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The virtual power principle in fluid mechanics

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A conceptual framework on analytical mechanics for continuous fluid medium, which connects the fluid motion and all of the (internal and external) forces with mechanical power, is proposed by using the virtual power and the virtual velocity. Based on this framework, it is found that the internal virtual power is equal to the external virtual power in fluid dynamics, which is called the virtual power principle. This framework is also proved to be equivalent to the vector dynamics (Cauchy's equation or Navier–Stokes equation). Furthermore, based on the virtual power principle, a theorem is introduced for continuous fluid medium, which indicates the relationship between the force (or torque) acting on a body immersed in a fluid and the specified virtual power. Subsequently, according to Galilean invariance, the detailed relationship for Newtonian fluids in incompressible flows is derived and used to illustrate the mechanisms on instantaneous forces: the added inertial effects, the boundary energy flux and dissipation effects, the vortex contribution, and the explicit body force contribution. As an application of the principle, the advantage of the V formation flight of geese is preliminarily discussed in the view of aerodynamics. Specifically, the total drag of the flock is reduced by contrast with the simple sum of the drag in solo fight and the optimal angle of V ranges from 60° to 120°. The principle could be a useful approach to reveal the contributions of the flow structures and the moving or deforming boundaries to the force and torque acting on a body, especially in a multibody system.

Key words: aerodynamics, general fluid mechanics, vortex dynamics

1. Introduction

There are two kinds of analysis methods in classical mechanics: one is vectorial mechanics and the other is analytical mechanics. The former was founded by Galileo and Newton in the 17th century, and the latter was constructed from the 18th century onwards by Lagrange, Hamilton, Jacobi and others in historical sequence. As we know, both methods have been used extensively to analyse mechanical systems idealized as collections of particles, rigid bodies and distributed parameter systems. Instead of using vector forces as in vector mechanics, the central content of analytical mechanics is to apply the scalar functions that are closely related to the mechanical kinetic and potential energies of dynamical systems, in order to analyse the relationships of mechanical system motion and to propose the efficient methods for accommodating

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displacement constraints and constraint forces, on the basis of the concepts of virtual displacements and virtual work.

In the field of fluid mechanics, there have been some famous works founded, such as the Cauchy equation, Euler equation and Navier–Stokes equation, which can be classified into vectorial mechanics. But up to now, the analytical mechanics has not been founded in fluid mechanics, although it would be more appropriate to investigate the relationship between the forces acting on the body/bodies and the fluid motion. We have noticed that there are some studies on the total hydrodynamic forces exerted on the rigid body in incompressible flows with a uniform density (e.g. Quartapelle & Napolitano 1983; Howe 1989; Chang 1992; Howe 1995; Wells 1996; Howe, Lauchle & Wang 2001). Chang & Lei (1996) has extended the study to compressible flows. In the book by Wu, Ma & Zhou (2006), these works are called 'projection theory'. More recently, the forces acting on a specified body in an arbitrary incompressible flow with many bodies have been given by Ragazzo & Tabak (2007), Chang, Yang & Chu (2008), Magnaudet (2011). These studies have a common feature, that the forces and torque can be expressed in a variety of scalar forms by the dot product of the Navier–Stokes equation and an auxiliary velocity, although the motivation and method of each work are different. For example, Chang (1992) used the associated potential flow (caused by a moving body) to extract the potential force from the real viscous flow so that the vorticity force appears naturally to make a convergent force representation, Howe (1995) expressed the component of the force by using the property of the Kirchhoff vector, and Magnaudet (2011) provided the expressions for the various forces and torques on the body especially in a non-uniform flow. Therefore, it is attractive to reform a conceptual framework on analytical mechanics for fluid mechanics.

In this article, the author will provide a unifying perspective on analytical mechanics to analyse the fluid motion and all of the (internal and external) forces. Compared with the rigid and elastic systems, fluids have several different features, such as the continuous deformation under the sustaining shear stress, the strain rate in the constitutive relation and the viscous dissipation rate. Consequently, some of the traditional concepts of rigid and elastic systems are not suitable for fluids. Therefore, a set of new concepts different from those in classical analytical mechanics will be introduced for fluids, for instance, the virtual velocity and the virtual power are used instead of the virtual displacement and the virtual work, respectively. In fact, the concept of virtual power has existed in continuum mechanics (Maugin 1980) and has been used to analyse the irreversible thermomechanical processes for a material body (Pao, Wang & Chen 2011).

This article is structured as follows. We first establish the forms of the virtual power principle in fluid mechanics according to the identical equation of the power done by the internal stress in § 2. Then, a theorem about the force and torque acting on a body immersed in a fluid is deduced in § 3, based on which a force formula for Newtonian fluids in incompressible flows and its general form based on Galilean invariance are provided in this section. Furthermore, we apply the present principle to analyse the thrust and lift acting on the leading and trailing geese in their V formation flight in § 4. Finally, concluding remarks are provided in § 5.

2. The formulation of the virtual power principle in fluid dynamics

2.1. *The virtual power of the stress tensor*

Fluid mechanics is a subdiscipline of the continuum mechanics, which simply means that its variation in properties is so smooth that the differential calculus can be used

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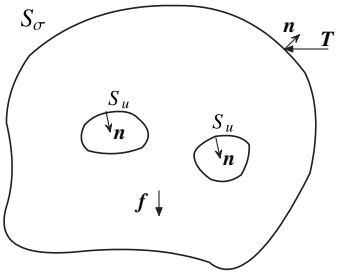


FIGURE 1. Flow domain to be analysed and notation, the given stress and velocity surface S_σ and S_u respectively, the mass body force vector $\boldsymbol{f}$, the stress vector $\boldsymbol{T}$ and the unit normal vector $\boldsymbol{n}$ pointing out of the fluid domain.

to analyse it. As a special case of Newton's second law, based on the mass-conserving assumption, the motion of fluids satisfies Cauchy's equation (also called the general differential form of the conservation of momentum, or the first Cauchy's law),

$$\rho \frac{\mathrm{d}u_i}{\mathrm{d}t} = \frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i, \tag{2.1}$$

where ρ is the fluid density, $\mathrm{d}/\mathrm{d}t$ is the material derivative, u_i and f_i, respectively, represent the velocity vector and the mass body force vector, and σ_{ji} is a tensor that represents the surface forces applied on a fluid particle (called the stress tensor), which denotes the magnitude of the ith component of the stress vector at the element of the coordinate plane with normal vector in the jth direction. At the boundary of the fluid, the appropriate boundary conditions are given by

$$\left.\begin{array}{ll} n_j\sigma_{ji} = T_i, & \text{on } S_\sigma, \\ u_i = (u_w)_i, & \text{on } S_u, \end{array}\right\} \tag{2.2}$$

where n_j is the unit outward normal vector, T_i is the stress vector on the stress surface S_σ according to the Euler–Cauchy stress principle, and $(u_w)_i$ is the velocity given on the velocity surface S_u. Here, the whole boundary of the flow field is described as $S = S_\sigma \cup S_u$, shown in figure 1.

In order to derive the formulation, a continuous velocity field u_i' that is a single-valued function and satisfies the velocity boundary condition on S_u is introduced. We call it 'the possible velocity'. The superscript $(\cdot)'$ emphasizes that the possible velocity field is independent of the stress field, which means that there is no need to specify whether the velocity u_i' is real or virtual.

If the internal stress σ_{ji} undergoes a strain rate $\partial u_i'/\partial x_j$ in the possible velocity field u_i', the total virtual power done by the stress tensor can be expressed,

$$\int_V \sigma_{ji}\frac{\partial u_i'}{\partial x_j}\,\mathrm{d}V = -\int_V \frac{\partial \sigma_{ji}}{\partial x_j}u_i'\,\mathrm{d}V + \int_S n_j\sigma_{ji}u_i'\,\mathrm{d}S \tag{2.3}$$

The left-hand side of (2.3) is called 'the virtual strain-rate power', and the first term of the right-hand side can be called 'the virtual complementary strain-rate power'. The strain-rate power and the complementary stain-rate power correspond respectively to the strain energy and the complementary strain energy in elasticity. Leaving the

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physics aside, equation (2.3) is only a mathematically identical equation, which further demonstrates that the possible velocity is surely independent of the stress.

Substituting (2.1) and the boundary conditions (2.2) into (2.3), we get

$$\int_V \sigma_{ji}\frac{\partial u_i'}{\partial x_j}\,\mathrm{d}V = \int_V \rho\left(f_i - \frac{\mathrm{d}u_i}{\mathrm{d}t}\right)u_i'\,\mathrm{d}V + \int_{S_\sigma} T_i u_i'\,\mathrm{d}S + \int_{S_u}(n_j\sigma_{ji})u_i'\,\mathrm{d}S, \tag{2.4}$$

where the first term of the right-hand side can be regarded as the power done by 'the generalized body forces' $(f_i - \mathrm{d}u_i/\mathrm{d}t)$ according to the Lagrangian mechanics, the second term by the surface stress on S_σ and the third term by the constraint forces $(n_j\sigma_{ji})$ on S_u. Therefore, the physics of (2.4) can be stated as: 'in flow fields, the total virtual strain-rate power is equal to the virtual power done by all of the external forces, including the generalized body forces, the surface forces and the constraint forces, although the stress (σ_{ji}) and the possible velocity (u_i') are unrelated'.

For convenience, the left-hand side of (2.4), the virtual strain-rate power, is called the internal virtual power P_i and the right-hand side of this equation is called the external virtual power P_e. Then, the principle of virtual power can be described as follows: 'the internal virtual power is equal to the external virtual power in fluid dynamics', which is the first form of the virtual power principle and can be written as

$$P_i = P_e, \tag{2.5}$$

where

$$\left.\begin{aligned} P_i &= \int_V \sigma_{ji}\frac{\partial u_i'}{\partial x_j}\,\mathrm{d}V, \\ P_e &= \int_V \rho\left(f_i - \frac{\mathrm{d}u_i}{\mathrm{d}t}\right)u_i'\,\mathrm{d}V + \int_{S_\sigma} T_i u_i'\mathrm{d}S + \int_{S_u}(n_j\sigma_{ji})u_i'\,\mathrm{d}S. \end{aligned}\right\} \tag{2.6}$$

2.2. *The virtual velocity principle, the second form of the virtual power principle*

As mentioned above, the possible velocity u_i' is independent of the stress σ_{ji}. Therefore, the variational principle can be employed to remove the effect of the unknown constraint forces acting on the velocity boundary, S_u. The virtual velocity u_i^* is specified as variation of the possible velocity by using variational notation,

$$u_i^* = \delta u_i'. \tag{2.7}$$

Then, the virtual velocity is zero on the surface S_u, where the velocity is prescribed, i.e.

$$u_i^* = 0 \quad \text{on } S_u \tag{2.8}$$

which is a homogeneous boundary condition. Thus, the power done by the constraint force becomes zero, and the virtual power (2.4) can be rewritten in terms of the virtual velocity

$$\int_V \sigma_{ji}\frac{\partial u_i^*}{\partial x_j}\,\mathrm{d}V = \int_V \rho\left(f_i - \frac{\mathrm{d}u_i}{\mathrm{d}t}\right)u_i^*\,\mathrm{d}V + \int_{S_\sigma} T_i u_i^*\,\mathrm{d}S. \tag{2.9}$$

This equation states that: 'for arbitrary virtual velocity u_i^* fitting the homogeneous boundary condition, the virtual power done by the internal stress is equal to that done by the external generalized body forces and the surface forces'. We call it the virtual velocity principle or the second form of the virtual power principle.

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Now, it is necessary to prove that (2.9) with the boundary condition (2.8) is equivalent to (2.1) with the conditions (2.2). To do this, we first have to prove that (2.1) with (2.2) is a sufficient condition to (2.9) with (2.8). According to the mathematically identical (2.3), the virtual velocity u_i^* satisfies

$$\int_V \sigma_{ji}\frac{\partial u_i^*}{\partial x_j}\,\mathrm{d}V = -\int_V \frac{\partial \sigma_{ji}}{\partial x_j}u_i^*\,\mathrm{d}V + \int_S n_j\sigma_{ji}u_i^*\,\mathrm{d}S. \tag{2.10}$$

Substituting it into (2.9), we obtain

$$\int_V \frac{\partial \sigma_{ji}}{\partial x_j}u_i^*\,\mathrm{d}V + \int_V \rho\left(f_i - \frac{\mathrm{d}u_i}{\mathrm{d}t}\right)u_i^*\,\mathrm{d}V - \int_S n_j\sigma_{ji}u_i^*\,\mathrm{d}S + \int_{S_\sigma} T_i u_i^*\,\mathrm{d}S = 0 \tag{2.11}$$

and then rewrite it as follows,

$$\int_V \left[\frac{\partial \sigma_{ji}}{\partial x_j} + \rho\left(f_i - \frac{\mathrm{d}u_i}{\mathrm{d}t}\right)\right]u_i^*\,\mathrm{d}V + \int_{S_\sigma}[T_i - n_j\sigma_{ji}]u_i^*\,\mathrm{d}S - \int_{S_u} n_j\sigma_{ji}u_i^*\,\mathrm{d}S = 0. \tag{2.12}$$

The first term of (2.12) becomes zero due to (2.1), while the second and third terms also take the values of zero due to the boundary condition (2.2). So (2.9) with (2.8) is true, if (2.1) with (2.2) is true.

Similarly, it is easy to prove that (2.9) with (2.8) is also the sufficient condition to (2.1) with (2.2). Considering that the virtual velocity u_i^* is arbitrary but fits the homogeneous boundary condition (2.8), we know that the only possible conclusion from (2.12) is that each $[\cdot]$ term must independently vanish. Therefore, equation (2.1) with the boundary conditions (2.2) is a necessary and sufficient condition of (2.9) with the homogenous condition (2.8). This means that the fluid motion expressed in scalar form as denoted by the virtual power is identical to that in vector form as described by Cauchy's equation. In other words, equation (2.9) can be regarded as another type of governing equation of the flow.

2.3. *The general form of the virtual power principle*

Compared with the first form of the virtual power principle, it is beyond doubt that the second, based on the variational principle, is more significant as the virtual velocity is introduced. In order to avoid the restriction of the homogeneous boundary condition (2.8) in (2.9), we apply a distributed 'Cauchy stress' instead of the velocity boundary constraint. The Cauchy stress is just the constraint force, which belongs to the generalized surface force,

$$T_i = n_j\sigma_{ji} \quad \text{on } S_u. \tag{2.13}$$

Then, the limit of the velocity boundary condition disappears, and (2.9) becomes

$$\int_V \sigma_{ji}\frac{\partial u_i^*}{\partial x_j}\,\mathrm{d}V = \int_V \rho\left(f_i - \frac{\mathrm{d}u_i}{\mathrm{d}t}\right)u_i^*\,\mathrm{d}V + \oint_S T_i u_i^*\,\mathrm{d}S. \tag{2.14}$$

It is the general form of the virtual power principle, which is stated as: 'for arbitrary virtual velocity u_i^* , the virtual power done by the internal stress is equal to that done by the external generalized body forces and all of the generalized surface forces'. To some extent, we prefer to call this general representation the virtual velocity principle.

Obviously, three expressions of the virtual power principle are in scalar forms, and the (internal or external) forces are related with each other by the virtual power they do. In other words, based on the virtual velocity u_i^*, the virtual power is a key bridge to connect the internal stress, the generalized body forces and the generalized surface forces together.

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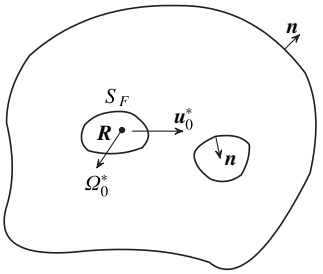


FIGURE 2. The specified body with the surface S_F, which translates with a constant virtual velocity $\boldsymbol{u}_0^*$ or rotates about the reference point $\boldsymbol{R}$ with a constant virtual angular velocity $\boldsymbol{\Omega}_0^*$.

3. The theorem of the hydrodynamic force and torque

In aerodynamics and hydrodynamics, it is a very important task to express the force and torque in a way that can precisely capture the key physical mechanisms contributing to the integrated performances. One can measure or calculate the stress on a body surface and apply the surface integral to get the force (or torque) in the various flow phenomena generated by the body and other boundaries, and even visualize the flow structures (streamlines, the contours of pressure or vorticity, etc.). However, it is still a confusing problem that what flow structures and boundaries have net contribution to the instantaneous force and torque. Fortunately, the principle of virtual power mentioned above provides an approach to analyse the contributions on the integrated performances of a body by the flow structures and the moving or deforming boundaries.

In the following subsections, the formulas of the hydrodynamic force and torque on a specified body immersed in a fluid (shown in figure 2) will be derived by using the given virtual flows.

3.1. *Force formula*

Based on the third form of the virtual power principle, the expression of the forces acting on a body considered can be derived by properly setting the virtual velocity distribution. Assume that the body translates with a 'constant virtual velocity' $\boldsymbol{u}_0^*$, while the other bodies or boundaries remain at rest, i.e.

$$(u_b^*)_i = \begin{cases} (u_0^*)_i & \text{on } S_F, \\ 0 & \text{on } S - S_F, \end{cases} \tag{3.1}$$

where S_F is the body boundary considered. Substituting the boundary condition (3.1) into (2.14), we get

$$\int_V \sigma_{ji} \frac{\partial u_i^*}{\partial x_j} \,\mathrm{d}V = \int_V \rho \left(f_i - \frac{\mathrm{d}u_i}{\mathrm{d}t} \right) u_i^* \,\mathrm{d}V + (u_0^*)_i \oint_{S_F} T_i \,\mathrm{d}S. \tag{3.2}$$

Furthermore, the hydrodynamic force $\boldsymbol{F}$ acting on the body is the reaction force according to Newton's third law and it is expressed as

$$F_i = -\oint_{S_F} T_i \,\mathrm{d}S. \tag{3.3}$$

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Therefore, the force on the body considered is obtained,

$$\frac{(u_0^*)_i}{u_0^*}F_i = -\frac{1}{u_0^*}\int_V \sigma_{ji}\frac{\partial u_i^*}{\partial x_j}\,\mathrm{d}V + \frac{1}{u_0^*}\int_V \rho\left(f_i - \frac{\mathrm{d}u_i}{\mathrm{d}t}\right)u_i^*\,\mathrm{d}V. \tag{3.4}$$

The physical meanings of the above equation can be described as: 'in a virtual velocity field caused by a virtual translating body, the component of the hydrodynamic force acting on the body along the virtual translational direction, is related with the virtual strain-rate power and the virtual power done by the generalized body forces'.

3.2. *Torque formula*

Similarly, we derive the torque on the body considered based on the virtual power principle. First, we assume that the body rotates about the reference point $\boldsymbol{R}$ with a 'constant virtual angular velocity' $\boldsymbol{\Omega}_0^*$ (shown in figure 2), while the other bodies or boundaries remain at rest, i.e.

$$(u_b^*)_i = \begin{cases} \varepsilon_{ist}(\Omega_0^*)_s(r_t - R_t) & \text{on } S_F, \\ 0 & \text{on } S - S_F, \end{cases} \tag{3.5}$$

where ε_{ist} is an alternating symbol in tensor analysis. Similarly, we have the expression of the hydrodynamic torque $\boldsymbol{M}$ about the reference point $\boldsymbol{R}$ acting on the body,

$$M_s = -\oint_{S_F} \varepsilon_{ist}(r_t - R_t)T_i\,\mathrm{d}S. \tag{3.6}$$

Then the following torque formula is acquired based on (2.14),

$$\frac{(\Omega_0^*)_s}{\Omega_0^*}M_s = -\frac{1}{\Omega_0^*}\int_V \sigma_{ji}\frac{\partial u_i^*}{\partial x_j}\,\mathrm{d}V + \frac{1}{\Omega_0^*}\int_V \rho\left(f_i - \frac{\mathrm{d}u_i}{\mathrm{d}t}\right)u_i^*\,\mathrm{d}V. \tag{3.7}$$

That means that in a virtual velocity field caused by a virtual rotating body, the component of the hydrodynamic torque acting on the body along the virtual rotational direction is also related with the virtual strain-rate power and the virtual power done by the generalized body forces.

3.3. *General formula of the generalized force*

According to (3.4) and (3.7), we can obtain

$$(u_0^*)_i F_i + (\Omega_0^*)_s M_s = \int_V \rho\left(f_i - \frac{\mathrm{d}u_i}{\mathrm{d}t}\right)u_i^*\,\mathrm{d}V - \int_V \sigma_{ji}\frac{\partial u_i^*}{\partial x_j}\,\mathrm{d}V. \tag{3.8}$$

which is a general formula of the force $\boldsymbol{F}$ (let $\Omega_0^* = 0$ and $u_0^* \neq 0$) and the torque $\boldsymbol{M}$ (let $u_0^* = 0$ and $\Omega_0^* \neq 0$) acting on a specified body. And the virtual boundary is rewritten as

$$(u_b^*)_i = \begin{cases} (u_0^*)_i + \varepsilon_{ist}(\Omega_0^*)_s(r_t - R_t) & \text{on } S_F, \\ 0 & \text{on } S - S_F. \end{cases} \tag{3.9}$$

Therefore, the following theorem is obtained.

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THEOREM. *Assuming a virtual flow caused by a body translating (or rotating) along a specified direction at a constant virtual velocity (or angular velocity), the virtual power done by the hydrodynamic force (or torque) acting on the body along this direction is equal to the difference between that done by the generalized body forces minus the virtual internal stress power.*

This theorem is only based on an assumption (the continuum fluid satisfying the Euler–Cauchy stress principle), so it can be applied to any flows with multiple bodies involving any continuum fluids (including both Newtonian and non-Newtonian fluids) that satisfy the Euler–Cauchy stress principle. Obviously, the formulas include more information than the surface integral of the stress.

3.4. *Expression of the theorem in incompressible flow involving Newtonian fluid*

Without loss of generality, the incompressible flow involving Newtonian fluid is taken as an example to further discuss the theorem presented above. For Newtonian fluid, the constitutive relation is

$$\sigma_{ij} = -p\delta_{ij} + \lambda S_{kk}\delta_{ij} + 2\mu S_{ij}, \tag{3.10}$$

where the stain-rate tensor $S_{ij} = 1/2(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$, p is the pressure and μ and λ are the first and second viscosity coefficients, respectively.

To simplify the virtual velocity field, it is natural to assume that the virtual flow is incompressible for a Newtonian fluid. Thus, the virtual velocity satisfies

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}^* = 0. \tag{3.11}$$

Then, substituting the constitutive relation into the internal virtual power in (3.8) yields

$$\int_V \sigma_{ji}\frac{\partial u_i^*}{\partial x_j}\,\mathrm{d}V = \int_V 2\mu S_{ji}\frac{\partial u_i^*}{\partial x_j}\mathrm{d}V. \tag{3.12}$$

where the right-hand side is called 'the virtual dissipation rate'. In an incompressible flow involving a uniform fluid, where the fluid density is constant, $\rho =$ constant, and the divergence of the velocity field is zero, $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$, the virtual dissipation rate can be rewritten as follows:

$$\int_V 2\mu S_{ji}\frac{\partial u_i^*}{\partial x_j}\,\mathrm{d}V = \int_V \mu\omega_k\omega_k^*\,\mathrm{d}V, \tag{3.13}$$

where ω_k and ω_k^* are the vorticity of the real and virtual flow, respectively. Therefore, equation (3.8) can be rewritten as

$$(u_0^*)_i F_i + (\Omega_0^*)_s M_s = \int_V \rho\left(f_i - \frac{\mathrm{d}u_i}{\mathrm{d}t}\right)u_i^*\,\mathrm{d}V - \int_V \mu\omega_k\omega_k^*\,\mathrm{d}V. \tag{3.14}$$

To further understand the force characteristics of the body, we will expand the integrals on the right-hand side of (3.14). To do this, the virtual flow caused by the virtual moving body is given first, and then the deduction can be continued.

3.4.1. *The virtual potential flow*

Since the virtual flow must satisfy the boundary condition (3.9) and the unique perturbation source comes from the surface, S_F, the perturbation in the virtual

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flow decays with increasing distance from the virtual moving body. In addition to assuming the virtual flow incompressible, we further assume it to be irrotational. So it is regarded as an incompressible potential flow with the following characteristics: (*a*) the flow is without history effects, (*b*) it is only determined by the boundary condition and (*c*) the perturbation decays rapidly with increasing distance from the virtual moving body. Thus, the governing equation of the virtual potential flow caused by a virtual moving body and the boundary conditions are expressed as

$$\left.\begin{aligned} &\nabla^2\varphi = 0 \quad \text{in } V, \\ &\nabla\varphi\cdot\boldsymbol{n} = [\boldsymbol{u}_0^* + \boldsymbol{\Omega}_0^*\times(\boldsymbol{r}-\boldsymbol{R})]\cdot\boldsymbol{n} \quad \text{on } S_F, \\ &\nabla\varphi\cdot\boldsymbol{n} = 0 \quad \text{on } S - S_F. \end{aligned}\right\} \tag{3.15}$$

where φ is the velocity potential function. And the virtual velocity is

$$\boldsymbol{u}^* = \nabla\varphi. \tag{3.16}$$

In a simply connected space, the virtual potential flow is uniquely determined. But if the fluid domain is multiply connected, the potential function φ becomes multivalent if there exists the circulation. In order to make the potential function unique, the additional condition is needed, that is 'no circulation' in the virtual flow. For example, to impose $\varphi = 0$ at infinity (Chang 1992) is a good choice as a kind of additional condition.

Furthermore, according to the boundary condition (3.9) and (3.15), it is noted that there is a 'virtual vortex layer' on the boundary, whose strength is

$$\boldsymbol{n}\times(\boldsymbol{u}_b^* - \nabla\varphi). \tag{3.17}$$

Because there is no dilatation and vorticity in the virtual incompressible irrotational flow, the analysis about the virtual power will be elegant in the next content.

3.4.2. *Expansion of the force term*

First, according to the formulation of the material derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}u_i = \frac{\partial}{\partial t}u_i + \frac{\partial}{\partial x_i}(u_j u_j/2) + \varepsilon_{ijk}\omega_j u_k, \tag{3.18}$$

the virtual power done by the inertial force consists of three parts,

$$\int_V \rho\left(-\frac{\mathrm{d}u_i}{\mathrm{d}t}\right)\frac{\partial\varphi}{\partial x_i}\,\mathrm{d}V = -\int_V \rho\left[\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_i}(u_j u_j/2) + \varepsilon_{ijk}\omega_j u_k\right]\frac{\partial\varphi}{\partial x_i}\,\mathrm{d}V, \tag{3.19}$$

where the first two terms of the right-hand side can be transformed into surface integrals,

$$\begin{aligned} \int_V \rho\frac{\partial u_i}{\partial t}\frac{\partial\varphi}{\partial x_i}\,\mathrm{d}V &= \int_V \rho\frac{\partial}{\partial x_i}(\varphi\frac{\partial u_i}{\partial t})\,\mathrm{d}V - \int_V \rho\varphi\frac{\partial}{\partial x_i}\frac{\partial u_i}{\partial x_j}\,\mathrm{d}V \\ &= \oint_S \rho\varphi n_i\frac{\partial u_i}{\partial t}\,\mathrm{d}S \\ &= \oint_S \rho\varphi n_i\frac{\mathrm{d}u_i}{\mathrm{d}t}\,\mathrm{d}S - \oint_S \rho\varphi n_i\left[(u_b)_j\frac{\partial u_i}{\partial x_j}\right]\mathrm{d}S \end{aligned} \tag{3.20}$$

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and

$$\int_V \rho \frac{\partial}{\partial x_i}(u_j u_j/2)\frac{\partial \varphi}{\partial x_i}\,\mathrm{d}V = \oint_{S_F} \rho n_i \frac{\partial \varphi}{\partial x_i}(u_j u_j/2)\,\mathrm{d}S. \tag{3.21}$$

In (3.20), the time derivative in the inertial system can be obtained by the chain rule:

$$\frac{\partial}{\partial t} = \left(\frac{\partial}{\partial t}\right)_{body} - (u_b)_j \left(\frac{\partial}{\partial x_j}\right), \tag{3.22}$$

where $(\partial/\partial t)_{body}$ is just the material (or substantial) derivative $\mathrm{d}/\mathrm{d}t$ on the body surface.

Second, due to the 'virtual vortex layer' the virtual dissipation rate can be also transformed into a surface integral,

$$\int_V \mu \omega_k \omega_k^*\,\mathrm{d}V = \oint_S \mu \omega_k \varepsilon_{kij} n_i \left[(u_b^*)_j - \frac{\partial \varphi}{\partial x_j}\right]\mathrm{d}S, \tag{3.23}$$

where $\boldsymbol{\omega}^*\,\mathrm{d}V = \boldsymbol{n} \times (\boldsymbol{u}_b^* - \nabla\varphi)\,\mathrm{d}S$, although the whole virtual flow is irrotational.

Finally, substituting (3.19)–(3.23) into (3.14), we obtain

$$\begin{aligned}(u_0^*)_i F_i + (\Omega_0^*)_s M_s \;=\; & \underbrace{-\oint_S \rho\varphi n_i \frac{\mathrm{d}u_i}{\mathrm{d}t}\,\mathrm{d}S}_{T_1} \underbrace{+\oint_S \rho\varphi n_i [(u_b)_j \frac{\partial u_i}{\partial x_j}]\,\mathrm{d}S - \oint_{S_F} \rho n_i \frac{\partial \varphi}{\partial x_i}(u_j u_j/2)\,\mathrm{d}S}_{T_2} \\ & \underbrace{-\oint_S \mu \omega_k \varepsilon_{kij} n_i [(u_b^*)_j - \frac{\partial \varphi}{\partial x_j}]\,\mathrm{d}S}_{T_3} \underbrace{-\int_V \rho(\varepsilon_{ijk}\omega_j u_k)\frac{\partial \varphi}{\partial x_i}\,\mathrm{d}V}_{T_4} \\ & \underbrace{+\int_V \rho f_i \frac{\partial \varphi}{\partial x_i}\,\mathrm{d}V}_{T_0},\end{aligned} \tag{3.24}$$

which is the general formula to calculate the force (let $\Omega_0^* = 0$ and $u_0 \neq 0$) and the torque (let $u_0^* = 0$ and $\Omega_0^* \neq 0$) acting on a specified body in the incompressible flow.

It is interesting that, among the five terms marked from T_0 to T_4, there are three boundary integrals (T_1, T_2 and T_3) and two volume integrals (one is dependent on the vortex distribution, the other on the real body force).

Also, it is noteworthy that (3.24) is in the absence of the pressure, which means that the force and torque can be determined completely by the flow field including the velocity field and the boundary acceleration.

3.5. *The inertial reference system and the force components*

It is well known that the force and torque acting on a body are Galilean invariant in arbitrary inertial reference systems. However, in (3.24), two terms (T_2 and T_4) depend on the reference system, while the sum of them ($T_2 + T_4$) is invariant.

When treating the time-dependent motion of the body, the selection of the coordinate systems becomes very important, because the velocity is relative to the reference system. Since the force (or torque) is acting on the body, it is useful to describe the motion of the surface, on which only the 'rotation and deformation' boundary condition is applied, in a body-fixed translating coordinate system. The

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relative motion of the reference frame is prescribed by the velocity of the system's origin, $\boldsymbol{U}_0$. So, we can get the vector form of (3.24) with $\boldsymbol{U}_0$,

$$\begin{aligned}\boldsymbol{u}_0^* \cdot \boldsymbol{F} + \boldsymbol{\Omega}_0^* \cdot \boldsymbol{M} = & \underbrace{-\oint_S \rho\varphi \boldsymbol{n} \cdot \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\,\mathrm{d}S}_{T_1} \\ & \underbrace{+\oint_S \rho\varphi \boldsymbol{n} \cdot [(\boldsymbol{u}_b - \boldsymbol{U}_0)\cdot\nabla](\boldsymbol{u} - \boldsymbol{U}_0)\,\mathrm{d}S - \oint_{S_F} \rho(\boldsymbol{n}\cdot\nabla\varphi)\frac{|\boldsymbol{u}-\boldsymbol{U}_0|^2}{2}\,\mathrm{d}S}_{T_2} \\ & \underbrace{-\oint_S \mu\boldsymbol{\omega}\cdot[\boldsymbol{n}\times(\boldsymbol{u}_b^* - \nabla\varphi)]\,\mathrm{d}S}_{T_3} \underbrace{-\int_V \rho[\boldsymbol{\omega}\times(\boldsymbol{u}-\boldsymbol{U}_0)]\cdot\nabla\varphi\,\mathrm{d}V}_{T_4} \\ & \underbrace{+\int_V \rho\boldsymbol{f}\cdot\nabla\varphi\,\mathrm{d}V}_{T_0}. \end{aligned} \tag{3.25}$$

Evidently, terms T_2 and T_4 of the generalized force in (3.25) become invariant in an arbitrary inertial reference system, for the relative velocity $(\boldsymbol{u} - \boldsymbol{U}_0)$ has been introduced.

Although every term of (3.25) has an important feature of the virtual power, we tend to call it the 'force component' considering that it describes the relationship between the virtual power and the force (or torque). Thus, the virtual power principle provides us with a perspective on the subject, how to understand the components of hydrodynamic force (or torque) influenced by the boundary movement or deformation, the flow structure, and the body force field.

We now further uncover the physics of the five integrals in (3.25).

3.5.1. *The added inertial effects,* T_1

Among the five integrals marked in (3.25), there is only one that does not contain any flow information (velocity or vorticity), T_1, which illustrates that this part of the force (or torque) acting on a moving or deforming body is directly caused by the accelerating boundary. This component is always called the 'added mass effects'. In the present work, we call it the 'added inertial effects', as it includes not only the added force but also the added torque.

The added inertial effects are unique instantaneous forces (or torque), and depend on the acceleration of all of the boundaries. This means that although a body may be at rest, the forces acting on it can be influenced by the adjacent accelerating bodies or walls. Obviously, the impact is also dependent on the value of φ, the virtual velocity potential, which becomes weak when the distance increases. In addition, the integral term T_1 reveals that the added inertial effects can be measured not only in inviscid flows, but also in viscous flows.

3.5.2. *The boundary energy flux effects,* T_2

The term T_2 in (3.25) consists of two boundary integrals, and each of them can be regarded as a 'boundary energy flux'. One is the virtual potential energy flux caused by the normal boundary convection. The value of φ, vanishing while the boundary $(S - S_F)$ is far enough from the body, could be regarded as a virtual potential energy although it does not have the energy dimension.

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The other is the virtual flux of the kinetic energy on the boundary of the body, S_F, and the flux is caused by the virtual movement. The effect of this integral is just like that of the kinetic energy term in the Bernoulli equation for inviscid flows. For viscous flows, it may be not null if the body undergoes rotation motion and deformation.

3.5.3. *The boundary dissipation effects,* T_3

The surface integral term T_3, the only one including the viscosity coefficient, comes from the volume integral of the 'virtual dissipation rate'. So we call it the 'boundary dissipation effect', which includes the virtual dissipation of energy due to all of the skin friction. In order to reveal this mechanism, we change the form of T_3 into

$$T_3 = \oint_S \mu(\boldsymbol{n} \times \boldsymbol{\omega}) \cdot \boldsymbol{u}_b^* \, \mathrm{d}S - \oint_S \mu(\boldsymbol{n} \times \boldsymbol{\omega}) \cdot \nabla\varphi \, \mathrm{d}S, \tag{3.26}$$

substitute the virtual boundary condition (3.9) into the above formulation and let the boundary shear stress $\tau = \mu(\boldsymbol{n} \times \boldsymbol{\omega})$, then we obtain

$$T_3 = \boldsymbol{u}_0^* \cdot \oint_{S_F} \tau \, \mathrm{d}S + \Omega_0^* \cdot \oint_{S_F} \boldsymbol{r} \times \tau \, \mathrm{d}S - \oint_S \tau \cdot \nabla\varphi \, \mathrm{d}S. \tag{3.27}$$

The first two integrals are the virtual power done by the friction drag and the friction torque, respectively.

The third integral in (3.27) shows that the skin friction can also affect the pressure distribution (normal stress) on the boundary, which contributes the pressure drag. For example, in the special case of a small sphere moving slowly through a viscous fluid (and thus at small Reynolds number), it is known that the drag consists of two parts, one is the skin friction drag, the other is the pressure drag. In this case, the expression of the third integral in (3.27) tells us the virtual power done by the pressure drag is $(-2\pi\mu U a u_0^*)$, which is half of that done by the friction drag $(-4\pi\mu U a u_0^*)$, where U and a are the velocity and the radius of the sphere, respectively. Then the total drag, $(-6\pi\mu U a)$, of a small sphere at small Reynolds number is derived by the virtual power principle.

3.5.4. *The vortex contribution,* T_4

It is well known that the vortex flow can influence the force (or torque) acting on the body. In general, the contribution of the vortex correlates with the Lamb vector, $\boldsymbol{l} = \boldsymbol{\omega} \times (\boldsymbol{u} - \boldsymbol{U}_0)$. Hamman, Klewicki & Kirby (2008) presented the relationship between the Lamb vector divergence and the pressure drag (or lift) on a stationary bluff body immersed in an incompressible flow (see Hamman *et al.* 2008, p. 267), which needs an added integration over the surface. Based on the principle in the present work, the expression about the vortex contribution can be easily deduced from the volume integral of (3.25), T_4, the virtual power done by the Lamb vortex.

The volume integral of (3.25), T_4, gives us an effective approach to measure the force (or torque) caused by the moving vortex. It can be applied to the multibody system and tells us about the influence of a vortex on the different bodies.

3.5.5. *The body force contribution,* T_0

Since the body force is always conservative, such as the gravity, $f_i = \partial(-gz)/\partial x_i$, its virtual power becomes

$$\begin{aligned} T_0 &= -\int_{S_F} [(u_0^*)_i + \varepsilon_{ist}(\Omega_0^*)_s(r_t - R_t)]\rho g z n_i \, \mathrm{d}S \\ &= \boldsymbol{u}_0^* \cdot (\rho g V_0 \boldsymbol{k}) + \Omega_0^* \cdot \left[(\boldsymbol{r}_c - \boldsymbol{R}) \times (\rho g V_0 \boldsymbol{k})\right], \end{aligned} \tag{3.28}$$

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where V_0 is the volume the body occupied, and $\boldsymbol{r}_c$ is the centre of buoyancy. This equation gives the buoyancy (or its torque about the reference point $\boldsymbol{R}$) along the virtual translational (or rotational) direction when $\Omega_0^* = 0$ (or when $u_0^* = 0$).

In short, the virtual velocity can be regarded as a bridge to connect all of the forces in the fluid and on the boundary, which indicates the intrinsic relationship among the bodies, walls and the fluid flow.

4. The application of the virtual power principle: 'vorticity force' in organized flight of geese

As mentioned in the introduction, some similar scalar forms of the hydrodynamic forces have been proposed and applied to analyse the force acting on the rigid bodies and bubbles (Howe 1995), the bluff bodies (Chang *et al.* 2008), the two-dimensional dragonfly wing–wing flapping (Hsieh *et al.* 2010) and the impulsively started plate (Lee *et al.* 2012).

Now, as an application of the virtual power principle, we propose a new perspective to discuss the organized flight of geese, which has been an interesting topic for almost a century.

The organized flight in birds has been reviewed extensively (see Lebar Bajec & Heppner 2009, for example). In long-distance migration, swans, cranes, large geese and other large birds often use the V formation, in which the rear birds can save energy in contrast with the leader (Hummel 1995; Weimerskirch *et al.* 2001). As Rayner mentioned (Rayner 2001), 'Wieselsberger (1914), a doctoral student under Ludwig Prandtl at Göttingen, realized that Prandtl's "lifting-line" principle of wing should apply to birds just as it did to aircraft, and that a group of birds could exploit the upwash velocity (outside of the wake) to fly more cheaply if they adopted a V-shaped formation in which each bird flies in the up-current generated by the one in front', which was the first suggestion of an aerodynamic advantage for line formation flight. Lissaman & Shollenberger (1970) produced the quantitative suggestion of exactly how much energy might be saved by a group of birds flying in a V formation. They proposed that a group of 25 (unspecified species) birds flying in a V formation would have 71 % more range than a single bird, based on the aerodynamic theory. However, not all ornithologists have accepted the aerodynamic explanation for this spectacular flight phenomenon, as mentioned by Rayner (2001). Fortunately, according to the experiments done by Weimerskirch *et al.* (2001), great white pelicans (a type of large birds) in a vortex wake had lower wing beat frequency, slower heart rate and longer glide time, and then their energy saving of 11.4–14 % was estimated. In this case, the prediction based on aerodynamics seems to work well. In fact, lack of analysis about the aerodynamic forces makes the aerodynamic mechanism of V formation flight unconvincing, although we believe it exists. Next, the aerodynamic forces will be discussed preliminarily with a simple model, and the analysis will provide us with a flight strategy for migrating geese from the view of aerodynamics.

4.1. *The wake model of a migrating goose*

Based on the experimental photographs of the wake of birds flying in clouds of helium bubbles (see Spedding 1987*a*,*b*), the wake consists of two trailing vortices. In addition, Hedrick, Tobalske & Biewener (2002) called the wake a continuous-vortex gait which is a popular phenomenon in fast flight ($\geqslant 7$ m s^{-1}) of cockatiels and ringed turtle-doves trained to fly in a wind tunnel. So, it is reasonable to model the wake of

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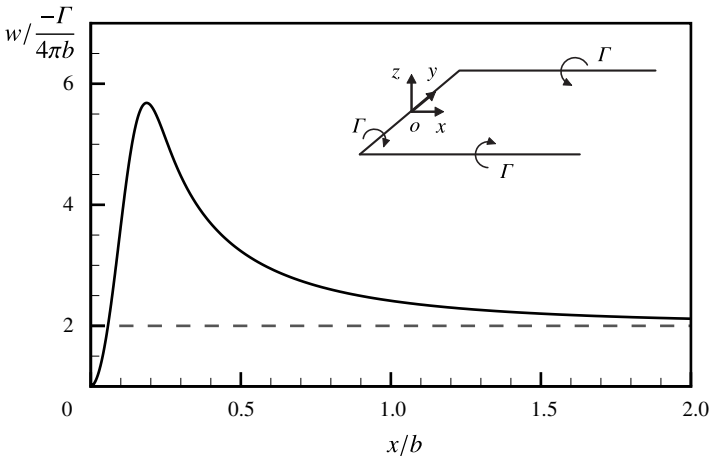


FIGURE 3. The induced downwash velocity distributed along one leg of the horseshoe, which is a wake model of a migrating goose, where the width between two legs of the horseshoe is b.

a migrating goose as a pair of counter-rotating vortices (wing-tip vortices) shedding from the wing tips, similar to that in the wake of an aircraft.

According to the lifting-line theory, a wing moving at a constant velocity, U, is conceptually replaced by a lifting line of circulation Γ, together with two trailing line vortices of equal circulation, as shown in the inset in figure 3. This structure is known as the horseshoe vortex. Approximately, if the lift force of circulation balances the weight, the circulation Γ is determined by using the Kutta–Joukowski theorem,

$$\Gamma = mg/(\rho U) \tag{4.1}$$

where mg is the weight of the goose and ρ is the air density. Furthermore, the downwash velocity distribution along one leg of the horseshoe vortex is induced according to the Biot–Savart law,

$$w(x) = -\frac{\Gamma}{4\pi b}\left[\frac{1}{x/b}\frac{1}{\sqrt{1+(x/b)^2}}\left(1-e^{-(x/\delta b)^3}\right)+\left(1+\frac{x/b}{\sqrt{1+(x/b)^2}}\right)\right], \tag{4.2}$$

where b is the width between wing tips and δ is a desingularization factor which can be regarded as the dimensionless core radius. The distribution of the downwash velocity distribution along the horseshoe vortex leg is shown in figure 3, where $\delta = 0.15$. It is found that there is a region of large induced velocity near the wing tip ($x/b < 1$), i.e. the downwash velocity of the wing-tip vortex is large when it is near the wing and has a characteristic of rapid decrease to a constant value with the increase of longitudinal distance from the wing tip.

A sketch of a goose and the wing-tip vortices is shown in figure 4. The strength of the vortex is Γ, and the downwash velocity induced by the wing circulation and the wing-tip vortices is $\boldsymbol{u}$. Then, the Lamb vector in the wake of the goose is $\boldsymbol{\Gamma} \times \boldsymbol{u}$, whose magnitude distributed along the longitudinal direction has the same feature as the induced downwash velocity. In addition, the Lamb vector directs outward along the span of the wing. According to the virtual power principle, the peak of the Lamb vector in the near wake of the wing tip could benefit the flight of the other geese.

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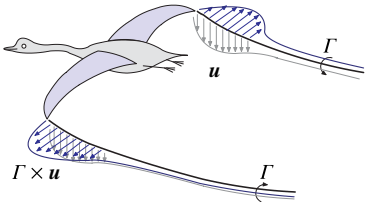


FIGURE 4. (Colour online) The sketch of a goose and the wing-tip vortex model, where Γ is the strength of the vortex, $\boldsymbol{u}$ is the downwash velocity induced by the wing circulation and the wing-tip vortices themselves and $\boldsymbol{\Gamma} \times \boldsymbol{u}$ is the Lamb vector.

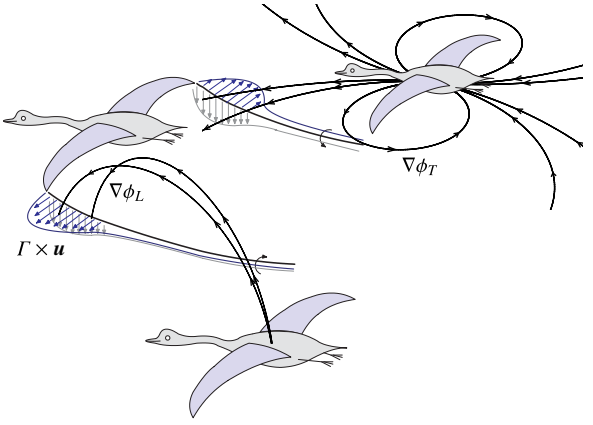


FIGURE 5. (Colour online) The organized flight of geese. Around the right (or left) trailing goose, the black curves with arrows are the streamlines of the virtual velocity $\nabla\phi_T$ (or $\nabla\phi_L$) caused by the virtual goose moving along the thrust (or lift) direction.

4.2. *The aerodynamical mechanisms of the V formation in geese flight*

In the migrating flight of V formation structure, the wake of the goose ahead may influence the flight of the geese behind. In this section, the mechanism of the V-shaped formulation is discussed in order to find the optimal angle where the trailing goose experiences high thrust or lift.

According to the vortex contribution T_4 in (3.25), the distributed vorticity in the flow can affect the aerodynamic forces on the adjacent geese (ahead/behind) in the formation flight, i.e.

$$T_4 = -\int_0^b \rho\,[\boldsymbol{\Gamma} \times \boldsymbol{u}] \cdot \nabla\varphi \,\mathrm{d}x, \tag{4.3}$$

where the Lamb vector $(\boldsymbol{\Gamma} \times \boldsymbol{u})$ is an objective existence, and the virtual velocity $(\nabla\varphi)$ needs to be determined.

Suppose that the position (distance and azimuthal angle) is sustained in the V-formation flight of the geese (as shown in figure 5), and the virtual velocity caused by a complex goose body can be modelled as a sphere in the analysis, which is reasonable when the distance between two geese is large enough.

First, we consider the thrust on the right rear goose moving virtually along the thrust (forward) direction. Here T_4 is employed to predict the contribution of the

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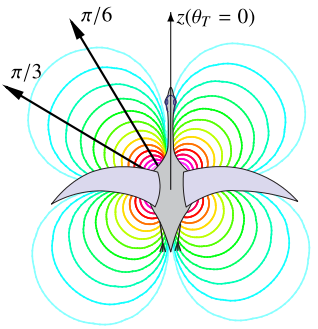


FIGURE 6. (Colour online) The contours of the spanwise component of the virtual velocity ($\nabla\varphi_T$) caused by the virtual moving right-rear goose along the thrust direction (z), where the curves approximate to circle arcs when $\theta_T \in [\pi/6, \pi/3]$.

leading goose's wake to the thrust. The virtual velocity potential excited by the forward virtual moving 'goose' (right-rear in figure 5) approximates to

$$\varphi_T = -u_0^* \frac{a^3}{2r^2} \cos\theta_T, \tag{4.4}$$

where a is the radius of the sphere, r is the radial distance from the centre of the goose's body to the point we considered and θ_T is the azimuthal angle between the zenith (forward) direction and the vector $\boldsymbol{r}$ (see figure 6). Then, the magnitude of the spanwise component of the virtual velocity is

$$v^* = \mp \frac{3u_0^*}{2} \left(\frac{a}{r}\right)^3 \sin\theta_T \cos\theta_T, \tag{4.5}$$

which is located in the horizontal plane. In the above formula, '$-$' and '$+$' indicate the left and right front regions of the goose, respectively.

As a preliminary study, substitute (4.5) into (4.3), and set the distance, d, between the right-rear goose and the near wake of the leading one to be constant, then we obtain

$$T_4 \propto \frac{\Gamma^2}{4\pi b} \frac{3u_0^*}{4} \left(\frac{a}{d}\right)^3 \sin(2\theta_T), \tag{4.6}$$

which is positive when $\theta_T \in (0, \pi/2)$. The above equation indicates that the virtual power caused by the wing-tip vortices of the leading goose is positive, which means that the trailing goose can receive induced thrust from the leading one. This part of the thrust is also called 'the drag reduction', D_r, which is

$$D_r \propto \frac{\Gamma^2}{b} \left(\frac{a}{d}\right)^3 \sin(2\theta_T). \tag{4.7}$$

According to the above equation, the optimal azimuthal angle is $\theta_T = \pi/4$, where the goose obtains the maximum drag reduction, D_{r0}. Furthermore, near $\pi/4$, the value of the drag reduction does not change a lot. Specifically, when $\pi/6 \leqslant \theta_T \leqslant \pi/3$, the drag reduction is more than 86 % of the maximum drag reduction D_{r0}, i.e.

$$\sqrt{3}/2 \leqslant \sin(2\theta_T) \leqslant 1, \quad \text{when } \theta_T \in [\pi/6, \pi/3] \tag{4.8}$$

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Thus, it can be concluded that the right-rear goose can receive the induced thrust (the drag reduction) when the front goose flies at the 10 to 11 o'clock position (see figure 6). According to the geometry relation, the angle of the V formation flight of geese, $\alpha = 2\theta_T$, is

$$\alpha \in [\pi/3, 2\pi/3]. \tag{4.9}$$

On the other hand, there may be negative effect for the leading goose if the trailing goose receives a positive effect. Certainly, by using the same analysis method, the virtual power T_4 is negative for the leading goose caused by the wake of the right-rear or the left-rear in the V formation. It reveals that the leading goose experiences larger drag than it flies solely. According to (4.7), the drag reduction decreases with the distance d. The distance from the leading goose to the wake of the trailing one is larger than that from its wake to the trailing. So the increased drag on the leading goose is less than the drag reduction on the trailing one. Compared with the simple sum of the drag in solo flight, the total drag of the flock is reduced.

Second, we consider the lift on the left-rear goose (left-bottom in figure 5), let it move virtually along the vertical direction, and generates the virtual potential φ_L. The streamlines are shown in figure 5. If the formation flight is in the horizontal plane, the magnitude of the spanwise component of the virtual velocity is zero, $v^* = 0$, which results in $T_4 = 0$. This means that there is no added lift for the trailing goose flying near the wake of the leading goose.

In short, the V formation flight may benefit the geese when they migrate a long distance from one place to another in the view of aerodynamics. There is an advantage of drag reduction for the trailing goose while the increased drag happens to the lead one, but the total drag is decreased for the flock. In addition, the optimal angle of the V formation flight is found to range from 60° to 120°. As a preliminary qualitative analysis, the model used may be too simple to resolve completely the 'V' organized flight of geese, but it provides a new perspective to discuss this topic.

5. Concluding remarks

In this work, we have first proposed a conceptual framework, named the virtual power principle in fluid mechanics, on analytical mechanics for fluids by using the virtual power and the virtual velocity. This principle indicates that the fluid motion and all of the (internal and external) forces are connected by a scalar quantity, the virtual mechanical power. The principle is stated as: 'the internal virtual power is equal to the external virtual power in fluid dynamics' (the first form). According to the variational principle, the second form of the virtual principle has been obtained from the first and proved to be equivalent with the vector dynamics, i.e. Cauchy's equation (or the Navier–Stokes equation). Furthermore, in order to remove the constraint of the velocity boundary conditions, the general virtual power principle, also called the virtual velocity principle, has been produced, which is expressed by (2.14): 'for arbitrary virtual velocity u_i^*, the virtual power done by the internal stress is equal to that done by the external generalized body forces and all of the generalized surface forces'.

In addition, on the basis of the general virtual power principle mentioned above, a theorem has been derived in terms of the specified virtual boundary conditions. This indicates that the relationship between the force (or torque) acting on a body immersed in a fluid and the specified virtual power. As expressed in (3.4) and (3.7) or (3.8), the force (or torque) component along a specified direction is equal to the virtual power divided by the constant virtual velocity (or angular velocity) of the body,

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where the virtual power is the difference between that done by the generalized body forces minus the virtual internal stress power in the virtual flow exerted by the body translating (or rotating) at the constant virtual velocity (or angular velocity) along the direction. Without loss of generality, the expression of the theorem for Newtonian fluids in incompressible flows is derived and used to illustrate the mechanisms on instantaneous forces. And according to Galilean invariance, the general formula (3.25) is proposed in an arbitrary inertial reference system. It uncovers the contribution divided into the added inertial effects, the boundary energy flux effects, the boundary dissipation effects, the vortex structures and the body force to the force and torque acting on a body.

Finally, as an application of the virtual power principle in a multibodies system, the organized flight of geese has been preliminarily discussed using a simple model. The results show that the V-formation flight can benefit the flock considering the total drag is reduced. Compared with the leading goose who experiences large drag, the trailing geese received the induced thrust from the wake of the leading goose. In addition, it is also found that the optimal angle of the V formation flight ranges from 60° to 120°.

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