1\} \cup \{-1, 1\}^{w+1}, \\
\Sigma_q &= \{c, a, x_1 - c, x_2 - c, x_3 - c, x_4 - c, c - y, 1\} \cup \{-1, 1\}^{w+1}, \\
\Sigma_r &= \{-a, b, \ldots\} \cup \{-1, 1\}^{w},
\end{aligned}$$

where $a, b, c, w, x_i, y$ are as in Lemma 7.1, and the remaining weights at $r$ are odd.

By Lemma 7.6, $x_i = c - x_j$ for some $i$ and $j$. By Lemma 6.4, there exist $x_i$ and $x_j$ where $i \neq j$ such that $2x_i = 2x_j = c$. Without loss of generality, let $2x_1 = 2x_2 = c$. Denote $x = x_1$. Lemma 6.4 also implies that $x_i \neq c - x_j$ for $i \neq 1$ and 2, and for all $j$. Therefore, $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i = 3$ and 4 by Lemma 7.2. Also, by Lemma 7.3, $y - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. Moreover, by Lemma 4.6(3) for $x$, none of $x_i$, $y$, $c - x_i$, and $c - y$ can be $x$ for $i \neq 1$ and 2. Hence, by Lemma 6.6, all of $x_i$, $y$, $c - x_i$, and $c - y$ are different for $i \neq 1$ and 2. We have the following cases:

(1) $y = 1$. By Lemma 6.6, none of $x_i$, $c - x_i$, and $c - y$ is 1 for all $i$. Then by Lemma 7.4, $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i = 3$ and 4. Moreover, $N_p(1) = N_p(-1) = N_q(1) = N_q(-1) + 1 = w + 2$. Hence, by Lemma 2.4 for 1, $N_r(-1) = N_r(1) + 1$. Considering Lemma 2.4 for each integer, one can show that

$$\Sigma_r = \{-a, b, -x_3, -x_4, c - x_3, c - x_4, 1 - c, -1, -f, f\} \cup \{-1, 1\}^w,$$

for some odd natural number $f$. By Lemma 6.8, since no additional multiples of $x$ should appear, $f = 1$, which contradicts the definition of $w$.

(2) $c - x_i = 1$ for some $i$. Since $2x_1 = 2x_2 = c \geq 4$, $c - x_1 = c - x_2 = x_1 \geq 2$. Hence, without loss of generality, assume that $c - x_4 = 1$. By Lemma 6.6, none of $x_i$, $y$, $c - x_j$, and $c - y$ can be 1 for all $i$ and $j \neq 4$. Thus $y \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ by Lemma 7.5 and $c - x_3 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ by Lemma 7.4. Considering Lemma 2.4 for each integer, one can show that

$$\Sigma_r = \{-a, b, -x_3, 1 - c, y, c - x_3, y - c, -1, -f, f\} \cup \{-1, 1\}^w,$$

for some odd natural number $f$. Then as above, $f = 1$, which contradicts the definition of $w$.

(3) $y \neq 1$ and $c - x_i \neq 1$, for all $i$. By Lemmas 7.5 and 7.4, $y, c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i = 3, 4$. Considering Lemma 2.4 for each integer, one can show that the

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weights are

$$\Sigma_p = \{-c, -b, x, x, x_3, x_4, -y, 1, -1, 1\} \cup \{-1, 1\}^w,$$
$$\Sigma_q = \{c, a, -x, -x, x_3 - c, x_4 - c, c - y, 1, -1, 1\} \cup \{-1, 1\}^w,$$
$$\Sigma_r = \{-a, b, -x_3, -x_4, c - x_3, c - x_4, y, y - c, -1, -1\} \cup \{-1, 1\}^w.$$

By Lemma 4.6(3) for $x$, no additional multiples of $x$ appear. Hence, $a \neq x$ and $b \neq x$. Since $2x = c = a + b$, either $a > c/2$ or $b > c/2$.

(a) $a > c/2$. By Lemma 4.6(1) for $a$, $\Sigma_q \equiv \Sigma_r \bmod a$. First, we can choose so that

$$\begin{aligned}\Sigma_q \supset\ & \{c = a + b, a, -1\} \cup \{-1, 1\}^w \\ & \equiv \{-a, b, -1\} \cup \{-1, 1\}^w \subset \Sigma_r \mod a.\end{aligned}$$

Next, $\{-x, -x\} \subset \Sigma_q$ and $-x \notin \Sigma_r$ imply that $\{a - x, a - x\} \subset \Sigma_r \setminus (\{-a, b, -1\} \cup \{-1, 1\}^w)$, $\{2a - x, 2a - x\} \subset \Sigma_r \setminus (\{-a, b, -1\} \cup \{-1, 1\}^w)$, or $\{a - x, 2a - x\} \subset \Sigma_r \setminus (\{-a, b, -1\} \cup \{-1, 1\}^w)$, since $|-x + ka| > c$ for $k < 0$ or $k > 2$. If the first or second case holds, it implies that two of $c - x_3$, $c - x_4$, and $y$ are equal since $c - x_3$, $c - x_4$, and $y$ are the only positive integers in $\Sigma_r \setminus (\{-a, b, -1\} \cup \{-1, 1\}^w)$, which contradicts Lemma 6.6. Hence $\{a - x, 2a - x\} \subset \Sigma_r \setminus (\{-a, b, -1\} \cup \{-1, 1\}^w)$.

Similarly, $N_q(1) = w + 2$, $N_r(1) = w$, and $\Sigma_q \equiv \Sigma_r \bmod a$ imply that either $\{1 + a, 1 + a\} \subset \Sigma_r \setminus (\{-a, b, -1\} \cup \{-1, 1\}^w)$, $\{1 - a, 1 - a\} \subset \Sigma_r \setminus (\{-a, b, -1\} \cup \{-1, 1\}^w)$, or $\{1 - a, 1 + a\} \subset \Sigma_r \setminus (\{-a, b, -1\} \cup \{-1, 1\}^w)$, since $|1 + ka| > c$ for $|k| \geq 2$. If the first case holds, $-2(a + 1) \in \Sigma_r$ by Lemma 4.6(3) for $a + 1$, which is a contradiction since $-2(a + 1) < -c$ where $-c$ is the smallest weight. If the second case holds, $2(a - 1) \in \Sigma_r$ by Lemma 4.6(3) for $1 - a$, which is a contradiction since $2(a - 1) \geq 2x = c$ but $c \notin \Sigma_r$. Hence, the third case must be the case.

To sum up, we have that $\{1 - a, 1 + a, a - x, 2a - x\} \subset \{-x_3, -x_4, y, c - x_3, c - x_4, y - c\} = \Sigma_r \setminus (\{-a, b, -1\} \cup \{-1, 1\}^w)$, i.e., $1 - a \in \{-x_3, -x_4, y - c\}$ and $\{1 + a, a - x, 2a - x\} = \{y, c - x_3, c - x_4\}$. For each $\alpha \in \{-x_3, -x_4, y - c\}$, we have that $\alpha + c \in \{y, c - x_3, c - x_4\}$. This implies that $1 - a + c \in \{y, c - x_3, c - x_4\} = \{1 + a, a - x, 2a - x\}$. If $1 - a + c = 1 + a$, then $1 - a + c = 1 - a + 2x = 1 + a$, hence $2x = 2a$, contradicting $a > x$. If $1 - a + c = 2a - x$, then $1 - a + c = 1 - a + 2x = 2a - x$, hence $3x + 1 = 3a$, which is a contradiction since $a > x$. Hence $1 - a + c = a - x$. Then $c + 1 = 2a - x$, which is a contradiction since $2a - x \in \{y, c - x_3, c - x_4\} \subset \Sigma_r$ but $2a - x = c + 1 > c$, where $c$ is the largest weight.

(b) $b > c/2$. By Lemma 4.6(1) for $b$, $\Sigma_p \equiv \Sigma_r \bmod b$. First, we can choose so that

$$\begin{aligned}\Sigma_p \supset\ & \{-c = -a - b, -b, -1\} \cup \{-1, 1\}^w \\ & \equiv \{-a, b, -1\} \cup \{-1, 1\}^w \subset \Sigma_r \mod b.\end{aligned}$$

Next, as above, one can show that $\{b - x, 2b - x, b + 1\} = \{y, c - x_3, c - x_4\}$. This implies that $b - x + 2b - x + b + 1 = 4b - 2x + 1 = 2c + y - x_3 - x_4 = y + c - x_3 + c - x_4$. Therefore, $4b - 2x + 1 = 2c + y - x_3 - x_4 = 4x + y - x_3 - x_4$, hence $4b + 1 = 6x + y - x_3 - x_4$. On the other hand, by Corollary 2.7, $c_1(M)|_p = b + x_3 + x_4 - y + 1 = 0$. Thus $y - x_3 - x_4 = b + 1$. Then we have that $4b + 1 = 6x + y - x_3 - x_4 = 6x + b + 1$, hence $3b = 6x$, which is a contradiction since $b < 2x = c$. □

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LEMMA 8.6. *In Lemma 7.1, $t = 1$ and $v = 2$ are impossible.*

*Proof.* The weights in this case are

$$\Sigma_p = \{-c, -b, x_1, x_2, x_3, x_4, -y, 1\} \cup \{-1, 1\}^{w+2},$$
$$\Sigma_q = \{c, a, x_1 - c, x_2 - c, x_3 - c, x_4 - c, c - y, 1\} \cup \{-1, 1\}^{w+2},$$
$$\Sigma_r = \{-a, b, \ldots\} \cup \{-1, 1\}^{w},$$

where $a, b, c, w, x_i, y$ are as in Lemma 7.1, and the remaining weights at $r$ are odd. By Lemma 7.3, $y - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. We have the following cases:

(1) $x_i = c - x_j$ for some $i$ and $j$. By Lemma 6.4, there exist $x_i$ and $x_j$ where $i \neq j$ such that $2x_i = 2x_j = c$. Without loss of generality, let $2x_1 = 2x_2 = c$. Denote $x = x_1$. Lemma 6.4 also implies that $x_i \neq c - x_j$ for $i \neq 1$ and 2, and for all $j$. Therefore, $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i = 3$ and 4 by Lemma 7.2. Moreover, by Lemma 4.6(3) for $x$, none of $x_i$, $y$, $c - x_i$, and $c - y$ can be $x$ for $i \neq 1$ and 2. Hence, by Lemma 6.6, all of $x_i$, $y$, $c - x_i$, and $c - y$ are different for $i \neq 1$ and 2.

We consider three cases: $y = 1$; $c - x_i$ for some $i$; and $y, c - x_i \neq 1$. By using Lemmas 7.5, 7.4, and 2.4 for each integer, one can show that in each case we have

$$\Sigma_r = \{-a, b, -x_3, -x_4, c - x_3, c - x_4, 1 - c, -1, -f, f, -h, h\} \cup \{-1, 1\}^w,$$
$$\Sigma_r = \{-a, b, -x_3, 1 - c, y, c - x_3, y - c, -1, -f, f, -h, h\} \cup \{-1, 1\}^w,$$
$$\Sigma_r = \{-a, b, -x_3, -x_4, y, c - x_3, c - x_4, y - c, -1, -1, -f, f\} \cup \{-1, 1\}^w,$$

for $y = 1$, for $c - x_i$ for some $i$, and for $y, c - x_i \neq 1$ respectively, for some odd natural numbers $f, h$. As in Lemma 8.5, we conclude that $f = 1$, which contradicts the definition of $w$.

(2) $x_i \neq c - x_j$, for all $i$ and $j$. By Lemmas 7.2 and 7.3, $-x_i, y - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$. By Lemma 6.6, it follows that $x_i \neq x_j$ for $i \neq j$. By Lemma 6.3, $x_i \neq c - y$ for all $i$ and $j$.

First, suppose that $y = 1$. By Lemma 6.6, none of $x_i$, $c - x_i$, and $c - y$ is 1. By Lemma 7.4, $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$. Moreover, $N_p(1) = N_p(-1) = N_q(1) = N_q(-1) + 1 = w + 3$. Hence, by Lemma 2.4 for 1, $N_r(-1) = N_r(1) + 1$. Hence, the weights are

$$\Sigma_p = \{-c, -b, x_1, x_2, x_3, x_4, -1, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w,$$
$$\Sigma_q = \{c, a, x_1 - c, x_2 - c, x_3 - c, x_4 - c, c - 1, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w,$$
$$\Sigma_r = \{-a, b\} \cup \{-x_i\}_{i=1}^4 \cup \{c - x_i\}_{i=1}^4 \cup \{1 - c, -1\} \cup \{-1, 1\}^w.$$

Then we have that $N_p(1) \geq w + 3$, $N_q(1) \geq w + 3$, and $N_r(1) = w$, which is a contradiction by Lemma 6.9.

Other cases—the case $c - x_i = 1$ for some $i$, and the case $y \neq 1$ and $c - x_i \neq 1$ for all $i$—are similar: considering Lemmas 6.6, 7.4, 7.5, and 2.4 for 1, we conclude that $N_p(1) \geq w + 3$, $N_q(1) \geq w + 3$, and $N_r(1) = w$, which contradicts Lemma 6.9. □

LEMMA 8.7. *In Lemma 7.1, $t = 2$ and $v = 1$ are impossible.*

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*Proof.* The weights in this case are

$$\Sigma_p = \{-c, -b, x_1, x_2, x_3, x_4, x_5, -y_1, -y_2, 1\} \cup \{-1, 1\}^{w+1},$$
$$\Sigma_q = \{c, a, x_1 - c, x_2 - c, x_3 - c, x_4 - c, x_5 - c, c - y_1, c - y_2, 1\} \cup \{-1, 1\}^{w+1},$$
$$\Sigma_r = \{-a, b, \ldots\} \cup \{-1, 1\}^{w},$$

where $a$, $b$, $c$, $w$, $x_i$, $y_i$ are as in Lemma 7.1, and the remaining weights at $r$ are odd.

By Lemma 7.6, $x_i = c - x_j$ for some $i$ and $j$. Then by Lemma 6.4, there exist $x_i$ and $x_j$ where $i \neq j$ such that $2x_i = 2x_j = c$. Without loss of generality, let $2x_1 = 2x_2 = c$. Denote $x = x_1$. Lemma 6.4 also implies that $x_i \neq c - x_j$ for $i \neq 1$ and 2, and for all $j$. Therefore, $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i \neq 1$ and 2 by Lemma 7.2. Also, by Lemma 7.3, $y_i - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$. Moreover, by Lemma 4.6(3) for $x$, none of $x_i$, $y_j$, $c - x_i$, and $c - y_j$ can be $x$ for $i \neq 1$ and 2, and for all $j$. Hence, by Lemma 6.6, all of $x_i$, $y_j$, $c - x_i$, and $c - y_j$ are different for $i \neq 1$ and 2, and for all $j$.

First, suppose that $y_i = 1$ for some $i$. Without loss of generality, let $y_2 = 1$. Then none of $x_i$, $c - x_i$, $y_1$, and $c - y_i$ is 1 for all $i$. Then we have that $c - x_i$, $y_1 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i \neq 1$ and 2 by Lemmas 7.4 and 7.5. By Lemma 2.4 for 1, $N_r(1) + 1 = N_r(-1)$. Then the weights are

$$\Sigma_p = \{-c, -b, x, x, x_3, x_4, x_5, -y_1, -1, 1, -1, 1\} \cup \{-1, 1\}^w,$$
$$\Sigma_q = \{c, a, -x, -x, x_3 - c, x_4 - c, x_5 - c, c - y_1, c - 1, 1, -1, 1\} \cup \{-1, 1\}^w,$$
$$\Sigma_r = \{-a, b, -x_3, -x_4, -x_5, y_1, c - x_3, c - x_4, c - x_5, y_1 - c, 1 - c, -1\} \cup \{-1, 1\}^w.$$

By Lemma 4.6(1) for $c - 1$, $\Sigma_q \equiv \Sigma_r \bmod c - 1$. First, $|1 + k(c - 1)| > c$ for $|k| \geq 2$. Also, $c \notin \Sigma_r$. Then, $N_q(1) = w + 2$, $N_r(1) = w$, and $\Sigma_q \equiv \Sigma_r \bmod c - 1$ imply that $N_r(2 - c) = 2$, which is a contradiction since $r$ has only one negative even weight.

Second, suppose that $c - x_i = 1$ for some $i$. Since $c - x_1 = c - x_2 = x_1 = c/2 \geq 2$, without loss of generality, let $c - x_5 = 1$. Then none of $x_i$, $c - x_j$, $y_i$, and $c - y_i$ is 1 for all $i$ and $j \neq 5$. Therefore, we have that $c - x_i$, $y_j \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $j = 3$ and 4 and for all $j$ by Lemmas 7.4 and 7.5. Moreover, by Lemma 2.4 for 1, $N_r(1) + 1 = N_r(-1)$. Then the weights are

$$\Sigma_p = \{-c, -b, x, x, x_3, x_4, c - 1, -y_1, -y_2, 1, -1, 1\} \cup \{-1, 1\}^w,$$
$$\Sigma_q = \{c, a, -x, -x, x_3 - c, x_4 - c, -1, c - y_1, c - y_2, 1, -1, 1\} \cup \{-1, 1\}^w,$$
$$\Sigma_r = \{-a, b, -x_3, -x_4, 1 - c, y_1, y_2, c - x_3, c - x_4, y_1 - c, y_2 - c, -1\} \cup \{-1, 1\}^w.$$

By Lemma 4.6(1) for $c - 1$, $\Sigma_p \equiv \Sigma_r \bmod c - 1$. As above, $N_p(1) = w + 2$, $N_r(1) = w$, and $\Sigma_p \equiv \Sigma_r \bmod c - 1$ imply that $N_r(2 - c) = 2$, which is a contradiction since $r$ has only one negative even weight.

Finally, suppose that $c - x_i \neq 1$ and $y_i \neq 1$ for all $i$. Then we have that $y_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$ by Lemma 7.5 and $c - x_j \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $j \neq 1$ and 2 by Lemma 7.4. Then

$$\Sigma_r = \{-a, b, -x_3, -x_4, -x_5, y_1, y_2, c - x_3, c - x_4, c - x_5, y_1 - c, y_2 - c\} \cup \{-1, 1\}^w$$

and hence $\lambda_r = \frac{1}{2} \dim M$, which contradicts $\lambda_r = \frac{1}{2} \dim M + 2$ in Lemma 3.3. □

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LEMMA 8.8. *In Lemma 7.1, $t = 2$ and $v = 2$ are impossible.*

*Proof.* In this case the weights are

$$\Sigma_p = \{-c, -b, x_1, x_2, x_3, x_4, x_5, -y_1, -y_2, 1\} \cup \{-1, 1\}^{w+2},$$
$$\Sigma_q = \{c, a, x_1 - c, x_2 - c, x_3 - c, x_4 - c, x_5 - c, c - y_1, c - y_2, 1\} \cup \{-1, 1\}^{w+2},$$
$$\Sigma_r = \{-a, b, \ldots\} \cup \{-1, 1\}^{w},$$

where $a$, $b$, $c$, $w$, $x_i$, $y_i$ are as in Lemma 7.1, and the remaining weights at $r$ are odd.

By Lemma 7.6, $x_i = c - x_j$ for some $i$ and $j$. Then by Lemma 6.4, there exist $x_i$ and $x_j$ where $i \neq j$ such that $2x_i = 2x_j = c$. Without loss of generality, let $2x_1 = 2x_2 = c$. Denote $x = x_1$. Lemma 6.4 also implies that $x_i \neq c - x_j$ for $i \neq 1$ and 2, and for all $j$. Therefore, $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i \neq 1$ and 2 by Lemma 7.2. Also, by Lemma 7.3, $y_i - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$. Moreover, by Lemma 4.6(3) for $x$, none of $x_i$, $y_j$, $c - x_i$, and $c - y_j$ can be $x$ for $i \neq 1$ and 2, and for all $j$. Hence, by Lemma 6.6, all of $x_i$, $y_j$, $c - x_i$, and $c - y_j$ are different for $i \neq 1$ and 2, and for all $j$.

First, suppose that $y_i = 1$ for some $i$. Without loss of generality, let $y_2 = 1$. Then none of $x_i$, $c - x_i$, $y_1$, and $c - y_i$ is 1 for all $i$. Therefore, $y_1 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ by Lemma 7.5 and $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i \neq 1$ and 2 by Lemma 7.4. Moreover, by Lemma 2.4 for 1, $N_r(1) + 1 = N_r(-1)$. Considering Lemma 2.4 for each integer, we have

$$\Sigma_r = \{-a, b\} \cup \{-x_i\}_{i=3}^5 \cup \{y_1\} \cup \{c - x_i\}_{i=3}^5 \cup \{y_i - c\}_{i=1}^2 \cup \{-1, -f, f\} \cup \{-1, 1\}^w,$$

for some odd natural number $f$. By Lemma 6.8, $f = 1$, which contradicts the definition of $w$.

Second, suppose that $c - x_i = 1$ for some $i$. Since $c - x_1 = c - x_2 = x_1 = c/2 \geq 2$, without loss of generality, let $c - x_5 = 1$. Then none of $x_i$, $c - x_j$, $y_i$, and $c - y_i$ is 1 for all $i$ and $j \neq 5$. Therefore, we have that $y_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$ by Lemma 7.5 and $c - x_j \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $j = 3$ and 4 by Lemma 7.4. Moreover, $N_p(1) = N_p(-1)$ and $N_q(1) = N_q(-1) + 1$. Hence, by Lemma 2.4 for 1, $N_r(1) + 1 = N_r(-1)$. Then the weights are

$$\Sigma_r = \{-a, b\} \cup \{-x_i\}_{i=3}^5 \cup \{y_i\}_{i=1}^2 \cup \{c - x_i\}_{i=3}^4 \cup \{y_i - c\}_{i=1}^2 \cup \{-1, -f, f\} \cup \{-1, 1\}^w,$$

for some odd natural number $f$. As above, we get a contradiction.

Finally, suppose that $c - x_i \neq 1$ and $y_i \neq 1$ for all $i$. Then we have that $y_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$ by Lemma 7.5 and $c - x_j \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $j \neq 1$ and 2 by Lemma 7.4. Since $N_p(1) \geq N_p(-1) + 1$ and $N_q(1) \geq N_q(-1) + 1$, by Lemma 2.4 for 1, $N_r(1) + 2 \leq N_r(-1)$. Then the weights are

$$\Sigma_p = \{-c, -b, x, x, x_3, x_4, x_5, -y_1, -y_2, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w,$$
$$\Sigma_q = \{c, a, -x, -x, x_3 - c, x_4 - c, x_5 - c, c - y_1, c - y_2, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w,$$
$$\Sigma_r = \{-a, b\} \cup \{-x_i\}_{i=3}^5 \cup \{y_i\}_{i=1}^2 \cup \{c - x_i\}_{i=3}^5 \cup \{y_i - c\}_{i=1}^2 \cup \{-1, -1\} \cup \{-1, 1\}^w.$$

Then we have that $N_p(1) \geq w + 3$, $N_q(1) \geq w + 3$, and $N_r(1) = w$, which is a contradiction by Lemma 6.9. □

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LEMMA 8.9. *In Lemma 7.1, $t = 3$ and $v = 2$ are impossible.*

*Proof.* In this case, the weights are

$$\Sigma_p = \{-c, -b\} \cup \{x_i\}_{i=1}^{6} \cup \{-y_i\}_{i=1}^{3} \cup \{1\} \cup \{-1, 1\}^{w+2},$$
$$\Sigma_q = \{c, a\} \cup \{x_i - c\}_{i=1}^{6} \cup \{c - y_i\}_{i=1}^{3} \cup \{1\} \cup \{-1, 1\}^{w+2},$$
$$\Sigma_r = \{-a, b, \ldots\} \cup \{-1, 1\}^{w},$$

where $a$, $b$, $c$, $w$, $x_i$, $y_i$ are as in Lemma 7.1, and the remaining weights at $r$ are odd.

By Lemma 7.6, $x_i = c - x_j$ for some $i$ and $j$. By Lemma 6.4, there exist $x_i$ and $x_j$ where $i \neq j$ such that $2x_i = 2x_j = c$. Without loss of generality, let $2x_1 = 2x_2 = c$. Denote $x = x_1$. Lemma 6.4 also implies that $x_i \neq c - x_j$ for $i \neq 1, 2$, and for all $j$. Therefore, $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i \neq 1, 2$ by Lemma 7.2. Also, by Lemma 7.3, $y_i - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$. Moreover, by Lemma 4.6(3) for $x$, none of $x_i$, $y_j$, $c - x_i$, and $c - y_j$ can be $x$ for $i \neq 1, 2$, and for all $j$. Hence, by Lemma 6.6, all of $x_i$, $y_j$, $c - x_i$, and $c - y_j$ are different for $i \neq 1, 2$, and for all $j$.

First, suppose that $y_i = 1$ for some $i$. Without loss of generality, let $y_3 = 1$. Then none of $x_i$, $c - x_i$, $y_j$, and $c - y_i$ is 1 for all $i$ and $j \neq 3$. Therefore, $y_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i \neq 3$ by Lemma 7.5 and $c - x_j \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $j \neq 1$ and 2 by Lemma 7.4. Moreover, $N_p(1) = N_p(-1)$ and $N_q(1) = N_q(-1) + 1$. Hence, by Lemma 2.4 for 1, $N_r(1) + 1 = N_r(-1)$. Then

$$\Sigma_r = \{-a, b\} \cup \{-x_i\}_{i=3}^{6} \cup \{y_i\}_{i=1}^{2} \cup \{c - x_i\}_{i=3}^{6} \cup \{y_i - c\}_{i=1}^{3} \cup \{-1\} \cup \{-1, 1\}^{w}$$

and hence $N_p(1) = w + 3$, $N_q(1) = w + 3$, and $N_r(1) = w$, which contradicts Lemma 6.9.

Second, suppose that $c - x_i = 1$ for some $i$. Since $c - x_1 = c - x_2 = x_1 = c/2 \geq 2$, without loss of generality, let $c - x_6 = 1$. Then none of $x_i$, $c - x_j$, $y_i$, and $c - y_i$ is 1 for all $i$ and $j \neq 6$. Therefore, $y_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$ by Lemma 7.5 and $c - x_j \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $j = 3$, 4, and 5 by Lemma 7.4. Moreover, by Lemma 2.4 for 1, $N_r(1) + 1 = N_r(-1)$. Then

$$\Sigma_r = \{-a, b\} \cup \{-x_i\}_{i=3}^{6} \cup \{y_i\}_{i=1}^{3} \cup \{c - x_i\}_{i=3}^{5} \cup \{y_i - c\}_{i=1}^{3} \cup \{-1\} \cup \{-1, 1\}^{w}$$

and thus $N_p(1) = w + 3$, $N_q(1) = w + 3$, and $N_r(1) = w$, which contradicts Lemma 6.9.

Finally, suppose that $c - x_i \neq 1$ and $y_i \neq 1$ for all $i$. Then we have that $y_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$ by Lemma 7.5 and $c - x_j \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $j \neq 1$ and 2 by Lemma 7.4. Then we have that $N_p(1) \geq w + 3$, $N_q(1) \geq w + 3$, and $N_r(1) = w$, which contradicts Lemma 6.9. □

LEMMA 8.10. *In Lemma 7.1, $t \geq v + 2$ is impossible.*

*Proof.* By Lemma 7.6, $x_i = c - x_j$ for some $i$ and $j$. Then by Lemma 6.4, there exist $x_i$ and $x_j$ where $i \neq j$ such that $2x_i = 2x_j = c$. Without loss of generality, let $2x_1 = 2x_2 = c$. Lemma 6.4 also implies that $x_i \neq c - x_j$ for $i \neq 1, 2$, and for all $j$. Therefore, $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i \neq 1, 2$ by Lemma 7.2. Also, by Lemma 7.3, $y_i - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$. Moreover, by Lemma 4.6(3) for $x$, none of $x_i$, $y_j$, $c - x_i$,

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and $c - y_j$ can be $x$ for $i \neq 1, 2$, and for all $j$. Hence, by Lemma 6.6, all of $x_i$, $y_j$, $c - x_i$, and $c - y_j$ are different for $i \neq 1, 2$, and for all $j$.

First, suppose that $c - x_i = 1$ for some $i$. Since $c - x_1 = c - x_2 = x_1 = c/2 \geq 2$, without loss of generality, let $c - x_3 = 1$. By Lemma 6.6, $c - x_i \neq 1$ for $i \neq 3$ and $y_j \neq 1$ for all $j$. Then $c - x_i \in \Sigma_r(\{-a, b\} \cup \{-1, 1\}^w)$ for $i \neq 1, 2, 3$ by Lemma 7.4, and $y_j \in \Sigma_(\{-a, b\} \cup \{-1, 1\}^w)$ for all $j$ by Lemma 7.3. Therefore, we have that $\{-x_i\}_{i=3}^{t+3} \cup \{y_i\}_{i=1}^{t} \cup \{c - x_i\}_{i=4}^{t+3} \cup \{y_i - c\}_{i=1}^{t} \subset \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$, which is a contradiction since there are $2t + 4 + 2u + 2v$ spaces in $\Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ but $4t + 1 > 4t \geq 2t + 4 + 2u + 2v$ by the assumption.

The other cases, where $y_i = 1$ for some $i$ and where $c - x_i \neq 1$ and $y_i \neq 1$ for all $i$, are similar: using Lemmas 7.3 and 7.4, we have $c - x_j$, $y_k \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $j \neq 1, 2$ and $k \neq i$, which contradicts the assertion that there are not enough spaces to fill in $\Sigma_r$. □

LEMMA 8.11. *In Lemma 7.1, $v \geq 3$ is impossible.*

*Proof.* This follows easily from the definition of $w$ and Lemma 6.9. □

## 9. *Proof of Theorem 1.1*

By quotienting out by the subgroup which acts trivially, without loss of generality we may assume that the action is effective. Then this is an immediate consequence of Propositions 4.3, 5.1 and 6.1.

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## REFERENCES

[**AB84**] M. Atiyah and R. Bott. The moment map and equivariant cohomology. *Topology* **23** (1984), 1–28.
[**F59**] T. Frankel. Fixed points and torsion on Kähler manifolds. *Ann. of Math. (2)* **70** (1959), 1–8.
[**K99**] Y. Karshon. Periodic Hamiltonian flows on four dimensional manifolds. *Mem. Amer. Math. Soc.* **672** (1999).
[**Ko74**] C. Kosniowski. Holomorphic vector fields with simple isolated zeros. *Math. Ann.* **208** (1974), 171–173.
[**MD88**] D. McDuff. The moment map for circle actions on symplectic manifolds. *J. Geom. Phys.* **5**(2) (1988), 149–160.
[**PT11**] A. Pelayo and S. Tolman. Fixed points of symplectic periodic flows. *Ergod. Th. & Dynam. Sys.* **31** (2011), 1237–1247.
[**TW00**] S. Tolman and J. Weitsman. On semifree symplectic circle actions with isolated fixed points. *Topology* **39**(2) (2000), 299–309.
[**T10**] S. Tolman. On a symplectic generalization of Petrie's conjecture. *Trans. Amer. Math. Soc.* **362**(8) (2010), 3963–3996.

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