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ARTICLE

# Disperse hypergraphs

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**Abstract**

For $\ell \geq 3$, an $\ell$-uniform hypergraph is *disperse* if the number of edges induced by any set of $\ell+1$ vertices is 0, 1, $\ell$, or $\ell+1$. We show that every disperse $\ell$-uniform hypergraph on $n$ vertices contains a clique or independent set of size $n^{\Omega_\ell(1)}$, answering a question of the first author and Tomon. To this end, we prove several structural properties of disperse hypergraphs.

**Keywords:** Erdos-Hajnal conjecture ; Homogeneous sets in hypergraphs

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## 1. Introduction

The Erdős-Hajnal conjecture [6] is a fundamental problem in extremal graph theory, stating that for every fixed graph $H$, every induced $H$-free $n$-vertex graph has a homogeneous set (i.e. a clique or independent set) of size at least $n^{c_H}$, where $c_H > 0$ is a constant depending only on $H$. This is in sharp contrast to general graphs, which may only have homogeneous sets of size $O(\log n)$. Despite significant recent progress (see, e.g. [4, 10, 11] and the references therein), the Erdős-Hajnal conjecture remains open in general.

It is natural to ask for analogues of the Erdős-Hajnal conjecture for hypergraphs. We will use the term *$\ell$-graph* as shorthand for $\ell$-uniform hypergraph. Let $\log_k(x)$ denote the $k$-times iterated logarithm, that is, $\log_0(x) = x$, $\log_1(x) = \log(x)$, and $\log_k(x) = \log(\log_{k-1}(x))$. Let $h_\ell(n)$ denote the maximum size of a homogeneous set guaranteed to exist in every $\ell$-vertex $n$-graph. It is well-known [7] that $h_\ell(n) \geq (\log_{\ell-1}(n))^{\Omega(1)}$, and it is conjectured that the number $\ell-1$ of iterated logarithms is best possible, that is, $h_\ell(n) \leq (\log_{\ell-1}(n))^{O(1)}$. Thus, a natural analogue of the Erdős-Hajnal conjecture for $\ell$-graphs would be that for every fixed $\ell$-graph $H$, every induced $H$-free $n$-vertex $\ell$-graph has a homogeneous set of size $(\log_{\ell-2}(n))^{c_H}$. However, there is some evidence that this might not be true for every $H$. Indeed, it is known that for uniformity $\ell \geq 4$, the so-called *stepping up construction* gives a tight lower bound for Ramsey numbers in terms of the number of iterated logarithms (though this number is not known, as for the critical case $\ell = 3$ we only know that the number of logarithms is at least 1 and at most 2; this corresponds to at least $\ell-2$ and at most $\ell-1$ logarithms for uniformity $\ell$). Conlon, Fox, and Sudakov [5] showed that the stepping up construction avoids certain $H$ as induced subgraphs. Thus, for $\ell \geq 4$, there exist $\ell$-graphs $H$ such that induced $H$-free $\ell$-graphs do not have significantly larger homogeneous sets than general $\ell$-graphs. Therefore, if $h_\ell(n) \leq (\log_{\ell-1}(n))^{O(1)}$ (as conjectured), then the above $\ell$-uniform analogue of the Erdős-Hajnal conjecture fails. See also [1, 12] for additional Erdős-Hajnal-type results for hypergraphs.

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Recently, there has been some interest in variants of the Erdős-Hajnal problem where one forbids *order-size pairs* instead of induced subgraphs $H$. To the best of our knowledge, this setting was first considered in [9]. Let us define the problem. For a set $Q$ of pairs of integers, we say that an $\ell$-graph $G$ is $Q$-*free* if for every $(m,f) \in Q$, there is no set of $m$ vertices in $G$ which induces exactly $f$ edges. What can we say about the size of homogeneous sets in $Q$-free $\ell$-graphs? It was recently shown by Arnold, the first author, and Sudakov [2] that this problem does in fact satisfy the natural hypergraph analogue of the Erdős-Hajnal conjecture, in the sense that for every $\ell \geq 2$, all but a finite number[1] of the sets $Q \neq \emptyset$ satisfy that every $Q$-free $\ell$-graph on $n$ vertices contains a homogeneous set of size at least $\left(\log_{\ell-2}(n)\right)^{c_Q}$.

The papers [3, 8] studied the special case where $\ell = 3$ and $Q$ consists of pairs of the form $(4,f)$; that is, we forbid certain numbers of edges on vertex-sets of size four. The first author and Tomon [8] showed that if an $n$-vertex 3-graph $G$ has no four vertices spanning exactly two edges, then $G$ has a homogeneous set of size $n^c$ (where $c > 0$ is an absolute constant). They also asked to show that this extends to $\ell$-graphs, in the following sense: If an $n$-vertex $\ell$-graph $G$, with $\ell \geq 3$, has no $\ell+1$ vertices spanning $2, 3, 4, \ldots, \ell-1$ edges, then $G$ has a homogeneous set of size $n^{c_\ell}$. Here we prove this conjecture. We call $\ell$-graphs with this property *disperse*.

**Definition 1.1.** *Let $G$ be an $\ell$-graph, $\ell \geq 3$. We say that $G$ is disperse if, for every $(\ell+1)$-set $X = \{v_1, \ldots, v_{\ell+1}\} \subseteq V(G)$, it holds that $e_G(X) \in \{0, 1, \ell, \ell+1\}$.*

Note that if $G$ is disperse then so is its complement $\overline{G}$.

**Theorem 1.2.** *Let $G$ be a disperse $\ell$-graph on $n$ vertices. Then,* $\max(\alpha(G), \omega(G)) \geq \Omega(n^{\frac{1}{3\ell-1}})$.

The constant $\frac{1}{3\ell-1}$ in Theorem 1.2 is best possible up to a factor of roughly $\frac{1}{3}$. Indeed, note that if an $\ell$-graph $G$ has no two edges intersecting in $\ell-1$ vertices, then $G$ is disperse (because every $(\ell+1)$-set of vertices spans 0 or 1 edges). Such $\ell$-graphs are known aspartial $(\ell, \ell-1)$-*Steiner systems*. It is known [13] that there exist $n$-vertex partial $(\ell, \ell-1)$-Steiner systems with independence number $\tilde{O}(n^{\frac{1}{\ell-1}})$. Hence, the constant in Theorem 1.2 cannot be improved beyond $\frac{1}{\ell-1}$. It is also worth noting that in the case $\ell = 3$, the constant $\frac{1}{8}$ given by Theorem 1.2 significantly improves the constant obtained in [8], which was much smaller and therefore not calculated explicitly. It would be interesting to determine the best possible constant in Theorem 1.2. For $\ell = 3$, the best upper-bound on this constant comes from [3], which constructed a disperse $n$-vertex 3-graph $G$ with $\max(\alpha(G), \omega(G)) \leq \tilde{O}(n^{1/3})$.[2]

As mentioned above, one example of disperse $\ell$-graphs is partial $(\ell, \ell-1)$-Steiner systems. Let us now describe another important example. A hypergraph $G$ is called a *cohypergraph* if either $|V(G)| = 1$, or there is a vertex partition $V(G) = X \cup Y$ such that $G[X], G[Y]$ are cohypergraphs and $E(G)$ contains all or none of the $\ell$-sets which intersect both $X$ and $Y$. It is easy to check (by induction) that every cohypergraph is disperse. Also, note that for $\ell = 2$, this recovers the definition of *cographs*. Recall also that a graph is a cograph if and only if it is induced $P_4$-free, where $P_4$ is the path with 4 vertices.

To prove Theorem 1.2, we show (implicitly) that every disperse $\ell$-graph is close to being a cohypergraph. To this end, we prove several structural properties of disperse hypergraphs. Let us now state the two key facts we will use, Theorems 1.4 and 1.5. We first introduce the basic definitions related to tight connectivity.

**Definition 1.3.** *Let $G$ be an $\ell$-graph. A tight walk is a sequence of edges $e_1, \ldots, e_m \in E(G)$ such that $|e_i \cap e_{i+1}| \geq \ell-1$ for all $1 \leq i \leq m-1$. For $A, B \in \binom{V(G)}{\ell-1}$, we say that $A, B$ are connected in $G$ if*

[1] It is conjectured in [2] that these exceptions are not necessary and the result in fact holds for every non-empty $Q$.

[2] In fact, this construction avoids not only four vertices spanning two edges, but also four vertices spanning three edges.

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*there is a tight walk $e_1, \dots, e_m$ with $A \subseteq e_1$ and $B \subseteq e_m$. This is an equivalence relation.*[3] *A tight component of $G$ is an equivalence class in this relation. Thus, the tight components partition $\binom{V(G)}{\ell-1}$. We say that $G$ is tightly connected if there is a single tight component, namely $\binom{V(G)}{\ell-1}$.*

In other words, $G$ is tightly connected if for every $A, B \in \binom{V(G)}{\ell-1}$, there is a tight walk $e_1, \dots, e_m$ with $A \subseteq e_1$ and $B \subseteq e_m$. Note that in the case $\ell = 2$, this coincides with the usual notion of graph connectivity.

For a hypergraph $G$, we use $\overline{G}$ to denote the complement of $G$. Our first theorem states that for a disperse $G$, either $G$ or $\overline{G}$ is not tightly connected. Our second theorem states that in a disperse $G$, each tight component is a complete hypergraph, that is, it consists of all $(\ell-1)$-tuples contained in some vertex set.

**Theorem 1.4.** *Let $G$ be a disperse $\ell$-graph. Then $G$ or $\overline{G}$ is not tightly connected.*

**Theorem 1.5.** *Let $G$ be a disperse $\ell$-graph. Then for every tight component $\mathcal{C}$ of $G$, there is $U \subseteq V(G)$ such that $\mathcal{C} = \binom{U}{\ell-1}$.*

Combining these two theorems, we conclude that for every disperse $\ell$-graph $G$, either in $G$ or in $\overline{G}$ we can find a vertex set $\emptyset \neq X \subsetneq V(G)$ such that there is no edge having $\ell - 1$ vertices in $X$ and one vertex in $Y := V(G) \setminus X$. Indeed, we simply take $X$ to be the vertex set corresponding to a tight component (via Theorem 1.5). One can then show, using the fact that $G$ is disperse, that there are only few edges intersecting both $X$ and $Y$. Repeating this inside $X$ and $Y$ (with some technicalities omitted here), we can show that $G$ is close to a cohypergraph.

Theorems 1.4 and 1.5 are proved in Section 2. We then prove Theorem 1.2 in Section 3. Section 4 contains some further remarks and open problems. We mostly use standard graph-theoretic notation. For a vertex-set $X$ in a hypergraph $G$, we use $e_G(X)$ to denote the number of edges of $G$ contained in $X$. Throughout the rest of the paper, we assume that the uniformity $\ell$ is at least 3, unless explicitly stated otherwise.

## 2. Structural results

In this section we study the structure of disperse hypergraphs, and in particular prove Theorems 1.4 and 1.5. Let us begin with some lemmas. For an $\ell$-graph $G$ and $v \in V(G)$, the *link* $L(v)$ is the $(\ell-1)$-graph on $V(G) \setminus \{v\}$ with edge set $\{e \in \binom{V(G)\setminus\{v\}}{\ell-1} : e \cup \{v\} \in E(G)\}$. A useful fact is that the links of a disperse hypergraph are also disperse.

**Lemma 2.1.** *Let $G$ be a disperse $\ell$-graph. Then for every $v \in V(G)$, $L(v)$ is disperse.*

***Proof.*** Suppose that there is $v \in G$ such that $L(v)$ is not disperse, and let $X = \{v_1, \dots, v_\ell\} \subseteq V \setminus \{v\}$ witness this, namely, $2 \leq |e_{L(v)}(X)| \leq \ell - 2$. Then $2 \leq e_G(X \cup \{v\}) \leq \ell - 1$, contradicting that $G$ is disperse. □

We will also use the following easy lemma.

**Lemma 2.2.** *Let $G$ be a disperse $\ell$-graph, let $f, g \in E(G)$ with $|f \cap g| \geq \ell - 1$, let $v \in f$ and let $A, B \subseteq g \setminus \{v\}$ with $|A| = |B| = \ell - 2$. Then $A, B$ are connected in $L(v)$.*

***Proof.*** If $v \in g$ then the assertion clearly holds, because $g \setminus \{v\}$ is a (one-edge) tight walk between $A$ and $B$ in $L(v)$. So suppose that $v \notin g$. Then we can write $f = \{v, v_1, \dots, v_{\ell-1}\}, g = \{v_1, \dots, v_{\ell-1}, w\}$ with $w \neq v$. As $G$ is disperse, the $(\ell+1)$-set $f \cup g$ must miss at most one edge. This means that there are $e_1, e_2 \in E(G)$ with $\{v\} \cup A \subseteq e_1$ and $\{v\} \cup B \subseteq e_2$. Now $e_1 \setminus \{v\}, e_2 \setminus \{v\}$ is a tight walk between $A$ and $B$ in $L(v)$. □

[3] Indeed, let $e_1, \dots, e_m$ be a tight walk with $A \subseteq e_1, B \subseteq e_m$, and let $f_1, \dots, f_k$ be a tight walk with $B \subseteq e_1, C \subseteq f_k$. Then $e_1, \dots, e_m, f_1, \dots, f_k$ is a tight walk connecting $A$ and $C$.

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The following is our main technical lemma. It shows that in a disperse hypergraph, if there is a tight walk of length 3 from a vertex $v$ to an $(\ell-1)$-set $B$, then there is also such a tight walk of length 2.

**Lemma 2.3.** *Let $G$ be a disperse $\ell$-graph, let $e,f,g$ be a tight walk in $G$, let $v \in e$, and let $B \subseteq g$ with $|B| = \ell - 1$. Then there is a tight walk $f', g'$ in $G$ such that $v \in f'$ and $B \subseteq g'$.*

***Proof.*** We first consider some easy degenerate cases. If $v \in f$ or $v \in g$ then we can take $(f', g') = (f, g)$ or $(f', g') = (g, g)$, respectively. If $B \subseteq f$ then take $(f', g') = (e, f)$, and if $|e \cap g| \geq \ell - 1$ then take $(f', g') = (e, g)$. Assuming that none of the above holds, without loss of generality we may write $e = \{v, w_1, \ldots, w_{\ell-1}\}, f = \{w_1, \ldots, w_\ell\}$ and $g = \{w_2, \ldots, w_{\ell+1}\}$ with the vertices $v, w_1, \ldots, w_{\ell+1}$ being distinct, and $B = \{w_2, \ldots, w_{\ell+1}\} \setminus \{w_i\}$ for some $i \neq \ell + 1$. Note that $|f \cap B| = \ell - 2$.

Suppose, for the sake of contradiction, that there is no tight walk $f', g'$ as in the lemma. We proceed via a series of claims which will eventually give a contradiction.

**Claim 2.4.** *For all $A \subseteq f \setminus \{w_1\}$ with $|A| = \ell - 2$, it holds that $\{v, w_1\} \cup A \in E(G)$.*

***Proof.*** Consider the $(\ell+1)$-set $X = \{v, w_1, \ldots, w_\ell\}$. As $e, f$ are both edges of $G$ contained in $X$, and as $G$ is disperse, $G$ misses at most one edge on $X$. Note that $h := \{v, w_2, \ldots, w_\ell\}$ is not an edge of $G$, as otherwise $f' = h$ and $g' = g$ satisfy $v \in f'$, $|f' \cap g'| = |\{w_2, \ldots, w_\ell\}| = \ell - 1$ and $B \subseteq g'$. So, $G$ contains all edges on $X$ besides $h$; in particular, $G$ contains $\{v, w_1\} \cup A$ for all $A \subseteq f \setminus \{w_1\}$ of size $\ell - 2$. □

**Claim 2.5.** *For all $A \subseteq g$ with $|A| = \ell - 1$, it holds that $\{v\} \cup A \notin E(G)$.*

***Proof.*** Otherwise we can take $f' = \{v\} \cup A$ and $g' = g$, which satisfy $v \in f'$, $|f' \cap g'| = |A| = \ell - 1$ and $B \subseteq g'$. □

**Claim 2.6.** *For all $A \subseteq g$ with $|A| = \ell - 1$, it holds that $\{w_1\} \cup A \in E(G)$ if and only if $A \neq B$.*

***Proof.*** Let $X = \{w_1, \ldots, w_{\ell+1}\}$. Note that $f, g$ are edges of $G$ contained in $X$, and thus, $G$ misses at most one edge on $X$. Moreover, if $\{w_1\} \cup B \in E(G)$, then letting $f' = \{v, w_1\} \cup (f \cap B)$ (which is an edge by Claim 2.4) and $g' = \{w_1\} \cup B$ gives the desired result. Thus, $\{w_1\} \cup B \notin E(G)$, implying that $G$ must contain all other edges on $X$ – in particular, all edges of the form $\{w_1\} \cup A$ with $A \subseteq g$, $|A| = \ell - 1$ and $A \neq B$. □

**Claim 2.7.** *For all $A \subseteq B$ with $|A| = \ell - 2$ and $A \neq f \cap B$, it holds that $\{v, w_1\} \cup A \notin E(G)$.*

***Proof.*** Let $X = \{v, w_1\} \cup B$, so $|X| = \ell + 1$. By Claim 2.5 we have $\{v\} \cup B \notin E(G)$, and by Claim 2.6 we have $\{w_1\} \cup B \notin E(G)$. Hence, as $G$ is disperse, it can contain at most one edge on $X$, which must be the edge $\{v, w_1\} \cup (f \cap B)$, as this is an edge by Claim 2.4. So, $G$ cannot contain any edge of the form $\{v, w_1\} \cup A$ for $A \subseteq B$ with $|A| = \ell - 2$ and $A \neq f \cap B$. □

Now, we use the above claims to derive a contradiction and hence prove Lemma 2.3. Fix an arbitrary $x \in f \cap B$ (this is possible because $|f \cap B| = \ell - 2 \geq 1$). Set $C = g \setminus \{x\}$, and note that $|C| = \ell - 1$; $C \neq B$ and hence $|C \cap B| = \ell - 2$; $w_{\ell+1} \in C$ (as $w_{\ell+1} \notin f$ and so $x \neq w_{\ell+1}$); and $C \cap B \neq f \cap B$ (as $x \in f \cap B$ while $x \notin C$). Consider the set $X = \{v, w_1\} \cup C$, so $|X| = \ell + 1$. We will show that there exist two edges and two non-edges on $X$. Indeed, by Claim 2.4, $\{v, w_1\} \cup (C \setminus \{w_{\ell+1}\}) \in E(G)$, and by Claim 2.6, $\{w_1\} \cup C \in E(G)$. On the other hand, by Claim 2.5, $\{v\} \cup C \notin E(G)$, and by Claim 2.7, $\{v, w_1\} \cup (C \cap B) \notin E(G)$. This contradicts the assumption that $G$ is disperse, as desired. □

Lemma 2.3 easily implies the following.

**Lemma 2.8.** *Let $G$ be a disperse $\ell$-graph, let $e_1, \ldots, e_m$ be a tight walk in $G$, and let $v \in e_1$. Then for all $1 \leq i \leq m-1$, there exists a tight walk $f, g$ such that $v \in f$ and $e_i \cap e_{i+1} \subseteq g$.*

***Proof.*** We prove this by induction on $i$. For $i = 1$ the claim is trivial by letting $f = g = e_1$. For the inductive step, let $f, g$ be a tight walk with $v \in f$ and $e_{i-1} \cap e_i \subseteq g$. Note that $f, g, e_i$ is a tight walk

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as $f, g$ is a tight walk and $|g \cap e_i| \geq |e_{i-1} \cap e_i| = \ell - 1$. Moreover, $v \in f$ and $B := e_i \cap e_{i+1} \subseteq e_i$, so we may apply Lemma 2.3 to obtain a tight walk $f', g'$ with $v \in f'$ and $e_i \cap e_{i+1} \subseteq g'$, as desired. □

We now use this result to prove the following key lemma.

**Lemma 2.9.** *Let $G$ be a disperse $\ell$-graph, let $v \in V(G)$ and let $A, B \subseteq V(G)\backslash\{v\}$ with $|A| = |B| = \ell - 2$. If $A \cup \{v\}$ and $B \cup \{v\}$ are connected in $G$, then $A$ and $B$ are connected in $L(v)$.*

***Proof.*** Let $e_1, \ldots, e_m$ be a tight walk in $G$ between $A \cup \{v\}$ and $B \cup \{v\}$. We will prove the lemma by induction on $m$. For $m \leq 2$ the lemma is trivial, by simply observing that $e_1\backslash\{v\}$, $e_m\backslash\{v\}$ is a tight walk between $A$ and $B$ in $L(v)$. So, suppose $m \geq 3$. First, consider the case where $v \in e_i$ for some $1 < i < m$. Fix any $C \subseteq e_i\backslash\{v\}$ of size $\ell - 2$. By the inductive hypothesis, $A$ and $C$ are connected in $L(v)$, because $e_1, \ldots, e_i$ is a tight walk between $A \cup \{v\}$ and $C \cup \{v\}$ in $G$. Similarly, $C$ and $B$ are connected in $L(v)$, by applying the inductive hypothesis to $e_i, \ldots, e_m$. By transitivity, $A$ and $B$ are connected in $L(v)$.

Now, suppose that $v \notin e_i$ for every $1 < i < m$. For $1 \leq i \leq m-2$, let $C_i := e_i \cap e_{i+1} \cap e_{i+2}$. Then $|C_i| \geq \ell - 2$ and we may assume, by passing to a subset if necessary, that $|C_i| = \ell - 2$. Note that $v \notin C_i$ for every $1 \leq i \leq m-2$. Set also $C_0 = A$ and $C_{m-1} = B$. It suffices to show that for every $1 \leq i \leq m-1$, $C_{i-1}, C_i$ are connected in $L(v)$. Indeed, this would imply, by transitivity, that $A = C_0$ and $B = C_{m-1}$ are connected in $L(v)$. Observe that $A$ and $C_1$ are connected in $L(v)$ because $e_1\backslash\{v\}$ is an edge of $L(v)$ containing $A, C_1$. Similarly, $B$ and $C_{m-2}$ are connected in $L(v)$ because $v \in e_m$ and $B, C_{m-2} \subseteq e_m\backslash\{v\}$. Now let $2 \leq i \leq m-2$. By Lemma 2.8, there is a tight walk $f, g$ with $v \in f$ and $e_i \cap e_{i+1} \subseteq g$. By definition, $C_i, C_{i-1} \subseteq e_i \cap e_{i+1}$, and hence $C_i, C_{i-1} \subseteq g$. Now, by Lemma 2.2, $C_i, C_{i-1}$ are connected in $L(v)$, as required. □

Lemma 2.9 immediately implies the following.

**Theorem 2.10.** *Let $G$ be a tightly connected disperse $\ell$-graph. Then $L(v)$ is tightly connected for all $v \in V(G)$.*

***Proof.*** Let $v \in V(G)$. Fix any $A, B \subseteq V(G) \setminus \{v\}$ of size $\ell - 2$. As $G$ is tightly connected, $A \cup \{v\}$ and $B \cup \{v\}$ are connected in $G$. Hence, by Lemma 2.9, $A$ and $B$ are connected in $L(v)$. This shows that $L(v)$ is tightly connected. □

Next, we need the following simple lemma proved by the first author and Tomon [8]. For completeness, we include the proof.

**Lemma 2.11.** *Let $G$ be a disperse 3-graph. Then $L(v)$ is a cograph for every $v \in V(G)$.*

***Proof.*** Let $v \in V(G)$, and suppose by contradiction that $a, b, c, d$ is an induced path in $L(v)$. In order for $v, a, b, c$ not to span two edges, we must have $\{a, b, c\} \in E(G)$. Similarly, $\{b, c, d\} \in E(G)$ and $\{a, b, d\}, \{a, c, d\} \notin E(G)$. But now $a, b, c, d$ span two edges, a contradiction. □

We now proceed to the proof of Theorem 1.4.

***Proof of Theorem 1.4.*** We use induction on $\ell$. When $\ell = 3$, the link of each vertex is a cograph by Lemma 2.11, and every cograph $H$ satisfies that $H$ or $\overline{H}$ is not connected. In particular, by the contrapositive of Theorem 2.10, this implies that $G$ or $\overline{G}$ is not tightly connected. Now, suppose $\ell > 3$. Fix any $v \in V(G)$. By Lemma 2.1, $L_G(v)$ is disperse. Hence, by the induction hypothesis, $L_G(v)$ or $\overline{L_G(v)} = L_{\overline{G}}(v)$ is not tightly connected. Thus, $G$ or $\overline{G}$ is not tightly connected by the contrapositive of Theorem 2.10, as desired. □

Next, we consider the structure of tight components in disperse hypergraphs, with the goal of proving Theorem 1.5. We begin a sequence of lemmas.

**Lemma 2.12.** *Let $G$ be a disperse $\ell$-graph and let $A_1, A_2 \subseteq V(G)$ with $|A_1| = |A_2| = \ell - 1$ and $|A_1 \cap A_2| = \ell - 2$. Suppose that $G$ has a tight walk between $A_1$ and $A_2$. Then there exists such a tight walk on two or less edges.*

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***Proof.*** We prove this claim by induction on the uniformity $\ell$. First, suppose $\ell = 3$ and write $A_1 = \{v, x\}$, $A_2 = \{v, y\}$. By Lemma 2.9, $x$ and $y$ are in the same component of $L(v)$. Since $L(v)$ is a cograph by Lemma 2.11, $L(v)$ is induced $P_4$-free, and so, the shortest path from $x$ to $y$ in $L(v)$ has length at most 2. Thus, $G$ contains a tight walk from $\{v, x\}$ to $\{v, y\}$ of length at most 2, as required. Now, suppose $\ell \geq 4$. Let $v \in A_1 \cap A_2$ be arbitrary. By Lemma 2.9, there is a tight walk from $A_1 \setminus \{v\}$ to $A_2 \setminus \{v\}$ in $L(v)$. Hence, by the inductive hypothesis, there exists such a walk of length at most 2. Denote this walk by $e_1, e_2$ (possibly $e_1 = e_2$). Then, $e_1 \cup \{v\}, e_2 \cup \{v\}$ is a tight walk of length at most 2 from $A_1$ to $A_2$ in $G$, as desired. □

**Lemma 2.13.** *Let $G$ be a disperse $\ell$-graph, let $B \subseteq V(G)$ of size $\ell$, and let $A_1, A_2 \subseteq B$ be distinct subsets of size $\ell - 1$. If $A_1, A_2$ belong to the same tight component $\mathcal{C}$ of $G$, then every $(\ell - 1)$-subset of $B$ also belongs to $\mathcal{C}$.*

***Proof.*** By assumption, $G$ has a tight walk between $A_1$ and $A_2$. By Lemma 2.12, there exists such a tight walk $W$ of length at most 2. If $W$ has length 1 then it consists of the single edge $e = A_1 \cup A_2 = B$, and thus, any $(\ell - 1)$-subset of $B$ clearly belongs to $\mathcal{C}$. So suppose that $W$ has length 2 and write $W = (e_1, e_2)$, where $A_1 \subseteq e_1$, $A_2 \subseteq e_2$, and $e_1 \neq e_2$. Since $G$ is disperse, $G$ misses at most one edge on the $(\ell + 1)$-set $C := e_1 \cup e_2$. This implies that for every $(\ell - 1)$-set $A \subseteq C$, there is an edge $e \in E(G)$ with $A \subseteq e \subseteq C$ (indeed, otherwise $C$ misses at least two edges). As $|e \cap e_1| \geq \ell - 1$, we have a tight walk $(e_1, e)$ between $A_1$ and $A$. Hence, $A$ belongs to $\mathcal{C}$, as required. □

**Lemma 2.14.** *Let $G$ be a disperse $\ell$-graph and let $e_1, \ldots, e_m$ be a tight walk in $G$. Then all $(\ell - 1)$-subsets of $\bigcup_{i=1}^{m} e_i$ belong to the same tight component of $G$.*

***Proof.*** We will prove this claim by induction on $m$. The claim is trivial when $m = 1$, as $e_1$ is a tight walk between any two $(\ell - 1)$-subsets of $e_1$. Now assume $m \geq 2$. By the induction hypothesis, there is a tight component $\mathcal{C}$ containing all $(\ell - 1)$-subsets of $W := \bigcup_{i=1}^{m-1} e_i$. If $e_m \subseteq \bigcup_{i=1}^{m-1} e_i$ then there is nothing to prove, so suppose otherwise. Let $x$ be the unique vertex in $e_m \setminus e_{m-1}$, and write $e_m = \{v_1, v_2, \ldots, v_{\ell-1}, x\}$. It suffices to show that for every $w_1, \ldots, w_{\ell-2} \in W$, the $(\ell - 1)$-set $\{w_1, w_2, \ldots, w_{\ell-2}, x\}$ belongs to $\mathcal{C}$. We now prove by induction on $j$ that for every $0 \leq j \leq \ell - 2$, $A_j := \{w_1, \ldots, w_j, v_{j+1}, \ldots, v_{\ell-2}, x\} \in \mathcal{C}$. For the base case $j = 0$, note that $A_0 = \{v_1, \ldots, v_{\ell-2}, x\}$ is in the same tight component as $\{v_1, \ldots, v_{\ell-1}\} \in \mathcal{C}$, because both of these $(\ell - 1)$-sets are contained in $e_m$. For the induction step, let $1 \leq j \leq \ell - 2$. By the inductive hypothesis, $A_{j-1} = \{w_1, \ldots, w_{j-1}, v_j, v_{j+1}, \ldots, v_{\ell-2}, x\} \in \mathcal{C}$. Also, $\{w_1, \ldots, w_j, v_j, v_{j+1}, \ldots, v_{\ell-2}\} \in \mathcal{C}$ because all vertices of this $(\ell - 1)$-set belong to $W$. Hence, by Lemma 2.13 for the set $B := \{w_1, \ldots, w_j, v_j, v_{j+1}, \ldots, v_{\ell-2}, x\}$, every $(\ell - 1)$-subset of $B$, and in particular $A_j$, also belongs to $\mathcal{C}$. This completes the induction step. Taking $j = \ell - 2$, we get that $A_{\ell-2} = \{w_1, \ldots, w_{\ell-2}, x\} \in \mathcal{C}$, as required. □

Using the above lemmas, we can now prove Theorem 1.5.

***Proof of Theorem 1.5.*** Let $\mathcal{C}$ be a tight component of $G$. Let $U$ be the set of all $v \in V(G)$ such that $v \in A$ for some $A \in \mathcal{C}$. We want to show that $\mathcal{C} = \binom{U}{\ell-1}$. So suppose for the sake of contradiction that there exists some $A \subseteq U$ of size $\ell - 1$ such that $A \notin \mathcal{C}$. Let $A'$ be a largest subset of $A$ such that $A' \subseteq B$ for some $B \in \mathcal{C}$. As $A \notin \mathcal{C}$, we have $|A'| < |A|$. So fix $x \in A \setminus A'$. As $x \in U$, there is some $C \in \mathcal{C}$ with $x \in C$ (by the definition of $U$). As $B, C \in \mathcal{C}$, there is a tight walk $e_1, \ldots, e_m$ in $G$ between $B$ and $C$. By Lemma 2.14, each $(\ell - 1)$-tuple of vertices in $\bigcup_{i=1}^{m} e_i$ also belongs to $\mathcal{C}$. As $A' \cup \{x\} \subseteq B \cup C \subseteq \bigcup_{i=1}^{m} e_i$, there is an $(\ell - 1)$-tuple $D \in \mathcal{C}$ with $A' \cup \{x\} \subseteq D$. However, this contradicts the maximality of $A'$, completing the proof. □

## 3. Proof of Theorem 1.2

The following is a standard bound on the independence number of hypergraphs, due to Spencer [14].

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**Lemma 3.1.** *Let $G$ be an $\ell$-graph with $n$ vertices and average degree $d$. Then*

$$\alpha(G) \geq \frac{\ell-1}{\ell} n \cdot \min\{1, d^{-1/(\ell-1)}\}.$$

***Proof.*** If $d < 1$ then $e(G) = \frac{nd}{\ell} < \frac{n}{\ell}$, and deleting one vertex per edge gives an independent set of size at least $\frac{\ell-1}{\ell} n$. Suppose now that $d \geq 1$. Sample a random subset $U \subseteq V(G)$ with probability $p := d^{-1/(\ell-1)} \leq 1$. Deleting one vertex per edge in $U$ gives an independent set of size at least $|U| - e(U)$. By linearity of expectation, we have $\mathbb{E}[|U| - e(U)] = pn - p^{\ell} \frac{dn}{\ell} = pn(1 - \frac{p^{\ell-1} d}{\ell}) = \frac{\ell-1}{\ell} pn = \frac{\ell-1}{\ell} n d^{-1/(\ell-1)}$. □

It is well-known that every $n$-vertex cograph contains a homogeneous set of size at least $n^{1/2}$. The same proof applies to cohypergraphs, as follows.

**Lemma 3.2.** *Every cohypergraph $G$ on $n$ vertices satisfies $\alpha(G) \cdot \omega(G) \geq n$ and hence* $\max(\alpha(G), \omega(G)) \geq n^{\frac{1}{2}}$.

***Proof.*** We prove this by induction on $n$. The case $n = 1$ is trivial, so suppose $n \geq 2$. By definition, we can write $V(G) = V(H_1) \cup V(H_2)$, where $H_1$ and $H_2$ are vertex-disjoint cohypergraphs, and $G$ has either all edges or no edges which intersect both $V(H_1)$ and $V(H_2)$. By the inductive hypothesis, $\alpha(H_i) \cdot \omega(H_i) \geq |V(H_i)|$ for $i = 1, 2$. Let us assume that $G$ has all edges which intersect both $V(H_1)$ and $V(H_2)$; the other case is symmetrical (by switching to $\overline{G}$). Then $\omega(G) = \omega(H_1) + \omega(H_2)$ and $\alpha(G) = \max(\alpha(H_1), \alpha(H_2))$, giving

$$\begin{aligned}
\alpha(G) \cdot \omega(G) &= \max(\alpha(H_1), \alpha(H_2)) \cdot (\omega(H_1) + \omega(H_2)) \\
&\geq \alpha(H_1) \cdot \omega(H_1) + \alpha(H_2) \cdot \omega(H_2) \\
&\geq |V(H_1)| + |V(H_2)| \\
&= |V(G)| = n,
\end{aligned}$$

as desired. □

For a tight component $\mathcal{C}$ of a disperse $\ell$-graph $G$, we denote by $V(\mathcal{C})$ the vertex set $U$ satisfying $\mathcal{C} = \binom{U}{\ell-1}$, using Theorem 1.5. The following lemma shows that if all tight components of $G$ are small, then $G$ has a large independent set.

**Lemma 3.3.** *Let $G$ be a disperse $\ell$-graph on $n$ vertices, and suppose that for every tight component $\mathcal{C}$ of $G$ it holds that $|V(\mathcal{C})| \leq m$. Then $\alpha(G) \geq \frac{\ell-1}{\ell} \cdot (n/m)^{\frac{1}{\ell-1}}$.*

***Proof.*** First, we will bound the number of edges of $G$. Note that each edge of $G$ is contained in $V(\mathcal{C})$ for some tight component $\mathcal{C}$. Thus, by summing over all tight components, we obtain that

$$\begin{aligned}
e(G) &\leq \sum_{\mathcal{C}} \binom{|V(\mathcal{C})|}{\ell} \\
&\leq \frac{m}{\ell} \sum_{\mathcal{C}} \binom{|V(\mathcal{C})|}{\ell-1} \\
&= \frac{m}{\ell} \binom{n}{\ell-1},
\end{aligned}$$

where the equality uses the fact that every $(\ell-1)$-subset of $V(G)$ is contained in $V(\mathcal{C})$ for exactly one tight component $\mathcal{C}$, as the tight components partition $\binom{V(G)}{\ell-1}$. Letting $d$ denote the average degree of $G$, we get

$$d = \frac{\ell \cdot e(G)}{n} \leq \frac{m\binom{n}{\ell-1}}{n} \leq m n^{\ell-2}.$$

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Thus, by Lemma 3.1,

$$\alpha(G) \geq \frac{\ell-1}{\ell} \cdot \frac{n}{m^{\frac{1}{\ell-1}} n^{\frac{\ell-2}{\ell-1}}} = \frac{\ell-1}{\ell} \cdot \left(\frac{n}{m}\right)^{\frac{1}{\ell-1}},$$

as desired. □

For two disjoint vertex-sets $X, Y$ in an $\ell$-graph $G$ and for $1 \leq i \leq \ell-1$, denote by $E_G(X^i, Y^{\ell-i})$ the set of edges of $G$ having $i$ vertices in $X$ and $\ell-i$ vertices in $Y$, and let $e_G(X^i, Y^{\ell-i})$ be the number of such edges. An edge in $\bigcup_{i=1}^{\ell-1} E_G(X^i, Y^{\ell-i})$ is said to *cross* $(X, Y)$.

**Lemma 3.4.** *Let $G$ be a disperse $\ell$-graph, and let $X, Y \subseteq V(G)$ be disjoint vertex-sets with $e_G(X^{\ell-1}, Y^1) = 0$. Let $A = \{v_1, \ldots, v_{\ell-1}\} \subseteq X \cup Y$, denote $i = |A \cap Y|$ and assume $i \geq 1$. Then there are less than $2^{i-1}$ edges in $E_G(X^{\ell-i}, Y^i)$ containing $A$.*

***Proof.*** We prove the claim by induction on $i$. The base case $i = 1$ holds by assumption as $e_G(X^{\ell-1}, Y^1) = 0$. Now let $2 \leq i \leq \ell-1$ and suppose that the result holds for $i-1$. Without loss of generality, assume that $v_1, v_{\ell-1} \in Y$ (note that $|A \cap Y| = i \geq 2$). Suppose, for the sake of contradiction, that there are at least $2^{i-1}$ edges in $E_G(X^{\ell-i}, Y^i)$ which contain $A$. As $|A \cap Y| = i$, each such edge consists of $A$ and a vertex from $X$. So let $X_A = \{x \in X \setminus A : \{x\} \cup A \in E(G)\}$ denote the set of vertices in $X$ such that $\{x\} \cup A \in E_G(X^{\ell-i}, Y^i)$. Then $|X_A| \geq 2^{i-1}$. Observe that for any $x_1, x_2 \in X_A$, the set of vertices $\{x_1, x_2\} \cup A$ contains at least two edges (namely, the edges $\{x_j\} \cup A$ for $j = 1, 2$), and thus, avoids at most one edge (as $G$ is disperse). Colour the pair $x_1x_2 \in \binom{X_A}{2}$ blue if $\{x_1, x_2, v_2, \ldots, v_{\ell-1}\} \in E(G)$, and red otherwise. So, if $x_1x_2$ is coloured red, then $\{x_1, x_2, v_1, \ldots, v_{\ell-2}\} \in E(G)$. Fix any $v \in X_A$. By pigeonhole, at least $\lceil \frac{|X_A|-1}{2} \rceil \geq 2^{i-2}$ of the edges touching $v$ have the same colour. Without loss of generality, suppose this colour is red. That is, $\{v, x, v_1, \ldots, v_{\ell-2}\} \in E(G)$ for at least $2^{i-2}$ vertices $x$. Note that the set $A' := \{v_1, \ldots, v_{\ell-2}, v\}$ satisfies $|A' \cap Y| = i-1$, because $v_{\ell-1} \in Y$ and $v \in X$. Hence, each edge $\{v, x, v_1, \ldots, v_{\ell-2}\}$ as above belongs to $E_G(X^{\ell-i+1}, Y^{i-1})$. Therefore, the number of edges in $E_G(X^{\ell-i+1}, Y^{i-1})$ containing $A'$ is at least $2^{i-2}$, a contradiction to the induction hypothesis. □

Finally, we prove Theorem 1.2.

***Proof of Theorem 1.2.*** Set $\epsilon = \frac{\ell+1}{3\ell-1}$, $\gamma = \frac{2}{3\ell-1}$. We first define an algorithm to maintain a partition $\mathcal{P}$ of $V(G)$ and a set of $\ell$-tuples $\mathcal{B} \subseteq \binom{V(G)}{\ell}$, as follows.

We first give the following claim, which will be used later.

**Claim 3.5.** *Let $U \subseteq V(G)$. If $U$ contains no $\ell$-tuple from $\mathcal{B}$, then $G[U]$ is a cohypergraph.*

***Proof.*** Observe that at each step of the algorithm, the hypergraph $H$ (Line 3) has no edges which cross $(X \cap U, Y \cap U)$ (because all such edges are added to $\mathcal{B}$). This implies that $G$ contains all or none of the edges crossing $(X \cap U, Y \cap U)$ (by the choice of $H$). Moreover, $|U \cap W| \leq \ell-1$ for every $W \in \mathcal{P}$ (here $\mathcal{P}$ is the partition at the end of the algorithm), because $U$ contains no $\ell$-tuple from $\mathcal{B}$, whereas $\binom{W}{\ell} \subseteq \mathcal{B}$. It follows that $G[U]$ is a cohypergraph obtained by starting with the empty hypergraphs $G[U \cap W]$, $W \in \mathcal{P}$, and repeatedly joining two of the parts (by backtracking the algorithm). □

Note that in Line 9 of the algorithm, we have $E_H(X^{\ell-1}, Y^1) = \emptyset$, because $X$ is the vertex set of a tight component $\mathcal{C}$ of $H$. Indeed, if there were an edge $\{x_1, \ldots, x_{\ell-1}, y\} \in E_H(X^{\ell-1}, Y^1)$, then we would have $y \in V(\mathcal{C}) = X$, a contradiction.

If the algorithm terminates in the first condition (Lines 4-5), then there is some set $W \subseteq V(G)$ with $|W| \geq n^{1-\gamma}$ such that in either $G[W]$ or $\overline{G[W]}$, all tight components have size at most $n^{1-\epsilon}$.

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**Algorithm 1** PARTITION-ALGORITHM

1: Initialise $\mathcal{P} = \{V(G)\}$
2: **while** $\exists W \in \mathcal{P}$ such that $|W| \geq n^{1-\gamma}$, **do**
3: Let $H \in \left\{G[W], \overline{G[W]}\right\}$ such that $H$ is not tightly connected (Theorem 1.4)
4: **if** All tight components $\mathcal{C}$ of $H$ satisfy $|V(\mathcal{C})| \leq n^{1-\epsilon}$, **then**
5: Terminate
6: **else**
7: Partition $W = X \cup Y$ where $|X| \geq n^{1-\epsilon}$ and $X = V(\mathcal{C})$ for a tight component $\mathcal{C}$ of $H$
8: Replace $W$ with $X, Y$ in $\mathcal{P}$
9: Add to $\mathcal{B}$ all edges in $E_H(X^i, Y^{\ell-i})$ for every $1 \leq i \leq \ell - 2$
10: **end if**
11: **end while**
12: **for** $W \in \mathcal{P}$ **do**
13: Add $\binom{W}{\ell}$ to $\mathcal{B}$
14: **end for**

Thus, by Lemma 3.3 (with $m = n^{1-\epsilon}$), we have

$$\begin{aligned}\max(\alpha(G), \omega(G)) &\geq \frac{\ell-1}{\ell} \cdot \left(\frac{|W|}{m}\right)^{\frac{1}{\ell-1}} \\ &\geq 0.5n^{\frac{\epsilon-\gamma}{\ell-1}} \\ &= 0.5n^{\frac{1}{3\ell-1}},\end{aligned}$$

using our choice of $\epsilon, \gamma$. Hence, we may assume from now on that the algorithm does not terminate on the first condition.

Thus, by Claim 3.5, our goal from now on is to find a large set $U \subseteq V(G)$ which is independent in the hypergraph with edge set $\mathcal{B}$. To this end, we upper-bound $|\mathcal{B}|$. Let $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) be the set of $\ell$-tuples added to $\mathcal{B}$ in Line 9 (resp. Line 13) of the algorithm.

**Claim 3.6.** $|\mathcal{B}_1| \leq 2^\ell n^{\ell-1+\epsilon}$.

***Proof.*** We will bound $|\mathcal{B}_1|$ as follows. For each pair of sets $(X, Y)$ obtained in Line 7 of the algorithm, and each edge of $H$ $e$ which crosses $(X, Y)$, we assign to $e$ an $(\ell-1)$-subset $A \subseteq e$ with $A \cap Y = e \cap Y$ (if there are several such $A$, namely if $|e \cap Y| \leq \ell - 2$, then choose one such $A$ arbitrarily). In other words, for any given $(\ell-1)$-set $A$, we will bound the number of edges $e$ of $H$ crossing $(X, Y)$, which contain $A$ and satisfy $A \cap Y = e \cap Y$. Summing over all $A \in \binom{V(G)}{\ell-1}$ and all choices for $(X, Y)$ will give us the desired bound on $|\mathcal{B}_1|$.

So, let $A \subseteq V(G)$ be an arbitrary set of size $\ell - 1$. Let $(X_1, Y_1), \ldots, (X_t, Y_t)$ be all pairs of sets $(X, Y)$ obtained in the course of the algorithm for which $A$ crosses $(X, Y)$, ordered according to when the algorithm processed them. Since any two sets handled by the algorithm are disjoint or nested, and since all $Y_i$ intersect $A$, we must have $Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_t$. Also, since $|X_i| \geq n^{1-\epsilon}$ for every $i$ (see Line 7 of the algorithm), we have $|Y_i| \leq |Y_{i-1}| - n^{1-\epsilon}$ for all $2 \leq i \leq t$, and thus, $t \leq n^\epsilon$. Now, fix any $1 \leq i \leq t$. As $e_H(X_i^{\ell-1}, Y_i^1) = 0$, Lemma 3.4 implies that the number of edges $e$ of $H$ crossing $(X_i, Y_i)$ and satisfying $A \cap Y_i = e \cap Y_i$ is less than $\sum_{i=1}^{\ell-1} 2^{i-1} = 2^{\ell-1} - 1$. Summing over

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all choices of $A \in \binom{V(G)}{\ell-1}$ and $i = 1, \dots, t$, we get that

$$\begin{aligned} |\mathcal{B}_1| &\leq \binom{n}{\ell-1} \cdot n^{\epsilon} \cdot 2^{\ell} \\ &\leq 2^{\ell} n^{\ell-1+\epsilon}, \end{aligned}$$

as desired. □

**Claim 3.7.** $|\mathcal{B}_2| \leq n^{\ell-(\ell-1)\gamma}$.

***Proof.*** By the definition of the algorithm, we have $|W| \leq n^{1-\gamma}$ for every $W \in \mathcal{P}$ (where $\mathcal{P}$ denotes the partition at the end of the algorithm). Hence,

$$\begin{aligned} |\mathcal{B}_2| &\leq \sum_{W \in \mathcal{P}} \binom{|W|}{\ell} \\ &\leq (n^{1-\gamma})^{\ell-1} \cdot \sum_{W \in \mathcal{P}} |W| \\ &= n^{\ell-(\ell-1)\gamma}, \end{aligned}$$

as desired. □

We now complete the proof of the theorem. Consider the auxiliary $\ell$-graph on the vertex set $V(G)$ and edge set $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. Let us denote this hypergraph by $\mathcal{B}$ as well. By Claims 3.6 and 3.7,

$$\begin{aligned} |E(\mathcal{B})| &\leq 2^{\ell} n^{\ell-1+\epsilon} + n^{\ell-(\ell-1)\gamma} \\ &= 2^{\ell} n^{\ell-1+\frac{\ell+1}{3\ell-1}} + n^{\ell-(\ell-1)\cdot\frac{2}{3\ell-1}} \\ &= 2^{\ell} n^{\frac{3\ell^2-3\ell-\ell+1+\ell+1}{3\ell-1}} + n^{\frac{3\ell^2-\ell-2\ell+2}{3\ell-1}} \\ &= (2^{\ell}+1) n^{\frac{3\ell^2-3\ell+2}{3\ell-1}}. \end{aligned}$$

Hence, the average degree of $\mathcal{B}$ is at most

$$O\left(n^{\frac{3\ell^2-3\ell+2}{3\ell-1}-1}\right) = O\left(n^{\frac{3\ell^2-6\ell+3}{3\ell-1}}\right) = O\left(n^{\frac{3(\ell-1)^2}{3\ell-1}}\right).$$

Thus, by Lemma 3.1,

$$\begin{aligned} \alpha(\mathcal{B}) &\geq \Omega\left(n^{1-\frac{3(\ell-1)^2}{3\ell-1}\cdot\frac{1}{\ell-1}}\right) \\ &= \Omega\left(n^{1-\frac{3\ell-3}{3\ell-1}}\right) \\ &= \Omega\left(n^{\frac{2}{3\ell-1}}\right). \end{aligned}$$

Thus, by Claim 3.5, $G$ contains an induced cohypergraph of size $\Omega(n^{\frac{2}{3\ell-1}})$. By Lemma 3.2,

$$\max\left(\alpha(G), \omega(G)\right) \geq \Omega\left(n^{\frac{1}{3\ell-1}}\right),$$

as desired. □

Note that the above proof in fact shows that every $n$-vertex disperse $\ell$-graph can be made into a cohypergraph by adding/deleting $O(n^{\frac{3\ell^2-3\ell+2}{3\ell-1}}) = O(n^{\ell-\frac{2\ell-2}{3\ell-1}}) = o(n^{\ell})$ edges. Indeed, changing the $\ell$-tuples in $\mathcal{B}$ in an appropriate way yields a cohypergraph.

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## 4. Concluding remarks

For a set $L \subseteq \{0, 1, \ldots, \ell+1\}$, let us call an $\ell$-graph*L-free* if it contains no induced $(\ell+1)$-vertex subgraph whose number of edges belongs to $L$. Thus, an $\ell$-graph is disperse if and only if it is $\{2, 3, \ldots, \ell-1\}$-free. Our main result is that disperse $\ell$-graphs have polynomial-size homogeneous sets.

**Problem 4.1.** *Is there any proper subset $L \subsetneq \{2, 3, \ldots, \ell-1\}$ for which $L$-free $\ell$-graphs have polynomial-size homogeneous sets?*

This trivially fails for $\ell = 3$, and can also be shown to fail for $\ell = 4$; but such an $L$ could exist for $\ell \geq 5$.

By combining Lemmas 2.1 and 2.11, we get that in a disperse $\ell$-graph $G$, for every $\ell-2$ distinct vertices $v_1, \ldots, v_{\ell-2}$, the link graph $L(v_1, \ldots, v_{\ell-2})$ is a cograph (this graph consists of all pairs $xy$ such that $\{v_1, \ldots, v_{\ell-2}, x, y\} \in E(G)$). Thus, the following would be a strengthening of Theorem 1.2.

**Problem 4.2.** *Is there $c_\ell > 0$ such that if $G$ is an $n$-vertex $\ell$-graph in which all links $L(v_1, \ldots, v_{\ell-2})$ are cographs, then $G$ has a homogeneous set of size at least $n^{c_\ell}$ ?*

Another well-studied graph class is that of split graphs. A graph is *split* if it has a vertex partition $X, Y$ such that $X$ is a clique and $Y$ is an independent set. One can ask for the size of homogeneous sets in 3-graphs in which every link is split. It turns out that such 3-graphs might only have homogeneous sets of logarithmic size, due to the following variant of a well-known construction. Take a random graph $F \sim G(n, \frac{1}{2})$, and define a 3-graph $G$ on $V(F)$ in which $xyz$ is an edge if $e_F(\{x, y, z\}) \geq 2$. It is easy to show, using standard arguments, that $\omega(G), \alpha(G) = O(\log n)$. Also, for every vertex $v$, $N_F(v)$ is a clique in the link of $v$, and $V(F) \backslash (N_F(v) \cup \{v\})$ is an independent set in the link of $v$. Hence each link is split.

We can also show that the above construction is inevitable, in the sense that if $G$ is a 3-graph where all links are split, then $G$ contains a large vertex-set which induces the above construction.

**Proposition 4.3.** *Let $G$ be an $n$-vertex 3-graph in which every link is split. Then there is $U \subseteq V(G)$, $|U| \geq n^{1/3}$, and a graph $F$ on $U$, such that for every distinct $x, y, z \in U$, $xyz \in E(G)$ if and only if $e_F(\{x, y, z\}) \geq 2$.*

***Proof.*** By assumption, for each $v \in V(G)$, there is a partition $V(G) \backslash \{v\} = A_v \cup B_v$ such that $A_v$ is independent in the link of $v$ and $B_v$ is a clique in the link of $v$. Call a pair of vertices $xy$ good if $x \in A_y \iff y \in A_x$ (i.e. $x \in A_y$ and $y \in A_x$, or $x \in B_y$ and $y \in B_x$). Otherwise, call $xy$ bad. We first claim that there is no $K_4$ consisting of bad pairs. Suppose otherwise; let $x_1, x_2, x_3, x_4$ such that $x_i x_j$ is bad for every $1 \leq i < j \leq 4$. Orient the edges $x_i x_j$ by letting $x_i \to x_j$ if $x_j \in A_{x_i}$ (note that exactly one of $x_j \in A_{x_i}, x_i \in A_{x_j}$ holds). Every tournament on 4 vertices has a transitive tournament on 3 vertices. Hence, without loss of generality, we can assume that $x_1 \to x_2, x_3$ and $x_2 \to x_3$. It follows that $x_2, x_3 \in A_{x_1}$, and hence $x_1 x_2 x_3 \notin E(G)$. On the other hand, $x_1, x_2 \in B_{x_3}$, and hence $x_1 x_2 x_3 \in E(G)$, giving a contradiction.

We showed that the graph of bad pairs has no $K_4$. As $R(K_4, K_m) \leq \binom{m+2}{3} \leq m^3$, there is a set $U \subseteq V(G)$, $|U| \geq n^{1/3}$, such that $U$ contains no bad pairs. Now define a graph $F$ on $U$ by letting $xy$ be an edge of $F$ if $x \in B_y$ and $y \in B_x$ (so $xy$ is not an edge of $F$ if $x \in A_y$ and $y \in A_x$). Let $x, y, z \in U$ be distinct. If $xy, xz \in E(F)$, then $y, z \in B_x$, so $xyz \in E(G)$. Similarly, if $xy, xz \notin E(F)$, then $y, z \in A_x$ and so $xyz \notin E(F)$. This proves the proposition. □

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## References


1. Amir, M., Shapira, A. and Tyomkyn, M. (2021) Two Erdős–Hajnal-type theorems in hypergraphs. *J. Combin. Theory, Ser. B* **146** 417–438.
2. Arnold, F., Gishboliner, L. and Sudakov, B. (2024) Two Erdős–Hajnal-type theorems for forbidden order-size pairs, arXiv preprint arXiv: 2406.04154.
3. Axenovich, M., Bradač, D., Gishboliner, L., Mubayi, D. and Weber, L. (2024) Large cliques or cocliques in hypergraphs with forbidden order-size pairs. *Combin., Probab. Comput.* **33**(3) 286–299.
4. Bucić, M., Nguyen, T., Scott, A. and Seymour, P. (2024) Induced subgraph density. I. A loglog step towards Erdős–Hajnal. *Int. Math. Res. Notices* **2024**(12) 9991–10004.
5. Conlon, D., Fox, J. and Sudakov, B. (2012) Erdős–Hajnal-type theorems in hypergraphs. *J. Combin. Theory, Ser. B* **102**(5) 1142–1154.
6. Erdös, P. and Hajnal, A. (1989) Ramsey-type theorems. *Discrete Appl. Math.* **25** 37–52.
7. Erdos, P. and Rado, R. (1952) Combinatorial theorems on classifications of subsets of a given set. *Proc. Lond. Math. Soc.* **3**(1) 417–439.
8. Gishboliner, L. and Tomon, I. (2022) On 3-graphs with no four vertices spanning exactly two edges. *Bull. Lond. Math. Soc.* **54**(6) 2117–2134.
9. Mubayi, D. and Suk, A. (2019) The Erdős–Szekeres problem and an induced Ramsey question. *Mathematika* **65**(3) 702–707.
10. Nguyen, T., Scott, A. and Seymour, P. (2023) Induced subgraph density. VI. Bounded VC-dimension, arXiv preprint arXiv: 2312.15572.
11. Nguyen, T., Scott, A. and Seymour, P. (2023) Induced subgraph density VII The five-vertex path, arXiv preprint arXiv: 2312.15333.
12. Rödl, V. and Schacht, M. (2012) Complete partite subgraphs in dense hypergraphs. *Random Struct. Algor.* **41**(4) 557–573.
13. Rödl, V. and Šinajová, E. (1994) Note on independent sets in Steiner systems. *Random Struct. Algor.* **5**(1) 183–190.
14. Spencer, J. (1972) Turán's theorem for *k*-graphs. *Discrete Math.* **2** 183–186.


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