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Subshifts of the three dot system

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Abstract. An example of a two-dimensional subshift is the set of arrays $\{x_{(i,j)} : (i, j) \in \mathbb{Z}^2\}$ of zeros and ones for which $x_{(i,j)}+x_{(i+1,j)}+x_{(i,j+1)} = 0$ for each $i, j \in \mathbb{Z}$, where addition is taken modulo 2. There are two shift operators, the horizontal shift and the vertical shift. This subshift is called the *Ledrappier example* or the *three dot dynamical system*. In this paper we give an internal description of all transitive subshifts for which one of the shift operators is sofic in the usual one-dimensional sense.

1. *Introduction*

If G is a finite group with the discrete topology then $G^{\mathbb{Z}}$ and $G^{\mathbb{Z}^2}$ with the product topologies and coordinatewise addition are compact topological groups. The shift transformation on $G^{\mathbb{Z}}$ is a continuous group automorphism. On $G^{\mathbb{Z}^2}$ the horizontal shift and the vertical shift are commuting group automorphisms.

A closed, shift invariant subgroup of either group is called a *Markov subgroup*. Given a Markov subgroup, we would like to describe the closed, shift-invariant subsets and then describe the invariant measures. Kitchens completely solved the one-dimensional case, $G^{\mathbb{Z}}$ (see [**K1**]). In the two-dimensional case it is a difficult and interesting problem. This paper is concerned only with Markov subgroups of the simplest such space $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$. In particular, we are going to focus on one particular Markov subgroup of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$. This Markov subgroup is usually called the *three dot dynamical system* or *Ledrappier's example*.

The paper is organized as follows. In §2 we describe the Markov subgroups of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$ and describe the three dot system and see how it can be viewed as the inverse limit of a very simple block map, τ, acting on the full two-shift. Section 3 contains several examples of transitive subshifts of the three dot system described in the inverse limit setting. In §4 we define algebraic and semi-algebraic subshifts in the inverse limit setting and examine some examples. It is conjectured that every subshift of the three dot system is either an algebraic or a semi-algebraic subshift. In §5 we state several facts about sofic systems which will be used in the proof of our main result. In §6 we prove the main result, which is that a (σ, τ)-transitive subshift of the full two-shift is algebraic if and only if it is sofic. In §7 we briefly review what is known about invariant measures on the three

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dot example, describe algebraic and semi-algebraic measures, and conjecture that every invariant measure is either algebraic or semi-algebraic. We conclude with a few remarks about how these results compare to similar results in other settings.

2. *Markov subgroups and the three dot example*

Let A be a finite set with the discrete topology. Then $A^{\mathbb{Z}}$ is the set of all bi-infinite sequences on A and $A^{\mathbb{Z}^2}$ is the set of all bi-infinite arrays on A. With the product topology, each is a compact, totally disconnected set. The shift transformation, σ, on $A^{\mathbb{Z}}$ is defined by $\sigma(x)_i = x_{i+1}$ and is a homeomorphism. There are two shift transformations on $A^{\mathbb{Z}^2}$: the horizontal shift transformation, σ_{H}, which is defined by $\sigma_{\mathrm{H}}(x)_{(i,j)} = x_{(i+1,j)}$; and the vertical shift transformation, σ_{V}, which is defined by $\sigma_{\mathrm{V}}(x)_{(i,j)} = x_{(i,j+1)}$. They are commuting homeomorphisms. The case of $A^{\mathbb{Z}}$ is referred to as the *one-dimensional* case and the case of $A^{\mathbb{Z}^2}$ is referred to as the *two-dimensional* case. In both cases a closed shift-invariant set is called a *subshift*. If X is a one-dimensional subshift, a *word* or *block* in X is any block $[x_0, \ldots, x_\ell]$ occurring in some $x \in X$. The collection of all words in a subshift X is denoted by $W(X)$. This paper assumes some familiarity with these spaces and a basic background in one-dimensional symbolic dynamics. The books [**K2**] and [**LM**] are sources for background material.

Let G be a finite group with the discrete topology. Then $G^{\mathbb{Z}}$ and $G^{\mathbb{Z}^2}$ with coordinatewise addition are compact, totally disconnected, topological groups and the shifts are continuous group automorphisms. A subshift which is also a subgroup is called a *Markov subgroup*. By Theorem 1 of [**K1**] every one-dimensional Markov subgroup is topologically conjugate to a subshift of finite type and if it contains a point whose σ-orbit is dense, then it is topologically conjugate to a full-shift. The structure of two-dimensional Markov subgroups is much more complicated.

This paper is concerned with Markov subgroups of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$. Given a finite subset $F \subseteq \mathbb{Z}^2$, define X_F to be the set of $x \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$ with $\sum_{(r,s)\in F} x_{(i+r,j+s)} = 0$ for all $(i, j) \in \mathbb{Z}^2$. X_F will be a Markov subgroup for every finite F. These subgroups are referred to as *shape spaces* because F can be thought of as a shape on a bi-infinite grid with the boxes on the grid indexed by $\mathbb{Z}^2$. The elements in X_F are those arrays which have the property that wherever one places the shape on the array, the sum of the entries in that shape is zero. Kitchens and Schmidt showed that each Markov subgroup of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$ is an intersection of a finite number of shape spaces [**KS**].

Example 2.1. Consider the shape space determined by the subset $L = \{(0, 0), (0, 1), (1, 0)\} \subseteq \mathbb{Z}^2$. The shape this subset determines is the L-shape below.

Given any sequence of zeros and ones one can 'build' a point $x \in X_L$ with this sequence as its zeroth row. The upper half plane will be determined and for each successive row below there will be two choices. X_L is called *Ledrappier's example* or the *three dot dynamical system*. It was first introduced by Ledrappier in [**L**] as an example of a $\mathbb{Z}^2$ action that is two mixing but not three mixing.

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Our goal is to investigate the subshifts of this example.

Define the map $\tau : (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}} \to (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ by $\tau(x_i) = x_i + x_{i+1}$. Note that τ is a continuous, onto, two-to-one group homomorphism that commutes with the shift. If $a \in \{0, 1\}$ is a symbol, we use $\bar{a}$ to denote the *opposite* symbol; that is, $\bar{1} = 0$ and $\bar{0} = 1$. If $x \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$, we use $\bar{x}$ to denote the *opposite* point; that is, $(\bar{x})_i = \bar{x_i}$, for all $i \in \mathbb{Z}$. Note that $\tau(x) = \tau(\bar{x})$. Furthermore, if E is the set of exponents of the non-zero terms in the polynomial $(1 + z)^m \in (\mathbb{Z}/2\mathbb{Z})[z]$ and $x \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$, then $\tau^m(x)_i = \sum_{j \in E} x_{i+j}$.

We form the *inverse limit* or *natural extension*, (X^*, τ^*), of τ acting on X. Since τ is a group homomorphism that commutes with the automorphism σ, X^* is a group, the induced map τ^* is a group automorphism and τ^* commutes with the induced group automorphism σ^*.

It is straightforward to see that there is a natural isomorphism between the dynamical systems (X^*, σ^*, τ^*) and $(X_L, \sigma_{\mathrm{H}}, \sigma_{\mathrm{V}})$.

A subshift of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ is *properly* τ-invariant if it is mapped by τ onto itself. We call a properly τ-invariant subshift a (σ, τ)-*subshift*. Using the isomorphism between (X^*, σ^*, τ^*) and $(X_L, \sigma_{\mathrm{H}}, \sigma_{\mathrm{V}})$ we see that the subshifts of X_L are in one-to-one correspondence with the (σ, τ)-subshifts of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$.

From this point on we concentrate on examining (σ, τ)-subshifts of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$. The interesting ones are those that have a point with a dense (σ, τ)-orbit. We call these (σ, τ)-*transitive subshifts*.

If X is a (σ, τ)-subshift it is easily seen that the topological entropy of σ on X is the same as the topological entropy of τ on X. It is shown in [**KS**] that if σ is mixing on X then either $X = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ or X is the single point of all zeros. Furthermore, Host has observed that if τ is one-to-one everywhere, then X is finite and if X is minimal it consists of a single finite τ-invariant σ-orbit [**H**].

3. *Examples*

Example 3.1. Let $x = \langle \ldots 011011.011011 \ldots \rangle$. Then $\tau(x) = \sigma^2(x)$, $\tau^2(x) = \sigma(x)$ and $\tau^3(x) = x$. So $\{x, \sigma(x), \sigma^2(x)\}$ is a (σ, τ)-transitive subshift.

The finite (σ, τ)-transitive subshifts correspond to points in X_L that are periodic with respect to both σ_{H} and σ_{V}. The periodic points are dense in X_L (see [**KS**]).

Example 3.2. Let $\tilde{x} = \langle \ldots 000.1000 \ldots \rangle$. Let $\tilde{X}$ be the closure of the (σ, τ)-orbit of $\tilde{x}$. This means that $\tilde{X}$ is σ-invariant and τ-invariant but we need to show $\tau(\tilde{X}) = \tilde{X}$. It is enough to show that $\tilde{x} = \lim_{i \to \infty} \sigma^{n_i} \tau^{m_i}(\tilde{x})$, where $1 < m_i$ for all i.

Since $\mathbb{Z}/2\mathbb{Z}$ has characteristic 2 as a field, $(1 + z)^{2^n} = 1 + z^{2^n}$. So coordinate zero and coordinate -2^n are the only non-zero coordinates of $\tau^{2^n}(\tilde{x})$ and $\tilde{x} = \lim_{n \to \infty} \tau^{2^n}(\tilde{x})$. Consequently, $\tilde{X}$ is a transitive (σ, τ)-subshift.

Because the coordinate rule defining τ is addition of consecutive coordinates, the τ-orbit of $\tilde{x}$ looks like Pascal's triangle modulo two. The word $[\tau^n(\tilde{x})_{-n}, \tau^n(\tilde{x})_{-n+1}, \ldots, \tau^n(\tilde{x})_0]$ is the nth row of Pascal's triangle and all other entries are zero. Because Pascal's triangle is easy to work with, we can describe $\tilde{X}$ in detail. We will make use of this description in Example 3.4. One can check that $p_n = [p_{n-2^m}, 0^{(2^{m+1}-n-1)}, p_{n-2^m}]$, where p_n is the nth row of Pascal's triangle and $2^m \leq n < 2^{m+1}$. If $p'_n = [0^{(2^{m+1}-n-1)}, p_{n-2^m}]$,

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then the σ-periodic points in the space are $\langle \dots p'_n, p'_n, p'_n \dots \rangle$. $\tilde{X}$ also has all of the points $\langle \dots p'_n, p'_n, p'_n, 0, 0, 0 \dots \rangle$ and the point whose entries are all zeros. If $x \in \tilde{X}$ has a word $[0, 1^n, 0]$ in it, then all of the words of the form $[1^m]$ that are found in x are subwords of $[0, 1^n, 0]$; that is, $m \leq n$. Hence, $\tilde{X}$ is a proper subshift of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$.

It is not the case that the closure of the (σ, τ)-orbit of any point $x \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ is properly τ-invariant. A trivial example of this is given by the point $x = \langle \dots 1, 1, 1, 1 \dots \rangle$. The closure of the (σ, τ)-orbit of x is the two element set

$$\{\langle \dots 1, 1, 1 \dots \rangle, \langle \dots 0, 0, 0 \dots \rangle\},$$

but $\langle \dots 1, 1, 1 \dots \rangle$ has no τ-preimage in this set.

Example 3.3. Let $x \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ be a point with finitely many of its coordinates equal to one. Let F be the set of coordinates in x that are one. We can write x as $\sum_{i \in F} \sigma^{-i}(\tilde{x})$, where $\tilde{x}$ is the point in Example 3.2. Since F is finite and σ and τ commute with addition,

$$\begin{aligned} \lim_{n\to\infty} \tau^{2^n}(x) &= \lim_{n\to\infty} \tau^{2^n}\left(\sum_{i\in F} \sigma^{-i}(\tilde{x})\right) = \lim_{n\to\infty} \sum_{i\in F} \sigma^{-i}\tau^{2^n}(\tilde{x}) \\ &= \sum_{i\in F} \sigma^{-i}\left(\lim_{n\to\infty} \tau^{2^n}(\tilde{x})\right) = \sum_{i\in F} \sigma^{-i}(\tilde{x}) = x. \end{aligned}$$

Therefore, x also determines a (σ, τ)-transitive subshift.

Another way to construct a (σ, τ)-subshift is to take the τ-limit set of a subshift. Such a set is usually difficult to describe but below is an example of such a construction.

Example 3.4. Let Y be the *Morse–Thue minimal set* [**HM**]. It can be constructed as follows. Define the one-sided Morse–Thue sequence $\langle a_i \rangle$ recursively by $a_0 = 0$ and

$$[a_{2^n}, \dots, a_{2^{n+1}-1}] = [\bar{a}_0, \dots, \bar{a}_{2^n-1}]$$

for $n \in \mathbb{N}$. Define a point $\tilde{y} \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ by $\tilde{y}_0 = a_0$, $\tilde{y}_i = a_i$ and $\tilde{y}_{-i} = \bar{a}_i$ for $i \in \mathbb{N}$. This is the two-sided Morse–Thue sequence.

$$\tilde{y} = \langle \dots 0110.01101001100101101001011001101001 \dots \rangle.$$

The Morse–Thue minimal set is the closure of the σ-orbit of $\tilde{y}$.

Let $\tilde{Y}$ be the τ-limit set of Y so

$$\tilde{Y} = \left\{x : x = \lim_{i\to\infty} \tau^{m^i}(y^i) \text{ for } m_i < m_{i+1} \text{ and } y^i \in Y\right\}.$$

The set $\tilde{Y}$ is a (σ, τ)-subshift. Recall from Example 3.2 the point $\tilde{x}$ and the set $\tilde{X}$. We will show $\tilde{X} = \tilde{Y}$.

First note that for all $m \geq n$, the word $[\tilde{y}_0, \dots, \tilde{y}_{2^n-1}]$ is the opposite of the word $[\tilde{y}_{2^m}, \dots, \tilde{y}_{2^m+2^n-1}]$ and the word $[\tilde{y}_0, \dots, \tilde{y}_{2^n-1}] = [\tilde{y}_{2^m+2^n}, \dots, \tilde{y}_{2^m+2^{n+1}-1}]$. So, for all $m \geq 1$,

$$[\tau^{2^m+1}(\tilde{y})_0, \dots, \tau^{2^m+1}(\tilde{y})_{2^{m+1}-2}] = [0^{2^m-1}, 1, 0^{2^m-1}].$$

Therefore, $\tilde{x} = \lim_{m\to\infty} \sigma^{2^m-1}\tau^{2^m+1}(\tilde{y})$ and hence $\tilde{X} \subseteq \tilde{Y}$.

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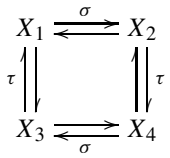
For all $k \in \mathbb{N}$, $[\tilde{y}_{k2^n}, \ldots, \tilde{y}_{(k+1)2^n-1}]$ is either $[\tilde{y}_0, \ldots, \tilde{y}_{2^n-1}]$ or it is the opposite of $[\tilde{y}_0, \ldots, \tilde{y}_{2^n-1}]$.

Consequently, $[\tau^{2^n}(\tilde{y})_{k2^n}, \ldots, \tau^{2^n}(\tilde{y})_{(k+1)2^n-1}] \in \{[0^{2^n}], [1^{2^n}]\}$. By the discussion of Pascal's triangle in Example 3.2, any word of length 2^{n+1} in $\tau^{2^n}(\tilde{y})$ is in $W(\tilde{X})$ and any word of length 2^n in $\tau^{2^n+i}(\tilde{y})$ for $0 \leq i < 2^n$ is in $W(\tilde{X})$. Therefore, $\tilde{Y} \subseteq \tilde{X}$ and $\tilde{Y} = \tilde{X}$.

4. *More examples: algebraic and semi-algebraic subshifts*

In this section we construct algebraic and semi-algebraic subshifts. All known examples of transitive (σ, τ)-subshifts fall into one of these two categories.

Example 4.1. This example exhibits the main ideas of Definition 4.2. Start with the subgroup $\{(0,0),(1,1)\} \subseteq (\mathbb{Z}/2\mathbb{Z})^2$. Then form the full-shift $\{(0,0),(1,1)\}^{\mathbb{Z}}$ on the subgroup. It is a Markov subgroup of $((\mathbb{Z}/2\mathbb{Z})^2)^{\mathbb{Z}}$. Embed $\{(0,0),(1,1)\}^{\mathbb{Z}}$ in $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ as a closed subgroup, X_1, by 'dropping the parentheses'. For each $x \in X_1$, $[x_{2n}, x_{2n+1}]$ equal to $[0,0]$ or $[1,1]$ for every integer n. This subgroup is neither σ-invariant nor τ-invariant. Let $X_2 = \sigma(X_1)$ and consider the union $X_1 \cup X_2$. This union is σ-invariant and is easily seen to be a strictly sofic system of period two. Let $X_3 = \tau(X_1)$ and $X_4 = \tau(X_2)$. Then $X_3 \cup X_4$ is also a strictly sofic system of period two. Since $\{(0,0),(1,1)\}^{\mathbb{Z}}$ is a Markov subgroup of $((\mathbb{Z}/2\mathbb{Z})^2)^{\mathbb{Z}}$, $X_1 = \tau(X_3)$ and $X_2 = \tau(X_4)$. Let $X = X_1 \cup X_2 \cup X_3 \cup X_4$. It is easily seen that X is (σ, τ)-transitive. The set X is illustrated by the following diagram.



Definition 4.2. Let $M \subseteq ((\mathbb{Z}/2\mathbb{Z})^{2^n})^{\mathbb{Z}}$ be a transitive Markov subgroup. Define X_1 to be the closed subgroup of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ constructed as in Example 4.1. Then $\sigma^{2^n}(X_1) = \tau^{2^n}(X_1) = X_1$. Define $X(M)$ to be the (σ, τ)-orbit of X_1. It is a transitive (σ, τ)-subshift of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ and is said to be the *algebraic subshift* determined by M. Subshifts of this type are called *purely algebraic*.

Example 4.3. Let $X(G)$ be the purely algebraic subshift defined in Example 4.1 and let $F = \{x, \sigma(x), \sigma^2(x)\}$, where x is the point from Example 3.1. Define $X(G, F)$ to be $\{y + z : y \in X(G) \text{ and } z \in F\}$. Since σ and τ are both homomorphisms and since F and $X(G)$ are both (σ, τ)-subshifts $X(G, F)$ is also a (σ, τ)-subshift. Since the σ-period of $X(G)$ is two and F consists of a single σ-orbit of period three, $X(G, F)$ is transitive.

Definition 4.4. If $X(M)$ is a purely algebraic subshift and F is a finite orbit with $\tau(F) = F$, then both $X(M)$ and $X(M, F) = \{y+z : y \in X(M) \text{ and } z \in F\}$ are said to be *algebraic subshifts*. We say that $X(M, F)$ is the *algebraic subshift determined by M and F*.

The classification of algebraic subshifts is a natural one in two dimensions. Let M' be a transitive (two-dimensional) Markov subgroup of $((\mathbb{Z}/2\mathbb{Z})^{2^n \times 2^n})^{\mathbb{Z}^2}$. Define $X(M')$ to be the union of the orbit of the subgroup of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$ obtained by dropping the parenthesis from the alphabet of M'. Then M' corresponds to a Markov subgroup, M, of $((\mathbb{Z}/2\mathbb{Z})^{2^n})^{\mathbb{Z}}$

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and the algebraic subshift $X(M)$ in $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ corresponds to $X(M')$ in the usual way. The converse also holds.

Definition 4.5. The nested intersection of algebraic subshifts is said to be a *semi-algebraic subshift*.

Example 4.6. Let $G_n = \{[0^{2^n}], [0^{2^n-1}, 1]\}$ and M_n be the full shift on G_n. Let X be the semi-algebraic subshift $\bigcap_{n\in\mathbb{N}} X(M_n)$. Recall from Examples 3.2 and 3.4 the point $\tilde{x} = \langle \ldots 000.1000 \ldots \rangle$ and its (σ, τ)-orbit closure $\tilde{X}$. We show $\tilde{X} = X$. The point $\tilde{x}$ is in $X(M_n)$ for each n. So $\tilde{X} \subseteq X$. Since any word of length 2^{n+1} in M_n is in $W(\tilde{X})$, any word of length 2^n in $W(X(M_n))$ is also in $W(\tilde{X})$. Therefore, $W(X) \subseteq W(\tilde{X})$ and $X \subseteq \tilde{X}$. So $\tilde{X}$ is semi-algebraic.

Algebraic and semi-algebraic subshifts are defined because (σ, τ)-subshifts seem to be rare. This leads to the following conjecture [**KS**].

CONJECTURE 4.7. *Any transitive (σ, τ)-subshift is algebraic or semi-algebraic.*

It is not hard to show that if α is between 0 and $\log 2$ there is a semi-algebraic subshift whose σ entropy is equal to α.

5. *Sofic background*

In this section we review facts about sofic systems that will be used in the proof of our main result. For a more detailed discussion of sofic systems see [**K2**] and [**LM**].

If S is a subshift and $w \in W(S)$ we define the *follower set* of w to be

$$\mathrm{fol}(w) = \{u \in W(S) : wu \in W(S)\}.$$

There are several equivalent definitions for a sofic system.

BACKGROUND 5.1. *If S a subshift, then the following statements about S are equivalent.*

(1) *There exists a subshift of finite type T and a shift-commuting continuous surjection $\phi : T \to S$.*
(2) *There exists a subshift of finite type T and a continuous, finite-to-one shift-commuting surjection $\phi : T \to S$.*
(3) *There is a directed graph G with labeled vertices such that S is the set of bi-infinite vertex walks on G.*
(4) *There is a directed graph G with labeled edges such that S is the set of bi-infinite edge walks on G.*
(5) *There are a finite number of distinct follower sets for the words in S.*

A subshift satisfying the conditions in Background 5.1 is said to be a *sofic system*.

A *Markov magic word* for a sofic system S is a word $m \in W(S)$ such that whenever $um \in W(S)$ and $mv \in W(S)$, $umv \in W(S)$. All sofic systems have Markov magic words.

A sofic system S is said to be *irreducible* if for every pair of words $w, v \in W(S)$ there is a word $u \in W(S)$ such that $wuv \in W(S)$. An irreducible sofic system has a period. To define the period of an irreducible sofic system S choose a Markov magic word m and define the *period* of S by

$$p(S) = \gcd\{p : \sigma^p(x) = x \text{ for some } x \in S \text{ which contains } m\}.$$

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In fact, the period of a sofic system is independent of one's choice for a Markov magic word used to define it. It is necessary to take into account a magic word since a sofic system with period two can have a fixed point.

BACKGROUND 5.2. *Let S be an irreducible sofic system with period p. Then there exist p subsets $\mathcal{C}_1^S, \ldots, \mathcal{C}_p^S$ satisfying:*

(1) $\bigcup_{i=1}^{p} \mathcal{C}_i^S = S$;

(2) $\sigma(\mathcal{C}_i^S) = \mathcal{C}_{i+1 \bmod p}^S$;

(3) *the $\mathcal{C}_i^S$ form a partition on the points in S containing a Markov magic word.*

Furthermore, up to a cyclic permutation of the indices, these sets are unique.

The sets referred to in Background 5.2 are called the *cyclic subsets* of S. If $\mathcal{C}_1^S, \ldots, \mathcal{C}_p^S$ are the cyclic subsets of S, let

$$\mathcal{C}_i^{S,\ell} = \{[x_{-\ell}, \ldots, x_\ell] : x \in \mathcal{C}_i^S\}.$$

BACKGROUND 5.3. *Let S be an irreducible sofic system with period p. Then for each ℓ there exists N such that for all $n > N$, $u \in \mathcal{C}_k^{S,\ell}$ and $v \in \mathcal{C}_{k+n \bmod p}^{S,\ell}$, there exists an $x \in \mathcal{C}_k^S$ with $[x_{-\ell}, \ldots, x_\ell] = u$ and $[x_{-\ell+n}, \ldots, x_{\ell+n}] = v$.*

If S is a sofic system, an *irreducible component* of S is a subshift of S that is irreducible and is not contained properly in any other irreducible subshift of S. A point $x \in S$ is said to be *wandering* (or σ-wandering) if x is not an element of any irreducible component of S; that is, x contains a word that is not seen at infinitely many coordinates of any element of S. A *connected* graph is a directed graph H for which if v_1 and v_2 are vertices in H, there is a path from v_1 to v_2. A subgraph of a directed graph G is called a *connected component* of G if it is connected and it is not a proper subgraph of another connected subgraph of G. If G is a graph representing S, then the irreducible components of S are represented by some connected components of G.

The definition below is not standard in the literature. It was developed in this paper to help us understand the behavior of wandering points under iterations of τ. If G is a labeled directed graph, we will let $S(G)$ denote the sofic system determined by G.

Definition 5.4. We will call a directed graph G with labeled vertices a *wandering graph* for the sofic system S if it satisfies the following statements.

(1) $S = S(G)$.

(2) For each irreducible component S' of S there is a connected component G' of G for which $S' = S(G')$. These connected components are completely disconnected from each other and the rest of the graph.

(3) The rest of the graph can be partitioned into 'chains': $C^1, \ldots, C^n$. Each C^i consists of a finite number of connected components: $C_1^i, \ldots, C_{k_i}^i$ for which there is one path from C_j^i to C_{j+1}^i that does not pass through another connected component, $1 \leq j < k_i$. Furthermore, for each i, C_1^i has no in-edges, $C_{k_i}^i$ has no out-edges, and

$$S(C^i) \not\subseteq S(C^i / C_1^i) \cup S(C^i / C_{k_i}^i) \cup \bigcup_{j \neq i} S(C^j) \cup I,$$

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where I is the union of the irreducible components of S, C^i/C^i_1 is the graph determined by deleting the vertices and edges of C^i_1 from C^i and $C^i/C^i_{k_i}$ is defined similarly.

We note that a wandering graph for a sofic subshift S contains non-empty chains if and only if S has σ-wandering points. It is not hard to construct a wandering graph from any graph which determines a given sofic system.

6. *Sofic systems as algebraic subshifts*

In this section we prove the following result.

THEOREM 6.1. *Let $X \subseteq (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ be a (σ, τ)-transitive subshift. Then X is algebraic if and only if it is sofic.*

To prove the theorem we first establish several lemmas. We use $|S|$ to denote the cardinality of a set S.

LEMMA 6.2. *Let G be an abelian group. If $A, B \subseteq G$ are finite subsets then $|A+B| \geq |B|$ and $|A + B| = |B|$ if and only if there is a subgroup $H \subseteq G$ such that B is a collection of cosets of H and A is a subset of a coset of H. If $|A + B| = |A| = |B|$ then A and B are both cosets of H.*

Proof. Given $A, B \subseteq G$, $A + B = \bigcup_{a\in A}(a + B)$. Since $|a + B| = |B|$ for each $a \in A$, $|A + B| \geq |B|$.

If there is a subgroup $H \subseteq G$ with $B = \bigcup_i (b_i + H)$ and $A \subseteq a + H$, then $A + B = \bigcup_i((a + b_i) + H)$ and $|A + B| = |B|$. Conversely, suppose $|A + B| = |B|$. Since $A + B = \bigcup_{a\in A}(a + B)$ all of the sets $a + B$ are the same. The stabilizer of the set B is the subgroup $S(B) = \{g : g + B = B\}$. Given $a \in A$ we see that A is a subset of the coset $a + S(B)$. Also, $B = B + S(B) = \bigcup_{b\in B}(b + S(B))$. This means that B is a collection of cosets of $S(B)$.

If $|A| = |B|$, then B must be a single coset and A must be all of one coset. □

For notational simplicity define the group $G_\gamma = (\mathbb{Z}/2\mathbb{Z})^{2^\gamma}$. Define a homomorphism τ_γ from $G_\gamma^{\mathbb{Z}}$ onto itself by $(\tau_\gamma(x))_i = x_i + x_{i+1}$. It is a slight generalization of the map τ defined in §2. To avoid confusion, we refer to the shift operator on G_γ as σ_γ.

Recall the definitions of cyclic subset and $\mathcal{C}_i^{S,\ell}$ from §5.

LEMMA 6.3. *If $T \subseteq (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ is an irreducible subshift of period p, then $\tau(T)$ is an irreducible subshift whose period divides p.*

Proof. First we show τ maps irreducible subshifts onto irreducible subshifts. Let $T \subseteq (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ be an irreducible subshift. Let u and v be two words in $\tau(T)$. Then there exist $u', v' \in W(T)$ with $\tau(u') = u$ and $\tau(v') = v$. Since T is irreducible, there exists $x \in T$ and $n \in \mathbb{N}$ such that x sees u' at its zeroth coordinate and x sees v' at its nth coordinate. However, then $\tau(x) \in \tau(T)$ and $\tau(x)$ sees u at its zeroth coordinate and v at its nth coordinate. So $\tau(T)$ is irreducible.

Now we show that the period of $\tau(T)$ divides p. Choose a Markov magic word $m \in W(\tau(T))$. Choose $w \in W(T)$ with $\tau(w) = m$. Every word is a subword of

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some Markov magic word. So there is a Markov magic word $m' \in W(T)$ that contains w as a subword. Then $\tau(m')$ contains m as a subword and every σ-periodic point $x \in T$ containing m' maps to a σ-periodic point containing m whose σ-period divides the σ-period of x. So

$$\gcd\{p : \sigma^p(x) = x \text{ for some } x \in S \text{ which contains } m\}$$

divides

$$\gcd\{p : \sigma^p(x) = x \text{ for some } x \in S \text{ which contains } m\}.$$

Therefore, the σ-period of $\tau(T)$ divides the σ-period of T. □

LEMMA 6.4. *Let S be an irreducible sofic system contained in $G_\gamma^{\mathbb{Z}}$. Suppose S has odd period p and $\tau_\gamma^q(S) = S$ for odd $q \geq 1$. Then for every ℓ there exists a subgroup $\mathcal{M}_\ell$ of $G_\gamma^{2\ell+1}$ such that each of the cyclic subsets $\mathcal{C}_k^{\tau_\gamma^j(S),\ell}$ is a coset of $\mathcal{M}_\ell$.*

Proof. By Lemma 6.4, since $\tau_\gamma^q(S) = S$, each $\tau_\gamma^j(S)$ is irreducible and has period p.

Fix $\ell \geq 0$. Then, by Background 5.3, since each $\tau_\gamma^j(S)$ is irreducible, there is an N so that for all $n \geq N$, $u \in \mathcal{C}_k^{\tau_\gamma^j(S),\ell}$ and $v \in \mathcal{C}_{k+n \bmod p}^{\tau_\gamma^j(S),\ell}$, there is an $x \in S$ with $[x_{-\ell}, \ldots, x_\ell] = u$ and $[x_{-\ell+n}, \ldots, x_{\ell+n}] = v$, for every j.

Choose r so $2^r \geq N$ and so that $2^r \equiv 1 \bmod pq$. Such an r exists because p and q are odd. Then $\tau_\gamma^{2^r}(S) = \tau_\gamma(S)$. For each j with $1 < j < q$, there is a t so that $\tau_\gamma^{2^r}(\mathcal{C}_k^{\tau_\gamma^{j-1}(S)}) = \mathcal{C}_{k+t \bmod p}^{\tau_\gamma^j(S)}$ for each k. Cyclically permute the subscripts on the cyclic subsets of $\tau_\gamma^j(S)$ so that $\tau_\gamma^{2^r}(\mathcal{C}_k^{\tau_\gamma^{j-1}(S)}) = \mathcal{C}_k^{\tau_\gamma^j(S)}$ for $1 < j < q$.

Let $u \in \mathcal{C}_k^{\tau_\gamma^{j-1}(S),\ell}$ and $v \in \mathcal{C}_{k+1 \bmod p}^{\tau_\gamma^{j-1}(S),\ell}$ for fixed $1 < j < q$ and $1 \leq k \leq p$. Then there exists $x \in \mathcal{C}_k^{\tau_\gamma^{j-1}(S)}$ with $[x_{-\ell}, \ldots, x_\ell] = u$ and $[x_{-\ell+2^r}, \ldots, x_{\ell+2^r}] = v$. However, $\tau_\gamma^{2^r}(x) \in \mathcal{C}_k^{\tau_\gamma^j(S)}$ and $[\tau_\gamma^{2^r}(x)_{-\ell}, \ldots, \tau_\gamma^{2^r}(x)_\ell] = u+v$. Furthermore, for every $w \in \mathcal{C}_k^{\tau_\gamma^j(S),\ell}$, $w = u + v$ for some $u \in \mathcal{C}_k^{\tau_\gamma^{j-1}(S),\ell}$ and $v \in \mathcal{C}_{k+1 \bmod p}^{\tau_\gamma^{j-1}(S),\ell}$. Therefore, for $1 < j < q$ and $1 \leq k \leq p$,

$$\mathcal{C}_k^{\tau_\gamma^j(S),\ell} = \mathcal{C}_k^{\tau_\gamma^{j-1}(S),\ell} + \mathcal{C}_{k+1 \bmod p}^{\tau_\gamma^{j-1}(S),\ell}.$$

Similarly, there is some t with $1 \leq t \leq p$ such that

$$\mathcal{C}_{k+t \bmod p}^{S,\ell} = \mathcal{C}_k^{\tau_\gamma^{q-1}(S),\ell} + \mathcal{C}_{k+1 \bmod p}^{\tau_\gamma^{q-1}(S),\ell}$$

for each k.

By Lemma 6.2, for $1 < j < q$ and $1 \leq k \leq p$,

$$\left|\mathcal{C}_k^{\tau_\gamma^j(S),\ell}\right| \geq \max\left\{\left|\mathcal{C}_k^{\tau_\gamma^{j-1}(S),\ell}\right|, \left|\mathcal{C}_{k+1 \bmod p}^{\tau_\gamma^{j-1}(S),\ell}\right|\right\}$$

and

$$\left|\mathcal{C}_k^{S,\ell}\right| \geq \max\left\{\left|\mathcal{C}_{k-t \bmod p}^{\tau_\gamma^{q-1}(S),\ell}\right|, \left|\mathcal{C}_{k+1-t \bmod p}^{\tau_\gamma^{q-1}(S),\ell}\right|\right\}$$

for each k.

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Combining these inequalities M times we get

$$\left|\mathcal{C}_k^{\tau_\gamma^j(S),\ell}\right| \geq \max\left\{\left|\mathcal{C}_{k-tm \bmod p}^{\tau_\gamma^{j-M \bmod q}(S),\ell}\right|, \left|\mathcal{C}_{k-tm+1 \bmod p}^{\tau_\gamma^{j-M \bmod q}(S),\ell}\right|, \ldots, \left|\mathcal{C}_{k-tm+M \bmod p}^{\tau_\gamma^{j-M \bmod q}(S),\ell}\right|\right\}$$

where m is the greatest integer less than M/q.

So, for all $M > p$ and for $1 \leq j < q$ and $1 \leq k \leq p$

$$\left|\mathcal{C}_k^{\tau_\gamma^j(S),\ell}\right| \geq \max\left\{\left|\mathcal{C}_1^{\tau_\gamma^{j-M \bmod q}(S),\ell}\right|, \left|\mathcal{C}_2^{\tau_\gamma^{j-M \bmod q}(S),\ell}\right|, \ldots, \left|\mathcal{C}_p^{\tau_\gamma^{j-M \bmod q}(S),\ell}\right|\right\}.$$

However, this implies that for $1 \leq j, j' < q$ and $1 \leq k, k' \leq p$

$$\left|\mathcal{C}_k^{\tau_\gamma^j(S),\ell}\right| = \left|\mathcal{C}_{k'}^{\tau_\gamma^{j'},\ell}\right|.$$

Lemma 6.2 now shows that these are all cosets of some subgroup $\mathcal{M}_\ell$ of $G_\gamma^{2\ell+1}$. □

LEMMA 6.5. *Let S be an irreducible sofic system contained in $G_\gamma^{\mathbb{Z}}$. Suppose that S has odd period and $\tau_\gamma^q(S) = S$ for odd $q \geq 1$. Then S is a Markov subgroup of $G_\gamma^{\mathbb{Z}}$ plus a finite $(\sigma_\gamma, \tau_\gamma^q)$-invariant set.*

Proof. By Lemma 6.4 there is a subgroup $\mathcal{M}_\ell$ of $G_\gamma^{2\ell+1}$ so that each cyclic subset, $\mathcal{C}_k^{\tau_\gamma^j(S),\ell}$, is one of its cosets.

For each ℓ let M_ℓ be the set of $x \in G_\gamma^{\mathbb{Z}}$ with $[x_{-\ell}, \ldots, x_\ell]$ in $\mathcal{M}_\ell$. Let M be the intersection of all the M_ℓ. It is a closed subgroup of $G_\gamma^{\mathbb{Z}}$ and each cyclic subset, $C_k^{\tau_\gamma^j(S)}$, is one of its cosets.

For any $x \in C_k^S$, $M = x + C_k^S$ so

$$\sigma_\gamma(M) = \sigma_\gamma(x) + \sigma_\gamma(C_k^S) = \sigma_\gamma(x) + C_{k+1}^S = M.$$

Consequently, M is a Markov subgroup of $G_\gamma^{\mathbb{Z}}$.

Let x be a σ_γ-periodic point in S. Then $\tau_\gamma^j(x)$ is a σ_γ-periodic point whose period is less than or equal to that of x. Because of this, we can choose x so that the σ_γ-orbit of x is mapped onto the σ_γ-orbit of x via τ_γ^q. Let F be the σ_γ-orbit of x. Then $S = M + F$. □

LEMMA 6.6. *Let $S \subseteq (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ be an irreducible sofic system that is invariant under some power of τ. Then for all sufficiently large γ, S is a finite union of σ^{2^γ}-sofic systems. Each is irreducible under σ^{2^γ}. All have the same odd σ^{2^γ}-period and are invariant under $(\tau^{2^\gamma})^q$ for the same odd q. Each can be thought of as an irreducible sofic system contained in $G_\gamma^{\mathbb{Z}}$ with odd period and invariant under an odd power of τ_γ.*

Proof. Let S be as stated, let $2^\delta p$ be the period of S and let $2^\xi q$ be such that $\tau^{2^\xi q}$ maps S onto itself and fixes each of the cyclic subsets of S. Here p and q are odd. Let γ be any integer larger than δ and ξ. Then S decomposes into 2^δ sets S_i for $i = 0, \ldots, 2^\gamma - 1$ given by $S_i = \bigcup_{n=0}^{p-1} C_{i+n2^\gamma \bmod p}^S$. Each (S_i, σ^{2^γ}) is an irreducible sofic system with period p and is $(\tau^{2^\gamma})^q$ invariant.

Think of S sitting in $G_\gamma^{\mathbb{Z}}$. Then it is composed of 2^δ irreducible sofic systems each invariant under τ_γ^q.

The entire (σ, τ)-orbit of S breaks up into a collection of σ^{2^γ}-irreducible sofic systems with period p each of which is fixed by the qth power of τ^{2^γ}. □

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Lemmas 6.5 and 6.6 together say that if $S \subseteq (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ is an irreducible sofic system that is invariant under some power of τ, then for all sufficiently large γ we can think of it as sitting in $G_\gamma^{\mathbb{Z}}$ where it breaks up into a finite collection of sets. Then there is a Markov subgroup and an odd q so that each of the sets is the Markov subgroup plus a finite (σ, τ^q)-invariant set.

LEMMA 6.7. *Let S be a τ-invariant sofic system. Let x be an element of S whose σ-orbit is dense in some irreducible subshift S'' of S and whose σ^{-1}-orbit is dense in some irreducible subshift S' of S. Suppose that m is the largest integer for which there exists $y \in S'$ with $y_i = x_i$ for all $i \leq m$ and n is the smallest integer for which there exists $y \in S''$ with $y_i = x_i$ for all $i \geq n$. Then:*

(1) *$\tau(S')$ and $\tau(S'')$ are irreducible subshifts of S;*
(2) *the σ-orbit of $\tau(x)$ is dense in $\tau(S'')$;*
(3) *the σ^{-1}-orbit of $\tau(x)$ is dense in $\tau(S')$;*
(4) *$m-1$ is the largest integer for which there exists $y \in \tau(S')$ with $y_i = \tau(x)_i$ for all $i \leq m-1$; and*
(5) *n is the smallest integer for which there exists $y \in \tau(S'')$ with $y_i = \tau(x)_i$ for all $i \geq n$.*

Proof. By Lemma 6.3, $\tau(S')$ and $\tau(S'')$ are irreducible.

Suppose that w is a word in $\tau(S')$. Then there exists a word u in S' for which $\tau(u) = w$. Since the σ^{-1}-orbit of x is dense in S', x sees u at infinitely many negative coordinates. So $\tau(x)$ sees w at infinitely many negative coordinates. Therefore, the σ^{-1}-orbit of $\tau(x)$ is dense in $\tau(S')$. Similarly, the σ-orbit of $\tau(x)$ is dense in $\tau(S'')$.

Choose $y \in S'$ with $y_i = x_i$ for all $i \leq m$. Then $\tau(y) \in \tau(S')$ and $\tau(y)_i = \tau(x)_i$ for all $i \leq m-1$. Suppose there exists $y \in \tau(S')$ with $y_i = \tau(x)_i$ for all $i \leq k$. Then there exists $z \in S'$ with $\tau(z) = y$. Either $z_i = x_i$ for all $i \leq k+1$ or $z_i = \bar{x}_i$ for all $i \leq k+1$. Suppose $z_i = \bar{x}_i$ for $i \leq k+1$. Since x sees every word in S' infinitely often in its σ^{-1}-orbit, $S' = \bar{S}'$. So $\bar{z} \in S'$ and $\bar{z}_i = x_i$ for all $i \leq k+1$. Therefore, $k \leq m-1$; that is, $m-1$ is the largest integer for which there exists a $y \in \tau(S')$ with $y_i = \tau(x)_i$ for all $i \leq m-1$. Similarly, n is the smallest integer for which there exists $y \in \tau(S'')$ with $y_i = \tau(x)_i$ for all $i \geq n$. □

Recall the definition of a wandering graph (Definition 5.4). Recall also that if G is a labeled directed graph, $S(G)$ is the subshift determined by G; that is, the set of bi-infinite walks on G.

LEMMA 6.8. *Let $S \subseteq (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ be a (σ, τ)-transitive sofic system. Then it contains no σ-wandering points.*

Proof. Let G be a wandering graph for S. Suppose S has σ-wandering points. Then G has a non-empty chain. Let $C = C^1$ be a chain in G whose connected components are $C_1, \ldots, C_n$. Since $S(C) \nsubseteq S(C/C_1) \cup S(C/C_n) \cup \bigcup_{j \neq 1} S(C^j) \cup I$, where I is the union of the irreducible components of S, there is a word w in S for which any walk which determines w begins in C_1 and ends in C_n. So, since there are no edges into C_1 and no edges out of C_n, the σ^{-1}-limit set of any point $x \in S$ that sees w must be a subset of $S(C_1)$ and the σ-limit set of such a point must be a subset of $S(C_n)$. Furthermore, we can choose w

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so that if $x \in S$ with $[x_k, \dots, x_\ell] = w$ then the largest integer m for which there exists $y \in S(C_1)$ with $y_i = x_i$ for all $i \leq m$ lies between k and ℓ and the smallest integer n for which there exists $y \in S(C_n)$ with $y_i = x_i$ for all $i \geq n$ lies between k and ℓ. In other words, $|m - n| \leq |w|$. Fix such a w.

If z is a (σ, τ)-transitive point in S, then, since τ maps S onto S, $\tau^r(z)$ is (σ, τ)-transitive for every $r > 0$. So there are infinitely many integers k for which $\tau^k(z)$ sees w. Choose two such integers, $k < k'$, with $k' - k > 2|w|$. Say

$$[\tau^k(z)_r, \dots, \tau^k(z)_s] = [\tau^{k'}(z)_{r'}, \dots, \tau^{k'}(z)_{s'}] = w.$$

Choose N larger than $\max\{|r|, |s|, |r'|, |s'|\} + (k' - k)$. Then for any point x with

$$x_i = \tau^k(z)_i \quad \text{for } |i| \leq N, [x_r, \dots, x_s] = [\tau^{k'-k}(x)_{r'}, \dots, \tau^{k'-k}(x)_{s'}] = w.$$

Since any path which determines the word w begins in C_1 and ends in C_n and since $\tau^k(z)$ sees w, there exists $x \in S$ with $x_i = \tau^k(z)_i$ for $|i| \leq |N|$, whose σ^{-1}-limit set is dense in $S(C_1)$, and whose σ-limit set is dense in $S(C_n)$. Fix such an x.

Let m be the largest integer for which there exists $y \in S(C_1)$ with $y_i = x_i$ for all $i \leq m$ and let n be the smallest integer for which there exists $y \in S(C_n)$ with $y_i = x_i$ for all $i \geq n$. Then $|n - m| \leq |w|$. By Lemma 6.7, $m - (k' - k)$ is the largest integer for which there exists $y \in S(C_1)$ with $y_i = \tau^{k'-k}(x)_i$ for all $i \leq m - (k' - k)$ and n is the largest integer for which there exists $y \in S(C_n)$ with $y_i = \tau^{k'-k}(x)_i$ for all $i \geq n$. However, $|(m-(k'-k))-n| = |(k-k')+(m-n)| \geq |k'-k|-|m-n| > 2|w|-|w| = |w|$. This contradicts the fact that $|(m - (k' - k)) - n| \leq |w|$. The contradiction arose from the supposition that S contains σ-wandering points. Therefore, S does not contain any σ-wandering points. □

Now we are ready to prove the main theorem.

Proof of Theorem 6.1. First we observe that a purely algebraic subshift is a sofic system. Let $X(M)$ be a purely algebraic subshift determined by a transitive Markov subgroup $M \subseteq ((\mathbb{Z}/2\mathbb{Z})^{2^k})^{\mathbb{Z}}$. By [**K1**], M is conjugate to a full shift in $((\mathbb{Z}/2\mathbb{Z})^{2^k})^{\mathbb{Z}}$. As it sits in $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ it is not invariant under σ but rather under σ^{2^k}. It follows that the irreducible subshift $\bigcup \sigma^i(M)$ has finitely many distinct follower sets and so is sofic. By the same reasoning, $X(M)$ has finitely many follower sets and is sofic. If F is a finite (σ, τ)-invariant set then the same reasoning shows that the algebraic subshift $X(M, F)$ has finitely many follower sets and is sofic.

Next we show a (σ, τ)-transitive sofic system is algebraic. Let $S \subseteq (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ be a sofic system. Suppose $\tau(S) = S$ and S is (σ, τ)-transitive. By Lemma 6.8, S contains no σ-wandering points. Since S is (σ, τ)-transitive τ acts transitively on the irreducible components of S. Pick one irreducible component S'. Use Lemma 6.6 to pick out one σ^{2^γ}-irreducible component of S' and think of it sitting in $((\mathbb{Z}/2\mathbb{Z})^{2^\gamma})^{\mathbb{Z}}$. Apply Lemma 6.5 to write it as $H + F'$ where H is a Markov subgroup of $((\mathbb{Z}/2\mathbb{Z})^{2^\gamma})^{\mathbb{Z}}$ and F' is a finite (σ, τ^q)-transitive set.

Now think of H as a σ^{2^γ}-invariant subgroup of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$. Also, go back to thinking of S and S' sitting in $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$.

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Consider the σ-periodic points of S. Let x be a point that is periodic under σ and τ and that has at least one point in its (σ, τ)-orbit in each of S irreducible components and in each corresponding cyclic subset. Let F be the (σ, τ)-orbit of x. Then $S = H + F$ and S is algebraic. □

Example 6.9. Let x be a point with finitely many of its coordinates equal to 1. Let F be the set of coordinates that are 1. As seen in Example 4.3, the closure of the (σ, τ)-orbit of this point, X, is properly (σ, τ)-invariant. Let $X(M_n)$ be the purely algebraic subshifts defined in Example 4.6. Then the $X(G_n)$ are sofic systems.

Let $Y_n = \left\{ \sum_{i \in F} \sigma^i(y) : y \in X(G_n) \right\}$. Observe that

$$\begin{aligned} \bigcap_{n \in \mathbb{N}} Y_n &= \{y : y \in Y_n \text{ for each } n\} \\ &= \left\{ \sum_{i \in F} \sigma^i(y) : y \in X(G_n) \text{ for each } n \right\} \\ &= \left\{ \sum_{i \in F} \sigma^i(y) : y \in \tilde{X} \right\} \\ &= X. \end{aligned}$$

(The second equality is true because the $X(M_n)$ are nested.)

We will show that if S is a sofic system and F is any finite subset of $\mathbb{Z}$, then $S' = \left\{ \sum_{i \in F} \sigma^i(x) : x \in S \right\}$ is also a sofic system. This will imply that Y_n is sofic and, hence, algebraic. So it will show that X is semi-algebraic.

Note that if $w \in W(S')$, then $w = \sum_{i \in F} w_i$, for some words $w_i \in W(S)$ that have the same length as w and

$$f(w) = \bigcup_{\sum w_i = w} \left\{ \sum_{i \in F} u_i : \text{the } u_i \text{ are the same length and } u_i \in f(w_i) \right\}.$$

The set of these sets is clearly finite since there are finitely many follower sets of elements in $X(M_n)$. Therefore, Y_n is sofic and hence algebraic. So X is semi-algebraic.

7. *Invariant measures on the three dot dynamical system*

The final goal of this investigation is to describe the (σ_H, σ_V)-invariant measures on the three dot dynamical system. As for the investigation of closed, invariant subsets we can reduce to describing (σ, τ)-invariant measures on $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$.

A (σ, τ)-invariant, Borel, probability measure on $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ is *ergodic* if the only (σ, τ)-invariant sets have measure zero or one.

Any compact group has a Haar measure that is preserved by automorphisms. The $(\frac{1}{2}, \frac{1}{2})$ Bernoulli measure is Haar measure on $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$. Since σ is an automorphism and τ is an onto homomorphism both maps preserve Haar measure and it is (σ, τ)-invariant. This leads to the first conjecture.

CONJECTURE 7.1. *The only ergodic, fully supported, (σ, τ)-invariant measure on $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ is Haar measure.*

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The *support* of a measure is the smallest closed set with measure one. If $F \subseteq (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ is a finite, (σ, τ)-transitive set then the equidistributed measure whose support is F is an ergodic, (σ, τ)-invariant measure.

Next we construct algebraic and semi-algebraic measures. Recall the construction of purely algebraic subshifts in §4. Start with a Markov subgroup $M \subseteq ((\mathbb{Z}/2\mathbb{Z})^{2^k})^{\mathbb{Z}}$. The group M has a Haar measure μ. The *purely algebraic* measure μ_M defined by μ is the Haar measure

$$\frac{1}{2^{2^k}}\left(\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^k-1}\sigma^i(\mu)\tau^j(\mu)\right).$$

It is the measure obtained by averaging μ over the orbit of M. This measure is (σ, τ)-invariant, ergodic and has support equal to $X(M)$. Given a finite (σ, τ)-invariant set F we defined the algebraic subshift $X(M, F)$. The purely algebraic measure defines an *algebraic* measure on $X(M, F)$ by averaging over the orbit of $X(M)$. If $X = \bigcap X(M_i)$ is a semi-algebraic subshift we define a *semi-algebraic* measure on X to be the limit of $\{\mu_{X(M_i)}\}$. These constructions produce a variety of invariant measures and lead to the next conjecture.

CONJECTURE 7.2. *Any (σ, τ)-invariant, ergodic measure on $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ is a Haar measure, has finite support or is an algebraic or semi-algebraic measure.*

It is easily seen that if μ is a measure on $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ that is mixing under μ, then it is invariant under group translation and so is a Haar measure. Lind in [**Li**] made some further observations. In particular, he showed that if μ_p is the Bernoulli measure $(p, 1-p)$, then

$$\lim \frac{1}{N}\sum_{j=0}^{N-1}\tau^j\mu_p$$

is a Haar measure.

A related problem is the *times two–times three* problem. Let f_2 be the angle doubling map on the circle and f_3 the angle tripling map on the circle. Each map has uncountably many ergodic, invariant measures. However, consider the $\mathbb{N}^2$ action generated by the two maps and ask to identify the closed invariant subsets and the invariant measures. Furstenberg posed this problem in [**F**] and showed that the only closed invariant subsets are finite or the entire circle. He conjectured that the only invariant measures are Lebesgue measure and measures whose support is a finite set. Rudolph showed in [**R**] that an invariant measure with positive entropy with respect to f_2 or f_3 is Lebesgue. Other work on this problem has been done by Lyons [**Ly**] and Johnson [**J**]. The complete answer is still unknown.

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