Note on some linear Diophantine Inequalities.

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Let A, B, C, D, P, Q be any real constants such that

$$\Delta = AD - BC \neq 0.$$

A well-known theorem * by Minkowski states that integer values of x, y exist such that

$$|Ax + By + P|\,|Cx + Dy + Q| \leqslant \tfrac{1}{4}|\Delta|. \quad \ldots\ldots\ldots(1)$$

It is an interesting question to see what can be said about the magnitude of each of these linear factors. This has led me to the following result which may perhaps be worth while noting, especially as it does not seem easy to prove it directly in what seems to be the obvious way.

Let $\Delta \neq 0$, $\Delta BC \geqslant 0$; *then the inequalities*

$$|Ax + By + P| \leqslant \tfrac{1}{2}|A|, \quad |Cx + Dy + Q| \leqslant \tfrac{1}{2}|D|, \quad \ldots(2)$$

are satisfied by integer values of x, y.

The result is obvious when $BC = 0$. Thus when $B = 0$, the first inequality gives an integer value for x (two values if $P \equiv 0 \pmod{\frac{1}{2}A}$ and $P \neq 0$) and then the second gives an integer value for y. It gives (1) only when $BC = 0$. For if $BC \neq 0$, we may without loss of generality take $B > 0$, $C > 0$ and then $\Delta BC > 0$ gives

$$0 < \Delta = AD - BC < AD.$$

We may suppose then in (2) that $BC \neq 0$.

For the proof, let a, b, h, f, g be any real numbers with $a > 0$, $ab > h^2$. Put

$$\phi(x, y) \equiv ax^2 + 2hxy + by^2 + 2fy + 2gx.$$

Then $\phi(x, y)$ has a minimum value for integers x, y since, for any real number M,

$$|\phi(x, y)| \leqslant M$$

is satisfied by at most a finite number of integers x, y, as is clear on writing $\phi(x, y)$ as a sum of squares of linear functions of x, y. Hence, if $\phi(x, y)$ assumes its minimum value for ξ, η,

$$\phi(\xi \pm 1, \eta) \geqslant \phi(\xi, \eta),$$

or
$$\pm 2(a\xi + h\eta) \pm 2g + a \geqslant 0;$$

and so
$$|a\xi + h\eta + g| \leqslant \tfrac{1}{2}a.$$

Similarly
$$|h\xi + b\eta + f| \leqslant \tfrac{1}{2}b.$$

* I have given a simple proof in the *Journal of the London Mathematical Society*, 3 (1927) 19–22.

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The condition $a > 0$ is not essential and we see that, if $ab > h^2$, the inequalities

$$|ax + hy + g| \leqslant \tfrac{1}{2}|a|, \quad |hx + by + f| \leqslant \tfrac{1}{2}|b|, \ldots\ldots(3)$$

are satisfied by integer values of x, y.

Multiply the second inequality by a constant $|\lambda|$, and the result (2) follows on putting

$$a, h, g = A, B, P,$$

and taking $$\lambda h, \lambda b, \lambda f = C, D, Q$$

respectively. Hence $\lambda = C/B$ and the condition $ab > h^2$ becomes

$$ADB/C > B^2$$

or $$BC(AD - BC) > 0;$$

as was to be shown.

The result (3) can obviously be extended to any number of variables, but this does not lead to a simple generalisation of (2).

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