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On bundle completion of parallelizable manifolds

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Abstract. In (5), Schmidt devised a procedure for completing a space-time M by adjoining a boundary ∂M called the b-boundary. Bosshard (1) and Johnson (4) have shown that certain Friedmann and Schwarzschild space-times have non-Hausdorff b-completions.

In (2), (3), the first author considered a modification of the b-completion which gives a Hausdorff completion in the Friedmann case. The modification used a parallelization of the space-time and can be given in terms of the linear connexion determined by the parallelization. We show that the completion obtained is the Cauchy completion in a particular Riemannian metric on the manifold and so is always Hausdorff.

Let M be a smooth n-manifold with frame bundle LM and natural projection Π: $LM \to M$. Let ∇ be a linear connexion on M. ∇ decomposes each tangent space

$$T_b LM = H_b \oplus V_b,$$

where $V_b = (D_b \Pi)^{-1}(\{0\})$ and H_b is determined by ∇.

A Riemannian structure is defined on LM using the following two forms:

(1) The canonical 1-form θ: $TLM \to \mathbb{R}^n$. It is defined on $T_b LM$ as $\rho_b \circ D_b \Pi$ where ρ_b takes a point of the tangent space to M at $\Pi(b)$ to its coordinates with respect to the frame b.

(2) The connexion form ω: $TLM \to \mathbb{R}^{n^2}$. Given $b \in LM$ define σ_b: $G \to LM$: $h \to R_h(b)$ where G is the general linear group $Gl(n; \mathbb{R})$, with identity e and R_h: $LM \to LM$, $h \in G$, is its action on LM. $D_e \sigma_b$ is one–one and its range is V_b. Identifying $T_e G$ with $\mathbb{R}^{n^2}$, define ω on $T_b LM$ by $\omega = (D_e \sigma_b)^{-1} \circ P_{V_b}$ where P_{V_b} is the projection $T_b LM \to V_b$ determined by the decomposition $T_b LM = H_b \oplus V_b$. The Riemannian metric tensor field g is defined on LM by

$$g(\cdot, \cdot) = \langle \theta(\cdot), \theta(\cdot) \rangle_1 + \langle \omega(\cdot), \omega(\cdot) \rangle_2,$$

where $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ are the usual inner products on $\mathbb{R}^n$ and $\mathbb{R}^{n^2}$ respectively.

The Riemannian manifold (LM, g) is completed in the topological metric determined by g to give $\overline{LM}$. The action of G on LM is uniformly continuous in this metric and so can be extended to a uniformly continuous action on $\overline{LM}$. The b-completion $\bar{M}$ of M

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with respect to ∇ is defined to be the topological space $\overline{LM}/G$ in which M is embedded by ϕ where the following diagram commutes:

$$\begin{array}{ccc} LM & \xrightarrow{i} & \overline{LM} \\ \downarrow \Pi & & \downarrow \Pi_1 \\ M & \xrightarrow{\phi} & \overline{LM}/G \end{array}$$

Note that $\bar{M}$ may fail to be a manifold and, indeed, need not be a Hausdorff space. Detailed proofs and examples are given in (3).

In (2), Dodson considered the situation in which a parallelization p is given on M. He worked with a parallelization in the (Lorentzian-) orthonormal bundle of a space–time but we shall suppose only that P is a C^1-section $M \to LM$. First define the linear connexion ∇_p on M by defining the horizontal subspace of the tangent space at $p(x)$ in LM to be $D_x p(T_x M)$ and that of the tangent space at $R_h(p(x))$ to be $D_{p(x)} R_h(H_{p(x)})$ for each $x \in M$. Let $\tilde{p}$ be the corresponding connexion form. In fact $\tilde{p}$ was defined by a different method in (2) but the two definitions are equivalent. A Riemannian structure is defined on LM by

$$\langle \cdot, \cdot \rangle_p = \langle \theta(\cdot), \theta(\cdot) \rangle_1 + \langle \omega(\cdot), \omega(\cdot) \rangle_2 + \langle \tilde{p}(\cdot), \tilde{p}(\cdot) \rangle_2.$$

The completion of LM in the resulting topological metric is denoted by $\widetilde{LM}$ and $\widetilde{LM}/G$ is called the p-completion of (M, ∇) with parallelization p.

The Riemannian metric $\langle \cdot, \cdot \rangle_p$ on LM is $g_p(\cdot, \cdot) + \langle \omega(\cdot), \omega(\cdot) \rangle_2$ where g_p is the metric on LM associated with the connexion ∇_p on M.

THEOREM. *The p-completion of (M, ∇) with parallelization p is homeomorphic to the Cauchy completion of M in the Riemannian metric induced by its embedding via p in $(LM, \langle \cdot, \cdot \rangle_p)$. In particular, the p-completion is Hausdorff.*

Proof. Consider the mapping $\Pi_p = p \circ \Pi : LM \to p(M)$. We extend Π_p to the boundary of $(LM, \langle \cdot, \cdot \rangle_p)$ as follows.

Let b be a boundary point. Then there is a curve $c : [0, 1) \to LM$ of finite length with b as its end-point. Define $c_1 : [0, 1) \to \Pi^{-1}(\Pi(c(0)))$ by setting $c_1(t)$ to be the parallel transport with respect to ∇_p of $c(t)$ along $\Pi \circ c$ to its initial point $\Pi(c(0))$. It may be shown that the tangent vectors to c and c_1 have the same image under

$$\tilde{p} : \tilde{p}(\dot{c}_1(t)) = \tilde{p}(\dot{c}(t)).$$

The length of c is finite in the metric $\langle \cdot, \cdot \rangle_p$ and so is finite in the metric g_p determined by ∇_p. Hence the length of c_1 is finite in this metric. But $\dot{c}_1(t)$ is vertical and so $\omega(\dot{c}_1(t)) = \tilde{p}(\dot{c}_1(t))$. Hence c_1 has finite length in the metric $\langle \cdot, \cdot \rangle_p$. It therefore has an end-point $c_1(1)$ in the complete submanifold $\Pi^{-1}(\Pi(c(0)))$.

The tangent vector to the curve $p \circ \Pi \circ c$ has the same length as that component of $\dot{c}$ horizontal with respect to ∇_p. By the triangle inequality, $p \circ \Pi \circ c$ has finite length in the metric $\langle \cdot, \cdot \rangle_p$ and so has end-point d, say. If

$$c_1(1) = R_k(c(0)) \quad \text{and} \quad p(\Pi(c(0))) = R_h(c(0))$$

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then $d = R_h(R_k^{-1}(b))$ and so b and d determine the same point of the p-boundary. The extension of Π_p to the boundary of $(LM, \langle \cdot, \cdot \rangle_p)$ is defined by setting $\Pi_p(b) = d$.

It may now be checked that Π_p is well-defined and determines a homeomorphism of the p-completion of (M, ∇) with the Cauchy completion of $p(M)$ in its metric as a submanifold of $(LM, \langle \cdot, \cdot \rangle_p)$. ∎

The procedure for obtaining the p-completion of (M, ∇) with respect to the parallelization p is one which makes use of the properties of the two linear connexions ∇ and ∇_p and could be generalized to use any linear connexion in place of ∇_p. The question to be answered is whether there is a natural choice for such a connexion on space-times.

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