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Fractal functional quantization of mean-regular stochastic processes

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Abstract

We investigate the functional quantization problem for stochastic processes with respect to $L^p(\mathbb{R}^d, \mu)$-norms, where μ is a fractal measure namely, μ is self-similar or a homogeneous Cantor measure. The derived functional quantization upper rate bounds are universal depending only on the mean-regularity index of the process and the quantization dimension of μ and as universal rates they are optimal. Furthermore, for arbitrary Borel probability measures μ we establish a (nonconstructive) link between the quantization errors of μ and the functional quantization errors of the process in the space $L^p(\mathbb{R}^d, \mu)$.

1. *Introduction*

Functional quantization is a way to discretize the path space of a stochastic process. In this paper, we investigate the functional quantization problem for mean-regular stochastic processes $X = (X_t)_{t\in T}$, $T \subset \mathbb{R}^d$ with respect to $L^p(\mu)$-norms (or quasi norms) of continuous fractal measures μ on $\mathbb{R}^d$ with $\operatorname{supp}(\mu) \subset T$.

The general framework can be stated as follows. Let $(E, ||\cdot||_E)$ denote a separable Banach space (or a complete quasi-normed space like an L^p-space with $0 < p < 1$) and let X be a random vector defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ taking values in E. Let $r \in (0, \infty)$ and assume $\mathbb{E}||X||_E^r < \infty$. For $N \in \mathbb{N}$, the $L^r(\mathbb{P})$-minimal Nth functional quantization error of X is defined by

$$e_{N,r}(X, E) := \inf\{(\mathbb{E}||X - \hat{X}||_E^r)^{1/r} : \hat{X} \text{ is an } N\text{-quantization of } X\} \tag{1·1}$$

where a random vector $\hat{X}\colon \Omega \to E$ is called an N-quantization of X if $\hat{X} = \pi(X)$ for some Borel measurable map $\pi\colon E \to E$ satisfying $\operatorname{card}(\pi(E)) \leqslant N$. Notice that $e_{N,r}(X, E)$ only

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depends on the distribution of X. In many situations of interest including all L^p-spaces, $p \in [1, \infty]$ the existence of $L^r(I\!P)$-optimal N-quantizations $\hat{X}$, i.e.

$$e_{N,r}(X, E) = (I\!E||X - \hat{X}||_E^r)^{1/r} = ||X - \hat{X}||_{L_E^r(I\!P)}$$

has been established (see [**12**]). Using Voronoi quantizations one checks that

$$e_{N,r}(X, E) = \inf\{(I\!E \min_{a\in\alpha} ||X - a||_E^r)^{1/r} : \alpha \subset E,\ \mathrm{card}(\alpha) \leqslant N\}. \tag{1·2}$$

In the sequel we consider the space $E = L^p(I\!R^d, \mu)$, where $p \in (0, \infty)$ and μ is a finite Borel measure on $I\!R^d$. Then an E-valued random vector X is a (product-) measurable process $X = (X_t)_{t\in T}$ with $T \subset I\!R^d$ and $\mathrm{supp}(\mu) \subset T$ whose sample paths (almost) all belong to $L^p(\mu)$ and the minimal quantization errors take the form

$$e_{N,r}(X, L^p(\mu)) = \inf\left\{\left(I\!E\left(\int |X_t - \hat{X}_t|^p d\mu(t)\right)^{r/p}\right)^{1/r} : \hat{X} \text{ is } N\text{-quantization of } X\right\}.$$

If μ is the Lebesgue measure on [0,1], then the convergence rate of $e_{N,r}(X, L^p(\mu))$ for $L^\rho(I\!P)$-Hölder regular processes with $\rho \geqslant 1$ say, has been established in [**21**] as

$$\forall p, r \in (0, \rho),\ e_{N,r}(X, L^p(\mu)) = O((\log N)^{-b}) \tag{1·3}$$

where b denotes the regularity index of X, i.e. the largest exponent $b > 0$ that satisfies

$$\forall s, t \in T,\ ||X_s - X_t||_{L^\rho(I\!P)} \leqslant C_\rho |s - t|^b.$$

Furthermore, for several classes of processes the upper bound derived from $L^\rho(I\!P)$-regularity provides the true rate of the quantization errors (see [**21**]). This extends to higher dimensions, that is to random fields. If μ is the d-dimensional Lebesgue measure on $[0, 1]^d$ and $I\!R^d$ is equipped with the l^2-norm $|\cdot|$, then

$$\forall p, r \in (0, \rho),\ e_{N,r}(X, L^p(\mu)) = O((\log N)^{-b/d}). \tag{1·4}$$

The same universal estimate holds for finite measures μ with nonvanishing absolutely continuous part w.r.t. d-dimensional Lebesgue measure satisfying a suitable moment condition (see [**7**] and Theorem 3).

The situation is different for measures which are singular with respect to the Lebesgue measure. Roughly, the space dimension d in (1·4) has to be replaced by the so called quantization dimension of μ of a certain order possibly depending on p. For an investigation of the quantization dimension we refer to [**7**]. We concentrate on two classes of fractal (continuous) measures: self-similar measures μ on $I\!R^d$ satisfying the open set condition and 1-dimensional generalizations of the classical Cantor measure. If $D_r = D_r(\mu) \in (0, d]$ denotes the quantization dimension of μ of order r, then

$$\forall p \in (0, \rho), \forall r \in (0, p],\ e_{N,r}(X, L^p(\mu)) = O((\log N)^{-b/D_{bp}}). \tag{1·5}$$

(see Theorems 1 and 2). We stress that for some classes of processes it is important to include the case $p < 1$. (E.g. in the Lévy process, Example (c) of Section 2, the estimate (2·13) holds for $p \in (0, \beta)$ and of course, $\beta \leqslant 1$ may happen.) It is further demonstrated that these rates are optimal as universal upper bounds. The rates are established using some explicit sequences of N-quantizations. This constructive aspect is of great interest for possible numerical applications. Our approach is suitable for processes which are not too smooth.

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The paper is organized as follows. Section 2 contains the precise statements of the main functional quantization rate results for $L^\rho(I\!P)$-regular processes X and fractal measures μ and some examples concerning d-parameter fractional Brownian motions, weakly stationary random fields and Lévy processes. Section 3 provides some general estimates based on special partitions of $\mathrm{supp}(\mu)$. Sections 4 and 5 are devoted to the proofs. In Section 6 we establish a (nonconstructive) link between the quantization errors of μ and the functional quantization errors $e_{N,r}(X, L^p(\mu))$ for arbitrary Borel probability measures μ on $I\!R^d$ in the Banach space case $p \geqslant 1$, which implies part of the preceding results on fractal measures. In particular, this link allows to deal with the case $p = \rho$ in (1·4) and (1·5) if $\rho \geqslant 1$. The link has the disadvantage that it does not provide explicit N-quantizations for which the error bounds hold.

It is convenient to write $a_N \approx b_N$ if $a_N = O(b_N)$ and $b_N = O(a_N)$. For $A \subset I\!R^d$, diam (A) denotes the diameter of A with respect to the l_2-norm. The 1-dimensional Lebesgue measure is denoted by λ. Let $\lfloor x \rfloor$ denote the integer part of $x \in I\!R$ and let $\lfloor x \rfloor^+ := \lfloor x \rfloor \vee 0$.

2. *Statement of main results*

In this section we formulate in full generality an asymptotic upper bound for the quantization error $e_{N,r}(X, L^p(\mu))$ based on the regularity of the mapping $t \mapsto X_t$ from T into $L^\rho(I\!P)$ and the quantization dimension of μ for fractal distributions μ.

Fix $\rho \in (0, \infty)$. Let $X = (X_t)_{t\in T}$, $T \subset I\!R^d$ be a measurable process defined on a probability space $(\Omega, A, I\!P)$ such that

$$X_t \in L^\rho(I\!P) \text{ for all } t \in T. \tag{2·1}$$

The following φ-Lipschitz assumption will be assumed throughout the paper:

$$\begin{aligned} &\forall s, t \in T, I\!E|X_s - X_t|^\rho \leqslant \varphi(|s-t|)^\rho \text{ if } \rho \geqslant 1, \\ &\forall t \in T, \forall h > 0, I\!E \sup_{\substack{s\in T \\ |s-t|\leqslant h}} |X_s - X_t|^\rho \leqslant \varphi(h)^\rho \text{ if } \rho < 1 \end{aligned} \tag{2·2}$$

where φ: $(0, \infty) \to (0, \infty)$ is an increasing function with $\varphi(0+) = 0$ and $|t|$ denotes the l_2-norm of $t \in I\!R^d$. Set $\varphi(0) := 0$. If the above supremum is not measurable its expectation is meant as outer integral. (For $\rho < 1$ the sample paths of X are typically assumed to be bounded and then the supremum is measurable w.r.t. the completion of $(\Omega, \mathcal{A}, I\!P)$.) Notice that φ may depend on T so that condition (2·2) may be satisfied for compact sets $T \subset I\!R^d$ say, but not for $T = I\!R^d$.

We assume that φ is regularly varying at 0 with index $b > 0$, i.e. for every $t > 0$

$$\lim_{u\to 0} \frac{\varphi(tu)}{\varphi(u)} = t^b.$$

This means that $u \mapsto \varphi(1/u)$ is regularly varying at infinity with index $-b$. The most prevalent form of φ is $\varphi(u) = cu^b$, $b > 0$. In this case conditions (2·1) and (2·2) impose some restrictions on ρ and b namely, $b \leqslant 1$ if $\rho \geqslant 1$ and $\rho b \leqslant 1$ if $\rho < 1$ (provided e.g. $T_{t_0} := \{s \in I\!R : (s, t_0) \in T\}$ has nonempty interior and $P(X_{(s_1,t_0)} \neq X_{(s_2,t_0)}) > 0$ for some $s_1, s_2 \in T_{t_0}$ and some $t_0 \in I\!R^{d-1}$).

Now let μ be a finite Borel measure on $I\!R^d$ with compact support such that $\mathrm{supp}(\mu) \subset T$. One checks that (2·1) and (2·2) imply $I\!E||X||^\rho_{L^\rho(\mu)} < \infty$ and therefore

$$\forall r, p \in (0, \rho], I\!E||X||^r_{L^p(\mu)} < \infty. \tag{2·3}$$

If $\rho < 1$ we always assume that the sample paths of X a.s. lie in $L^1(\mu)$.

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2·1. *Self-similar distributions*

A particulary interesting class of fractal measures on $\mathbb{R}^d$ is the class of self-similar measures. Self-similar measures are continuous and most of them are singular with respect to Lebesgue measure. For the background we refer the reader to [**5**].

Let $K \in \mathbb{N}$, $K \geqslant 2$ and let $S_1, \ldots, S_K$ be contractive similarities on $\mathbb{R}^d$, that is $|S_i(x) - S_i(y)| = s_i|x - y|$ with contraction numbers $s_i \in (0, 1)$, $i = 1, \ldots, K$. The attractor of the iterated function system $(S_1, \ldots, S_K)$ is the unique non-empty compact set $A \subset \mathbb{R}^d$ with

$$A = \bigcup_{i=1}^{K} S_i(A).$$

We will assume throughout that $(S_1, \ldots S_K)$ satisfies the open-set-condition (OSC), that is, there exists a nonempty open set $U \subset \mathbb{R}^d$ with $S_i(U) \subset U$ for all i and $S_i(U) \bigcap S_j(U) = \varnothing$ for all i, j with $i \neq j$.

Let $(p_1, \ldots, p_K) \in (0, \infty)^K$ be a probability vector. Then there exists a unique Borel probability measure μ on $\mathbb{R}^d$ which satisfies the equation

$$\mu = \sum_{i=1}^{K} p_i \mu \circ S_i^{-1}. \tag{2·4}$$

μ is called the self-similar probability corresponding to $(S_1, \ldots, S_K, p_1, \ldots, p_K)$ and its support equals A. Moreover, μ is continuous.

It was shown in [**9**] that the quantization dimension

$$D_r = D_r(\mu) := \lim_{N\to\infty} \frac{\log N}{-\log e_{N,r}(\mu, \mathbb{R}^d)} \tag{2·5}$$

of μ of order $r \in (0, \infty)$ exists in $(0, \infty)$ and is the unique $D_r \in (0, \infty)$ satisfying

$$\sum_{i=1}^{K} \left(p_i s_i^r\right)^{D_r/(D_r+r)} = 1. \tag{2·6}$$

Moreover,

$$e_{N,r}(\mu, \mathbb{R}^d) \approx N^{-1/D_r} \text{ as } N \to \infty \tag{2·7}$$

(see [**8**]) and if $(\log p_1 s_1^r, \ldots, \log p_K s_K^r)$ is not arithmetic, then even $\lim_{N\to\infty} N^{1/D_r} e_{N,r}(\mu, \mathbb{R}^d)$ exists (see [**10**]). The similarity dimension of $(S_1, \ldots, S_K)$ is defined as the unique solution $D_\infty \in (0, \infty)$ of

$$\sum_{i=1}^{K} s_i^{D_\infty} = 1 \tag{2·8}$$

and also equals the Hausdorff dimension of A (see [**14**]). The Hausdorff dimension of μ is given by

$$D_0 = D_0(\mu) = \frac{\sum_{i=1}^{K} p_i \log p_i}{\sum_{i=1}^{K} p_i \log s_i}.$$

If $(p_1, \ldots, p_K) = (s_1^{D_\infty}, \ldots, s_K^{D_\infty})$, then $D_r = D_\infty$ for every $r \in [0, \infty]$ and μ is the

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normalized D_∞-dimensional Hausdorff measure on A. If $(p_1, \dots p_K) \neq (s_1^{D_\infty}, \dots s_K^{D_\infty})$, then the function $[0, \infty] \to (0, \infty), r \mapsto D_r$ is continuous and strictly increasing and in particular,

$$\lim_{r\to 0} D_r = D_0, \text{ and } \lim_{r\to\infty} D_r = D_\infty \tag{2·9}$$

(see [**8**]). We have $D_r \in (0, d]$ for every $r \in [0, \infty]$.

THEOREM 1. *Assume that* $X = (X_t)_{t\in T}$ *satisfies* (2·1) *and* (2·2) *for some* $\rho \in (0, \infty)$ *where* φ *is regularly varying at zero with index* $b > 0$. *Let* μ *be a self-similar distribution on* $\mathbb{R}^d$ *defined by* (2·4) *satisfying OSC such that supp*$(\mu) \subset T$ *with quantization dimensions* D_r.

(a) *Assume that the slowly varying part* ψ *of* φ *given by* $\psi(u) = u^{-b}\varphi(u)$ *is increasing or decreasing on* (0,1) *and moreover,* $\psi \circ \exp(-\cdot)$ *is regularly varying at infinity. Then*

$$\forall p \in (0, \rho), \forall r \in (0, p], e_{N,r}(X, L^p(\mu)) = O(\varphi((\log N)^{-1/D_{bp}})) \text{ as } N \to \infty.$$

If $\rho \geqslant 1$, *then this estimate is also true for* $p = \rho$ *and* $r \in (0, \rho)$.

(b) *Assume* $p_1 = \dots = p_K$ *and* $s_1 = \dots = s_K$. *Then*

$$\forall r, p \in (0, \rho), e_{N,r}(X, L^p(\mu)) = O(\varphi((\log N)^{-1/D_\infty})) \text{ as } N \to \infty.$$

If $\rho \geqslant 1$, *then this estimate is also true for* $p = \rho$.

Remarks. (a) The assumption of Theorem 1(a) is satisfied for functions φ of the form

$$\varphi(u) = cu^b |\log u|^a, a \in \mathbb{R}, c > 0$$

for small u.

(b) Notice that $b \mapsto b/D_{bp}$ is increasing so that more regularity of the process improves the rate and that $p \mapsto b/D_{bp}$ is decreasing.

(c) For the subclass of self-similar measures with $(p_1, \dots, p_K) = (s_1^{D_\infty}, \dots s_K^{D_\infty})$ we obtain in the situation of Theorem 1(a) ($D_{bp} = D_\infty$ and)

$$\forall r, p \in (0, \rho), e_{N,r}(X, L^p(\mu)) = O(\varphi((\log N)^{-1/D_\infty})).$$

If $(p_1, \dots, p_K) \neq (s_1^{D_\infty}, \dots, s_K^{D_\infty})$, the above discussion of quantization dimensions D_r of μ shows the dependence of the quantization rate of X on the number p.

Two special cases are as follows. Let $K = 2^d$, $p_1 = \dots = p_K = 1/K$, $y_1, \dots, y_K$ enumeration of $\{0, 1/2\}^d$ and $S_i : \mathbb{R}^d \to \mathbb{R}^d$, $S_i x = x/2 + y_i$. Then $D_\infty = d$ and μ is the Lebesgue measure on $[0, 1]^d$. We obtain from Theorem 1(b)

$$\forall r, p \in (0, \rho), e_{N,r}(X, L^p(\mu)) = O(\varphi((\log N)^{-1/d}))$$

in accordance with known results for $d = 1$ (see [**21**]). If $p_1 = p_2 = 1/2$ and $S_1, S_2 : \mathbb{R} \to \mathbb{R}$, $S_1 x = x/3$, $S_2 x = x/3 + 2/3$, then μ is the classical middle-third Cantor measure. Here $D_\infty = \log 2/\log 3$ and Theorem 1(b) yields

$$\forall r, p \in (0, \rho), e_{N,r}(X, L^p(\mu)) = O(\varphi((\log N)^{-\log 3/\log 2})).$$

We emphasize that the preceding theorem provides the true rate e.g. for the d-parameter fractional Brownian fields in case $p \geqslant 1$ (see the subsequent examples) so that for any self-similar measure μ on $\mathbb{R}^d$ satisfying the OSC the above quantization rate is optimal as

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universal rate. A further question concerns the existence of

$$\lim_{N\to\infty} (\log N)^{b/D_{bp}} e_{N,r}(X, L^p(\mu))$$

in $(0, \infty)$. This is true for several Gaussian processes on $T = [0, 1]$ (like Brownian motion, Brownian bridge and Ornstein–Uhlenbeck processes) and $p = 2$ in the non-arithmetic case, i.e. $(\log p_1 s_1^{2b}, \ldots, \log p_K s_K^{2b})$ is not arithmetic and appears as a consequence of results from spectral theory in [**23**], results in [**20**] and the strong equivalences of $e_{N,2}(X, L^2(\mu))$ and $e_{N,r}(X, L^2(\mu))$ (see [**3**]).

Of course, Theorem 1 does not always provide the true rate since e.g. it does not take into account C^k-smoothness with $k \geqslant 1$. For instance, for the integrated Brownian motion and $T = [0, 1]$, Theorem 1(a) applies with $\varphi(u) = cu$ and yields the upper bound $(\log N)^{-1/D_p}$ while the true rate is $(\log N)^{\frac{-3/2}{D_{p3/2}}}$ (see [**17**] and [**11**]).

We present a few examples. Further examples can be treated along the lines of [**21**].

Examples. Let μ be a self-similar distribution on $\mathbb{R}^d$ satisfying OSC with quantization dimensions D_r.

(a) The d-parameter fractional Brownian motion W^H on $\mathbb{R}^d$ with Hurst exponent $H \in (0, 1)$ is a centered continuous (isotropic) Gaussian field having the covariance function

$$\mathbb{E} W_s^H W_t^H = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |s - t|^{2H})$$

and thus satisfies for every $\rho \in (0, \infty)$

$$\mathbb{E}\left|W_s^H - W_t^H\right|^\rho = C_{H,\rho}|s - t|^{\rho H}, s, t \in T = \mathbb{R}^d.$$

It follows from Theorem 1 and [**11**]

$$\begin{aligned} &\forall p \in (0, 1), \forall r \in (0, p] \text{ and } \forall p \in [1, \infty), \forall r \in (0, \infty), \\ &\quad e_{N,r}(W^H, L^p(\mu)) = O((\log N)^{-H/D_{Hp}}). \end{aligned} \tag{2·10}$$

From the tight relationship between the functional quantization problem and the small deviation behaviour of Gaussian processes (see [**4**], [**11**]) one may deduce that this is the true rate in case $p \geqslant 1$. In fact, [**18**, theorem 5·7] gives

$$-\log \mathbb{P}(||W^H||_{L^p(\mu)} \leqslant \varepsilon) \approx \varepsilon^{-D_{Hp}/H} \text{ as } \varepsilon \to 0$$

and the tight connection between small deviation and functional quantization asymptotics yields

$$\forall p \in [1, \infty), \forall r \in (0, \infty), e_{N,r}(W^H, L^p(\mu)) \approx (\log N)^{-H/D_{Hp}}. \tag{2·11}$$

(b) Let $X = (X_t)_{t\in\mathbb{R}^d}$ be a d-parameter centered weakly stationary (square integrable) field with $\mathbb{E} X_s X_t = \mathrm{Var} X_0 R(s - t)$ where $R(t)$ denotes the correlation between X_t and X_0 so that

$$\mathbb{E}|X_s - X_t|^2 = 2\mathrm{Var} X_0(1 - R(s - t)), s, t \in \mathbb{R}^d.$$

Assume that there exist finite constants $C > 0, b > 0$ and $\varepsilon > 0$ such that

$$\forall t \in \mathbb{R}^d \text{ with } |t| \leqslant \varepsilon, 1 - R(t) \leqslant C|t|^{2b}$$

Then X is stochastically continuous and hence has a measurable modification. Furthermore,

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(2·2) holds for bounded sets T. Consequently,

$$\forall p \in (0,2), \forall r \in (0,p], e_{N,r}(X, L^p(\mu)) = O((\log N)^{-b/D_{bp}}), \qquad (2\cdot12)$$
$$\forall r \in (0,2), e_{N,r}(X, L^2(\mu)) = O((\log N)^{-b/D_{2b}}).$$

(c) Let $X = (X_t)_{t\geqslant 0}$ be a real càdlàg Lévy process without Brownian component and with Blumenthal–Getoor index

$$\beta := \inf\{p > 0 : \int_{\{|x|\leqslant 1\}} |x|^p d\nu(x) < \infty\}$$

where ν denotes the Lévy measure of X. Necessarily, $\beta \in [0,2]$. Assume $\beta > 0$ and $I\!E|X_1|^\rho < \infty$ for every $\rho \in (0,\beta)$. Assume further that the tail function $\underline{\nu}$ of the Lévy measure defined on $(0,\infty)$ by $\underline{\nu}(u) := \nu([-u,u]^c)$ satisfies

$$\exists c \in (0,1], \exists C \in (0,\infty), \underline{\nu}(u) \leqslant Cu^{-\beta} \text{ on } (0,c].$$

Then (2·1) and (2·2) are satisfied (for the process $(X_t - at)_{t\geqslant 0}$ with suitable $a \in I\!R$) on bounded sets $T \subset I\!R_+$ for every $\rho \in (0,\beta)$ with functions φ_β given by

$$\varphi_\beta(u) = c_1 u^{1/\beta} \text{ if } \beta \neq 1 \text{ or } \beta = 1, \nu \text{ symmetric}$$

and

$$\varphi_1(u) = c_1 u|\log u| \text{ if } \beta = 1$$

for small u (see [**22**]). Consequently, if $\mathrm{supp}(\mu) \subset I\!R_+$,

$$\forall p \in (0,\beta), \forall r \in (0,p], e_{N,r}(X, L^p(\mu)) = O(\varphi_\beta((\log N)^{-1/D_{p/\beta}})). \qquad (2\cdot13)$$

The case of fixed $\rho \in (0,\beta)$ can be treated similarly and for the case $\rho \in [\beta,\infty)$ we refer to [**22**].

2·2. *Homogeneous Cantor distributions*

Let $(a_k)_{k\in I\!N}$ be a sequence in $(0,1/2]$. The generalized (dyadic) Cantor set $\mathbf{C} = \mathbf{C}((a_k))$ is constructed inductively. Start with $I_0 = [0,1]$, $\mathcal{D}_0 = \{I_0\}$ and $C_0 = I_0$. Given a collection $\mathcal{D}_n$ of intervals in $[0,1]$, remove an open interval of length $(1-2a_{n+1})\prod_{i=1}^{n} a_i$ ($\prod_{i=1}^{0} a_i = 1$) in the middle of each interval from $\mathcal{D}_n$ so that 2 card $(\mathcal{D}_n)$ intervals $I_1, \ldots, I_{2|\mathcal{D}_n|}$ remain. Set $\mathcal{D}_{n+1} = \{I_1, \ldots, I_{2card(\mathcal{D}_n)}\}$ and $C_{n+1} = \bigcup_{j=1}^{2\,\mathrm{card}\,(\mathcal{D}_n)} I_j$. Then for $n \geqslant 0$, $\mathcal{D}_n$ consists of 2^n intervals with disjoint interior of length $\prod_{i=1}^{n} a_i$. Let $\mathbf{C} = \bigcap_{n=0}^{\infty} C_n$. The corresponding homogeneous Cantor measure $\mu = \mu((a_k))$ is the unique Borel probability measure on $I\!R$ with

$$\mu(I) = 2^{-n} \text{ for every } I \in \mathcal{D}_n \qquad (2\cdot14)$$

and

$$\mu(A) = \inf\left\{\sum_j \mu(I_j) : A \cap \mathbf{C} \subset \bigcup_j I_j, I_j \in \bigcup_{n\geqslant 0} \mathcal{D}_n\right\}.$$

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It has support $\mathbf{C}((a_k))$ and it is obvious that μ is continuous. Furthermore, $\lambda(C_n) = 2^n \prod_{i=1}^{n} a_i$ so that

$$\lambda(\mathbf{C}) = \lim_{n\to\infty} 2^n \prod_{i=1}^{n} a_i.$$

If $\lambda(\mathbf{C}) > 0$, then one checks that

$$\mu = \lambda(\cdot \cap \mathbf{C})/\lambda(\mathbf{C}).$$

An overlap with self-similar distributions occurs in the case that $a_k = a \in (0, 1/2]$ for every k. Then $\mu((a_k))$ coincides with the self-similar measure corresponding to $p_1 = p_2 = 1/2$, $S_1, S_2 : \mathbb{R} \to \mathbb{R}$, $S_1 x = ax$ and $S_2 x = ax + 1 - a$.

The upper quantization dimension of μ (of order $r \in [1, \infty)$) is defined by

$$D = D(\mu) := \limsup_{N\to\infty} \frac{\log N}{-\log e_{N,r}(\mu, \mathbb{R})}. \tag{2·15}$$

Then $D \in [0, 1]$ (see [**7**, corollary 11·4]) and

$$D = \limsup_{n\to\infty} \frac{\log 2}{-\frac{1}{n}\sum_{i=1}^{n} \log a_i} \tag{2·16}$$

(see [**15**]). In particular, we see that D does not depend on r. If $\lim_{N\to\infty} \log N/-\log e_{N,r}(\mu, \mathbb{R})$ exists then we say that the quantization dimension of μ exists. In this case $\lim_{n\to\infty} \sum_{i=1}^{n} \log(a_i)/n$ exists and the quantization dimension equals

$$\lim_{n\to\infty} \frac{\log 2}{-\frac{1}{n}\sum_{i=1}^{n} \log a_i}.$$

We will employ the condition

$$\prod_{i=1}^{n} a_i = O(2^{-n/D}) \tag{2·17}$$

for which $D > 0$ is necessary. At least under certain restrictions on (a_k) this condition is related to the quantization problem for μ as follows (see [**16**, proposition 5·3]): Assume $\sup a_k \leqslant 1/3$ and that the quantization dimension exists. Then for every $r \in [1, \infty)$,

$$e_{N,r}(\mu, \mathbb{R}) = O(N^{-1/D}) \quad \text{if and only if (2·17).}$$

THEOREM 2. *Assume that $X = (X_t)_{t\in T}$, $T \subset \mathbb{R}$ satisfies* (2.1) *and* (2.2) *for some $\rho \in (0, \infty)$ where φ is regularly varying at 0 with index $b > 0$. Let $\mu = \mu((a_k))$ be a homogeneous Cantor distribution on $\mathbb{R}$ with upper quantization dimension $D > 0$ satisfying* (2·17) *such that $\mathrm{supp}(\mu) \subset T$. Then*

$$\forall r,\ p \in (0, \rho),\ e_{N,r}(X, L^p(\mu)) = O(\varphi((\log N)^{-1/D})) \text{ as } N \to \infty.$$

If $\rho \geqslant 1$, then this estimate is also true for $p = \rho$.

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Remarks. (a) Observe that in the self-similar case $a_k = a \in (0, 1/2]$ for every k, we have $D = -\log 2/\log a$, $D = D_\infty$ and $\prod_{i=1}^{n} a_i = 2^{-n/D}$ so that in this case Theorem 2 reduces to Theorem 1(b).

(b) A broad subclass of homogeneous Cantor measures with $D > 0$ correspond to sequences (a_k) with $\inf a_k > 0$ and (2.17) is satisfied provided $\lim a_k$ exists in $(0, \infty)$.

(c) If $D = 0$, then $\prod_{i=1}^{n} a_i = O(2^{-n/\eta})$ for every $\eta > 0$ and the proof in Section 5 shows that in this case

$$\forall r, p \in (0, \rho), e_{N,r}(X, L^p(\mu)) = O(\varphi((\log N)^{-1/\eta})) \ \forall \eta > 0.$$

3. *Time filtration and the functional quantization problem*

In this section we describe general tools which will be used to prove the results announced above. Using countable filtration of finite (almost) partitions for the underlying time space $\mathbb{R}^d$ a general upper estimate for the functional quantization errors in $L^p(\mu)$ for Borel probabilities μ on $\mathbb{R}^d$ with compact support will be formulated.

First introduce some notations. Let $\{1, \ldots, K\}^*$ denote the set of finite words over the alphabet $\{1, \ldots, K\}$ including the empty word $\varnothing$, $K \geqslant 2$. For $\sigma \in \{1, \ldots, K\}^*$ and $(t_1, \ldots, t_K) \in \mathbb{R}^K$, define

$$t_\sigma := \begin{cases} 1, & \sigma = \varnothing \\ \prod_{i=1}^{n} t_{\sigma_i}, & \sigma = \sigma_1 \ldots \sigma_n. \end{cases}$$

Let $|\sigma|$ be the length of σ where $|\varnothing| = 0$ and for $0 \leqslant m \leqslant |\sigma|$ let

$$\sigma_{|m} := \begin{cases} \varnothing, & m = 0 \\ \sigma_1 \ldots \sigma_m, & m \geqslant 1 \end{cases}$$

be the restriction of σ to the first m letters. Set $\sigma^- = \sigma_{||\sigma|-1}$ if $|\sigma| \geqslant 1$ and for $j \in \{1, \ldots, K\}$, let $\sigma * j = \sigma_1 \cdots \sigma_n j$, if $\sigma = \sigma_1 \cdots \sigma_n$ and $\sigma * j = j$ if $\sigma = \varnothing$. A natural partial order for words is defined by

$$\sigma \leqslant \tau \text{ iff } |\sigma| \leqslant |\tau| \text{ and } \tau_{||\sigma|} = \sigma. \tag{3·1}$$

In particular, $\varnothing \leqslant \sigma$ for every $\sigma \in \{1, \ldots, K\}^*$. For an infinite sequence $\eta \in \{1, \ldots, K\}^{\mathbb{N}}$ the restriction $\eta_{|m}$ is defined in an analogous way.

A finite nonempty set $\Gamma \subset \{1, \ldots, K\}^*$ is called a finite antichain if any two elements of Γ are incomparable with respect to (3·1). A finite antichain Γ is called maximal if, for every finite antichain $\Gamma' \subset \{1, \ldots, K\}^*$ with $\Gamma \subset \Gamma'$, we have $\Gamma = \Gamma'$. Note that a finite antichain Γ is maximal if and only if for every $\eta \in \{1, \ldots, K\}^{\mathbb{N}}$ there exists a (unique) $\sigma \in \Gamma$ such that $\eta_{||\sigma|} = \sigma$. If $(q_1, \ldots, q_K) \in [0, \infty)^K$ is a probability vector and Γ a maximal finite antichain then

$$\sum_{\sigma \in \Gamma} q_\sigma = 1. \tag{3·2}$$

A maximal finite antichain Γ' is called a refinement of the maximal finite antichain Γ if for every $\sigma \in \Gamma'$ there exists a (unique) $\tau \in \Gamma$ such that $\sigma \geqslant \tau$. Then the induced restriction

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mapping

$$v\colon \Gamma' \longrightarrow \Gamma,\ v(\sigma) = \tau \tag{3·3}$$

is surjective.

Observe that for a probability vector $(q_1, \ldots, q_K) \in (0, \infty)^K$ and $\varepsilon \in (0, 1]$, the set

$$\Gamma(\varepsilon) := \{\sigma \in \{1, \ldots, K\}^* : q_{\sigma^-} \geqslant \varepsilon > q_\sigma\}$$

is a maximal finite antichain and we have:

LEMMA 1. $1/\varepsilon < \mathit{card}\,(\Gamma(\varepsilon)) \leqslant (1/\min q_j)(1/\varepsilon)$.

Proof. Using (3·2), we obtain

$$1 = \sum_{\sigma\in\Gamma(\varepsilon)} q_\sigma \geqslant \sum_{\sigma\in\Gamma(\varepsilon)} q_{\sigma^-} \min q_j \geqslant \varepsilon \operatorname{card}(\Gamma(\varepsilon)) \min q_j$$

and

$$1 = \sum_{\sigma\in\Gamma(\varepsilon)} q_\sigma < \varepsilon \operatorname{card}(\Gamma(\varepsilon)).$$

Now let μ be a Borel probability measure on $\mathbb{R}^d$ with compact support and

$$A := \operatorname{supp}(\mu).$$

A filtration for μ is a family $(A_\sigma)_{\sigma\in\{1,\ldots,K\}^*}$ of Borel subsets of A with the following properties:

$$\begin{aligned} &A_\varnothing = A, \\ &\forall \sigma \in \{1, \ldots, K\}^*, \mu(A_\sigma) > 0, \\ &\forall \sigma \in \{1, \ldots, K\}^*, A_\sigma = \bigcup_{i=1}^{K} A_{\sigma * i} \text{ and } \mu(A_{\sigma*i} \cap A_{\sigma*j}) = 0 \text{ if } i \neq j. \end{aligned} \tag{3·4}$$

One checks that then for every finite antichain Γ,

$$\mu(A_\sigma \cap A_\tau) = 0 \text{ for all } \sigma, \tau \in \Gamma, \sigma \neq \tau \tag{3·5}$$

and for every maximal finite antichain Γ,

$$\bigcup_{\sigma\in\Gamma} A_\sigma = A. \tag{3·6}$$

For a maximal finite antichain $\Gamma \subset \{1, \ldots, K\}^*$, consider the subspace

$$V_\Gamma := \operatorname{span}\{1_{A_\sigma} : \sigma \in \Gamma\}$$

of $L^p(\mu)$ and for $f \in L^1(\mu)$ set

$$f^\Gamma := \sum_{\sigma\in\Gamma} \frac{\int_{A_\sigma} f d\mu}{\mu(A_\sigma)} 1_{A_\sigma}. \tag{3·7}$$

Since $f^\Gamma = E_\mu(f|\mathcal{B}_\Gamma)\mu$-a.s. with $\mathcal{B}_\Gamma$ the σ-algebra generated by $A_\sigma, \sigma \in \Gamma$, the linear operator $f \mapsto f^\Gamma$ is a norm 1 projection of $L^p(\mu)$ onto $V_\Gamma = L^p(\mathcal{B}_\Gamma, \mu)$, $p \geqslant 1$.

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The following proposition provides the main tool for the proof of Theorems 1 and 2.

PROPOSITION 1. *Assume that the measurable process* $X = (X_t)_{t\in T}$, $T \subset \mathbb{R}^d$ *satisfies* (2·1) *and* (2·2) *for some* $\rho \in (0, \infty)$ *where* φ *is regular varying at zero. Let* μ *be a Borel probability measure on* $\mathbb{R}^d$ *with compact support* A *with* $A \subset T$. *Let* $(A_\sigma)_{\sigma\in\{1,\dots,K\}^*}$ *be a filtration for* μ *and let* $(\Gamma_n)_{n\geqslant 0}$ *be a sequence of maximal finite antichains such that* $\Gamma_0 = \{\varnothing\}$ *and* Γ_n *is a refinement of* Γ_{n-1}. *Let* $f : \{1, \dots, K\}^* \to (0, \infty)$ *be a decreasing function such that* $diam(A_\sigma) \leqslant f(\sigma)$ *for every* σ. *Assume*

$$\exists C \geqslant 1,\ \forall n \geqslant 1\, \forall \sigma \in \Gamma_n,\ \ f(v_n(\sigma)) \leqslant C f(\sigma) \text{with}$$
$$v_n : \Gamma_n :\longrightarrow \Gamma_{n-1} \text{ the restriction map from (3·3)} \tag{3·8}$$

and

$$\exists \gamma > 1,\ card(\Gamma_n) \approx \gamma^n \text{ as } n \to \infty. \tag{3·9}$$

Then for every $N \in \mathbb{N}$,

$$\forall p \in (0, \rho), \forall r \in (0, p], e_{N,r}(X, L^p(\mu)) \leqslant C_1 \left(\sum_{\sigma\in\Gamma_{n(N)}} \mu(A_\sigma)\varphi(f(\sigma))^p \right)^{1/p}$$

where

$$n(N) = \left\lfloor \frac{\log\log N}{\log \gamma} + C_2 \right\rfloor^+$$

for a suitable real constant C_2.

For several fractal measures μ there is a natural choice of the filtration and the function f while one has to optimize the sequence of maximal antichains.

For the proof we need the following 1-dimensional feature (see [**7**], [**21**]).

LEMMA 2. *Let* $r, \delta > 0$. *There exists a real constant* $C_{r,\delta}$ *such that, for every real random variable* ξ *and every* $N \geqslant 1$,

$$e_{N,r}(\xi, \mathbb{R}) \leqslant C_{r,\delta} ||\xi||_{L^{r+\delta}(\mathbb{P})} N^{-1}.$$

Since $e_{N,r}(X, L^p(\mu))$ is increasing in r and in p, it is enough to consider the case $p = r$. Condition (2·3) implies that the paths of X a.s. lie in $L^\rho(\mu)$ and thus in $L^1(\mu)$ if $\rho \geqslant 1$. When $\rho < 1$ the sample paths of X lie in $L^1(\mu)$ by assumption.

For a maximal finite antichain $\Gamma \subset \{1, \dots, K\}^*$, let X^Γ denote the projection of X onto V_Γ as defined in (3·7). Notice that condition (2·2) holds for any $p \in (0, \rho]$ instead of ρ. Let $E = L^p(\mu)$.

The proof of Proposition 1 relies on the following two estimates.

LEMMA 3. *Let* $p \leqslant \rho$. *Then under conditions* (2·1) *and* (2·2),

$$||X - X^\Gamma||_{L^p_E(\mathbb{P})} \leqslant 2 \left(\sum_{\sigma\in\Gamma} \mu(A_\sigma)\varphi(diam(A_\sigma))^p \right)^{1/p}.$$

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Proof. Using (3·5) and (3·6), we obtain

$$||X - X^\Gamma||_E^p = \int \left| X_t - \sum_{\sigma\in\Gamma} \frac{\int_{A_\sigma} X_s d\mu(s)}{\mu(A_\sigma)} 1_{A_\sigma}(t) \right|^p d\mu(t)$$

$$= \sum_{\sigma\in\Gamma} \int_{A_\sigma} \left| X_t - \frac{\int_{A_\sigma} X_s d\mu(s)}{\mu(A_\sigma)} \right|^p d\mu(t)$$

$$= \sum_{\sigma\in\Gamma} \int_{A_\sigma} \left| \int_{A_\sigma} (X_t - X_s) d\mu(s) \mu(A_\sigma)^{-1} \right|^p d\mu(t).$$

If $p \geqslant 1$, then by Jensen's inequality

$$||X - X^\Gamma||_E^p \leqslant \sum_{\sigma\in\Gamma} \int_{A_\sigma} \int_{A_\sigma} |X_t - X_s|^p d\mu(s) \mu(A_\sigma)^{-1} d\mu(t)$$

so that by (2·2), using the monotony of φ

$$\begin{aligned} I\!E||X - X^\Gamma||_E^p &\leqslant \sum_{\sigma\in\Gamma} \mu(A_\sigma)^{-1} \int_{A_\sigma} \int_{A_\sigma} I\!E|X_t - X_s|^p d\mu(s) d\mu(t) \\ &\leqslant \sum_{\sigma\in\Gamma} \mu(A_\sigma)^{-1} \int_{A_\sigma} \int_{A_\sigma} \varphi(\text{diam } (A_\sigma))^p d\mu(s) d\mu(t) \\ &\leqslant \sum_{\sigma\in\Gamma} \mu(A_\sigma) \varphi(\text{diam}(A_\sigma))^p. \end{aligned}$$

Now let $p < 1$. If $\rho \geqslant 1$, then again by Jensen's inequality

$$\begin{aligned} I\!E||X - X^\Gamma||_E^p &= \sum_{\sigma\in\Gamma} \int_{A_\sigma} I\!E \left| \int_{A_\sigma} (X_t - X_s) d\mu(s) \mu(A_\sigma)^{-1} \right|^p d\mu(t) \\ &\leqslant \sum_{\sigma\in\Gamma} \int_{A_\sigma} (I\!E \int_{A_\sigma} |X_t - X_s| d\mu(s) \mu(A_\sigma)^{-1})^p d\mu(t) \\ &\leqslant \sum_{\sigma\in\Gamma} \mu(A_\sigma) \varphi(\text{diam}(A_\sigma))^p. \end{aligned}$$

If $\rho < 1$, using

$$||X - X^\Gamma||_E^p \leqslant \sum_{\sigma\in\Gamma} \mu(A_\sigma) \sup_{s,t\in A_\sigma} |X_t - X_s|^p \leqslant 2^p \sum_{\sigma\in\Gamma} \mu(A_\sigma) \sup_{s\in A_\sigma} |X_s - X_{t^\sigma}|^p$$

where $t^\sigma \in A_\sigma$, we obtain from (2·2)

$$I\!E||X - X^\Gamma||_E^p \leqslant 2^p \sum_{\sigma\in\Gamma} \mu(A_\sigma) \varphi(\text{diam}(A_\sigma))^p.$$

LEMMA 4. *Let $s \in (0, \infty)$, $(b_n)_{n\geqslant 0}$ be a decreasing sequence in $(0, \infty)$ and $(l_n)_{n\geqslant 0}$ an increasing sequence in $(0, \infty)$. Assume*

$$\exists \gamma > 1, l_n \approx \gamma^n \text{ as } n \to \infty \tag{3·10}$$

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and

$$\exists C \geqslant 1, \forall n \geqslant 0, b_n \leqslant C b_{n+1}. \tag{3·11}$$

Then for every $N \in \mathbb{N}$,

$$\begin{aligned}\inf\left\{\sum_{j=0}^{n} b_j N_j^{-s} + b_n : n \geqslant 0, N_0, \ldots, N_n \in \mathbb{N}, \prod_{j=0}^{n} N_j^{l_j} \leqslant N\right\} \\ \leqslant \min\left\{\sum_{j=0}^{n(N)} b_j N_j^{-s} + b_{n(N)} : N_0, \ldots, N_{n(N)} \in \mathbb{N}, \prod_{j=0}^{n(N)} N_j^{l_j} \leqslant N\right\} \\ \leqslant C_1 b_{n(N)}\end{aligned}$$

where

$$n(N) = \left\lfloor \frac{\log\log N}{\log \gamma} + C_2 \right\rfloor^{+} \tag{3·12}$$

for some suitable real constant C_2 .

Proof. For a given $N \in \mathbb{N}$, we may first optimize the integer allocation given by the $N_j's$ for fixed n and then optimize n. To this end, note that the continuous allocation problem reads

$$\inf\left\{\sum_{j=0}^{n} b_j y_j^{-s} : y_j > 0, \prod_{j=0}^{n} y_j^{l_j} \leqslant N\right\} = \sum_{j=0}^{n} b_j z_j^{-s} = L N^{-s/L}\left(\prod_{i=0}^{n}\left(\frac{b_i}{l_i}\right)^{l_i}\right)^{1/L}$$

where $L = L(n) = \sum_{j=0}^{n} l_j$,

$$z_j = z_j(n) = N^{1/L}\left(\frac{b_j}{l_j}\right)^{1/s}\left(\prod_{i=0}^{n}\left(\frac{b_i}{l_i}\right)^{l_i}\right)^{-1/sL}, \; j \in \{1, \ldots, n\}$$

and $z_1 \geqslant \cdots \geqslant z_n$ since $(b_j/l_j)_{j\geqslant 0}$ is decreasing. One can produce an (approximate) integer solution by setting

$$N_j = N_j(n) := \lfloor z_j \rfloor = \left\lfloor N^{1/L}\left(\frac{b_j}{l_j}\right)^{1/s}\left(\prod_{i=0}^{n}\left(\frac{b_i}{l_i}\right)^{l_i}\right)^{-1/sL}\right\rfloor, \; j \in \{1, \ldots, n\} \tag{3·13}$$

provided $z_n \geqslant 1$. Then

$$\begin{aligned}\sum_{j=0}^{n} b_j N_j^{-s} &= \sum_{j=0}^{n} b_j (N_j + 1)^{-s}\left(\frac{N_j + 1}{N_j}\right)^{s} \\ &\leqslant 2^s \sum_{j=0}^{n} b_j z_j^{-s} \\ &= 2^s L N^{-s/L}\left(\prod_{i=0}^{n}\left(\frac{b_i}{l_i}\right)^{l_i}\right)^{1/L} \\ &\leqslant 2^s L \frac{b_n}{l_n}.\end{aligned}$$

In the sequel C denotes a constant whose value may change from line to line. By (3·10),

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$L(n) \approx l_n$ since

$$\frac{L(n)}{l_n} \leqslant C\gamma^{-n}\sum_{i=0}^{n}\gamma^i = C\gamma^{-n}\frac{L(n)}{l_n} \leqslant C\gamma^{-n}\sum_{i=0}^{n}\gamma^i = C\gamma^{-n}\frac{\gamma^{n+1}-1}{\gamma-1} \leqslant C\frac{\gamma}{\gamma-1}$$

so that

$$\sum_{j=0}^{n} b_j N_j^{-s} \leqslant C b_n. \tag{3·14}$$

The constraint on n reads $n \in I(N)$ with

$$I(N) := \{n \geqslant 0 : z_n(n) \geqslant 1\} = \{n \geqslant 0 : a_n \leqslant N\}$$

where

$$a_n := \left(\frac{l_n}{b_n}\right)^{L(n)/s}\left(\prod_{i=0}^{n}\left(\frac{b_i}{l_i}\right)^{l_i}\right)^{1/s} = \left(\prod_{i=0}^{n}\left(\frac{b_i/l_i}{b_n/l_n}\right)^{l_i}\right)^{1/s}.$$

The sequence $(a_n)_{n\geqslant 0}$ is increasing and

$$a_n \geqslant \left(\frac{b_0/l_0}{b_n/l_n}\right)^{l_0/s} \longrightarrow \infty \text{ as } n \to \infty$$

so that $I(N) = \{0, \ldots, n^*(N)\}$ is finite. Thus we arrive at the following upper bound for the integer allocation

$$\min_{n\in I(N)} \min\left\{\sum_{j=0}^{n} b_j N_j^{-s} + b_n : \prod_{j=0}^{n} N_j^{l_j} \leqslant N\right\} \leqslant C \min_{n\in I(N)} b_n = C b_{n^*(N)}. \tag{3·15}$$

One checks that

$$a_{n+1} = \left(\frac{l_{n+1}b_n}{b_{n+1}l_n}\right)^{L(n)/s} a_n, a_0 = 1.$$

By (3·10) and (3·11), this implies

$$a_{n+1} \leqslant e^{CL(n)}a_n$$

for some real constant $C > 0$ and induction yields

$$a_n \leqslant e^{C\sum_{i=0}^{n-1} L(i)}.$$

Again by (3·10), $\sum_{i=0}^{n-1} L(i) \leqslant C\gamma^n$ so that

$$a_n \leqslant e^{C\gamma^n}, C > 0.$$

Consequently, $n \in I(N)$ provided $\exp(C\gamma^n) \leqslant N$ and this inequality is equivalent to

$$n \leqslant \frac{\log\log N}{\log\gamma} - \frac{\log C}{\log\gamma}.$$

This completes the proof.

Proof of Proposition 1. For $n \geqslant 0$, let $X^n := X^{\Gamma_n}$. It follows from Lemma 3 that

$$\|X - X^n\|_{L_E^p(\mathbb{P})}^p \leqslant 2^p \sum_{\sigma\in\Gamma_n} \mu(A_\sigma)\varphi(\operatorname{diam}(A_\sigma))^p \leqslant 2^p \sum_{\sigma\in\Gamma_n} \mu(A_\sigma)\varphi(f(\sigma))^p. \tag{3·16}$$

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Obviously

$$X^n = X^0 + \sum_{k=1}^{n} (X^k - X^{k-1}).$$

Using (3·5), surjectivity of $v_k : \Gamma_k \to \Gamma_{k-1}$, and

$$A_\tau = \bigcup_{\substack{\sigma \in \Gamma_k \\ v_k(\sigma) = \tau}} A_\sigma, \tau \in \Gamma_{k-1}$$

we obtain for $k \geqslant 1$,

$$\begin{aligned} X^k - X^{k-1} &= \sum_{\sigma \in \Gamma_k} \mu(A_\sigma)^{-1} \int_{A_\sigma} X_t d\mu(t) 1_{A_\sigma} - \sum_{\tau \in \Gamma_{k-1}} \mu(A_\tau)^{-1} \int_{A_\tau} X_t d\mu(t) 1_{A_\tau} \\ &= \sum_{\sigma \in \Gamma_k} (\mu(A_\sigma)^{-1} \int_{A_\sigma} X_t d\mu(t) - \mu(A_{v_k(\sigma)})^{-1} \int_{A_{v_k(\sigma)}} X_t d\mu(t)) 1_{A_\sigma} \\ &= \sum_{\sigma \in \Gamma_k} \xi_{\sigma,k} 1_{A_\sigma} \end{aligned}$$

and

$$X^0 = \xi_{\varnothing,0} 1_A$$

where, for $\sigma \in \Gamma_k, k \geqslant 1$,

$$\begin{aligned} \xi_{\sigma,k} &:= \frac{1}{\mu(A_\sigma)} \int_{A_\sigma} X_t d\mu(t) - \frac{1}{\mu(A_{v_{k(\sigma)}})} \int_{A_{v_{k(\sigma)}}} X_t d\mu(t) \\ &= \frac{1}{\mu(A_\sigma)\mu(A_{v_{k(\sigma)}})} \int_{A_\sigma} \int_{A_{v_k(\sigma)}} (X_t - X_s) d\mu(s) d\mu(t), \qquad (3\cdot17) \\ \xi_{\varnothing,0} &:= \int X_t d\mu(t) \end{aligned}$$

so that

$$X^n = \sum_{k=0}^{n} \sum_{\sigma \in \Gamma_k} \xi_{\sigma,k} 1_{A_\sigma}.$$

Let $l_k := \mathrm{card}(\Gamma_k)$. One considers a product N-quantization

$$\hat{X} = \hat{X}^{n, N_0, \ldots, N_n} := \sum_{k=0}^{n} \sum_{\sigma \in \Gamma_k} \hat{\xi}_{\sigma,k} 1_{A_\sigma} \qquad (3\cdot18)$$

with $L^p(I\!P)$-optimal N_k-quantization $\hat{\xi}_{\sigma,k}$ of $\xi_{\sigma,k}$ satisfying $\prod_{k=0}^{n} N_k^{l_k} \leqslant N$. Then

$$\begin{aligned} e_{N,p}(X, L^p(\mu)) &\leqslant ||X - \hat{X}||_{L_E^p(I\!P)} \leqslant ||X - X^n||_{L_E^p(I\!P)} + ||X^n - \hat{X}||_{L_E^p(I\!P)} \text{ if } p \geqslant 1, \\ e_{N,p}(X, L^p(\mu))^p &\leqslant ||X - \hat{X}||^p_{L_E^p(I\!P)} \leqslant ||X - X^n||^p_{L_E^p(I\!P)} + ||X^n - \hat{X}||^p_{L_E^p(I\!P)} \text{ if } p < 1 \end{aligned} \qquad (3\cdot19)$$

using that $||\cdot||_{L_E^p(I\!P)}$ is a norm if $p \geqslant 1$ and one relies on the inequality

$$||U + V||^p_{L_E^p(I\!P)} \leqslant ||U||^p_{L_E^p(I\!P)} + ||V||^p_{L_E^p(I\!P)}$$

for $E = L^p(\mu)$-valued random vectors if $p < 1$.

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One obtains from Lemma 2 for $p < \rho$ in case $p \geqslant 1$,

$$\begin{aligned}
||X^n - \hat{X}||_{L_E^p(\mathbb{P})} &= \left|\left| \sum_{k=0}^{n} \sum_{\sigma \in \Gamma_k} (\xi_{\sigma,k} - \hat{\xi}_{\sigma,k}) 1_{A_\sigma} \right|\right|_{L_E^p(\mathbb{P})} \\
&\leqslant \sum_{k=0}^{n} \left|\left| \sum_{\sigma \in \Gamma_k} (\xi_{\sigma,k} - \hat{\xi}_{\sigma,k}) 1_{A_\sigma} \right|\right|_{L_E^p(\mathbb{P})} \\
&= \sum_{k=0}^{n} \left(\sum_{\sigma \in \Gamma_k} ||\xi_{\sigma,k} - \hat{\xi}_{\sigma,k}||^p_{L_E^p(\mathbb{P})} \mu(A_\sigma) \right)^{1/p} \\
&= \sum_{k=0}^{n} \left(\sum_{\sigma \in \Gamma_k} e_{N_k,p}(\xi_{\sigma,k}, \mathbb{R})^p \mu(A_\sigma) \right)^{1/p} \\
&\leqslant C \sum_{k=0}^{n} \left(\sum_{\sigma \in \Gamma_k} ||\xi_{\sigma,k}||^p_{L^\rho(\mathbb{P})} \mu(A_\sigma) \right)^{1/p} N_k^{-1}.
\end{aligned}$$

We have for $k \geqslant 1$,

$$||\xi_{\sigma,k}||^\rho_{L^\rho(\mathbb{P})} \leqslant E\left(\int_{A_\sigma} \int_{A_{v_k(\sigma)}} |X_t - X_s| d\mu(s) \mu(A_{v_{k(\sigma)}})^{-1} d\mu(t) \mu(A_\sigma)^{-1} \right)^\rho.$$

and one argues as in Lemma 3 to show that

$$||\xi_{\sigma,k}||_{L^\rho(\mathbb{P})} \leqslant 2\varphi(\operatorname{diam}(A_{v_k(\sigma)})).$$

Using (3·8) and regular variation of φ this yields

$$||\xi_{\sigma,k}||_{L^\rho(\mathbb{P})} \leqslant C\varphi(f(\sigma)).$$

Note that this inequality is also true for $k = 0$. Therefore, if $p \geqslant 1$

$$||X^n - \hat{X}||_{L_E^p(\mathbb{P})} \leqslant C \sum_{k=0}^{n} \left(\sum_{\sigma \in \Gamma_k} \mu(A_\sigma) \varphi(f(\sigma))^p \right)^{1/p} N_k^{-1}$$

and if $p < 1$,

$$||X^n - \hat{X}||^p_{L^p(\mathbb{P})} \leqslant C \sum_{k=0}^{n} \left(\sum_{\sigma \in \Gamma_k} \mu(A_\sigma) \varphi(f(\sigma))^p \right) N_k^{-p}.$$

Consequently, using (3·16) and (3·19)

$$\begin{aligned}
||X - \hat{X}||_{L^p(\mathbb{P})} &\leqslant C \left(\sum_{k=0}^{n} b_k N_k^{-1} + b_n \right) \text{ if } p \geqslant 1, \\
||X - \hat{X}||^p_{L^p(\mathbb{P})} &\leqslant C \left(\sum_{k=0}^{n} b_k^p N_k^{-p} + b_n^p \right) \text{ if } p < 1
\end{aligned} \tag{3·20}$$

where for $k \geqslant 0$,

$$b_k := \left(\sum_{\sigma \in \Gamma_k} \mu(A_\sigma) \varphi(f(\sigma))^p) \right)^{1/p}.$$

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We deduce

$$e_{N,p}(X, L^p(\mu)) \leqslant C \inf\left\{\sum_{k=0}^{n} b_k N_k^{-1} + b_n : n \geqslant 0, N_0, \ldots, N_n \in \mathbb{N}, \prod_{k=0}^{n} N_k^{l_k} \leqslant N\right\} \quad \text{if } p \geqslant 1,$$

$$e_{N,p}(X, L^p(\mu))^p \leqslant C \inf\left\{\sum_{k=0}^{n} b_k^p N_k^{-p} + b_n^p : n \geqslant 0, N_0, \ldots, N_n \in \mathbb{N}, \prod_{k=0}^{n} N_k^{l_k} \leqslant N\right\} \quad \text{if } p < 1. \tag{3·21}$$

The sequence $(b_k)_{k\geqslant 0}$ is decreasing since

$$\begin{aligned} b_{k+1} &= \left(\sum_{\sigma\in\Gamma_{k+1}} \mu(A_\sigma)\varphi(f(\sigma))^p\right)^{1/p} \\ &= \left(\sum_{\tau\in\Gamma_k}\sum_{\substack{\sigma\in\Gamma_{k+1}\\ v_{k+1}(\sigma)=\tau}} \mu(A_\sigma)\varphi(f(\sigma))^p\right)^{1/p} \\ &\leqslant \left(\sum_{\tau\in\Gamma_k}\sum_{\substack{\sigma\in\Gamma_{k+1}\\ v_{k+1}(\sigma)=\tau}} \mu(A_\sigma)\varphi(f(\tau))^p\right)^{1/p} \\ &= \left(\sum_{\tau\in\Gamma_k} \mu\left(\bigcup_{\substack{\sigma\in\Gamma_{k+1}\\ v_{k+1}(\sigma)=\tau}} A_\sigma\right)\varphi(f(\tau))^p\right)^{1/p} \\ &= \left(\sum_{\tau\in\Gamma_k} \mu(A_\tau)\varphi(f(\tau))^p\right)^{1/p} \\ &= b_k. \end{aligned}$$

Moreover, $(b_k)_{k\geqslant 0}$ satisfies the weak regularity condition (3·11) since by (3·8), monotony and regular variation of φ,

$$\begin{aligned} b_{k+1} &= \left(\sum_{\tau\in\Gamma_k}\sum_{\substack{\sigma\in\Gamma_{k+1}\\ v_{k+1}(\sigma)=\tau}} \mu(A_\sigma)\varphi(f(\sigma))^p\right)^{1/p} \\ &\geqslant C\left(\sum_{\tau\in\Gamma_{k+1}}\sum_{\substack{\sigma\in\Gamma_{k+1}\\ v_{k+1}(\sigma)=\tau}} \mu(A_\sigma)\varphi(f(\tau))^p\right)^{1/p} \\ &= Cb_k. \end{aligned}$$

Finally, the sequence $(l_k)_{k\geqslant 0}$ is increasing since the mappings $v_{k+1}\colon \Gamma_{k+1} \to \Gamma_k$ are surjective. Now the assertion follows from Lemma 4.

4. *Proof of Theorem* 1 *for* $p < \rho$

In this section we will use Proposition 1 to prove the main part of Theorem 1.

Let μ be the self-similar probability on $\mathbb{R}^d$ corresponding to $(S_1, \ldots, S_K, p_1, \ldots, p_K)$ satisfying the OSC, where $(p_1, \ldots, p_K) \in (0, \infty)^K$ is a probability vector. For $\sigma \in$

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$\{1, \dots, K\}^*$ set

$$S_\sigma := \begin{cases} Id, & \sigma = \varnothing \\ S_{\sigma_1} \circ \cdots \circ S_{\sigma_n}, & \sigma = \sigma_1 \cdots \sigma_n \end{cases}$$

and

$$A_\sigma := S_\sigma(A) \tag{4·1}$$

where $A = \operatorname{supp}(\mu)$. Then $(A_\sigma)_{\sigma \in \{1,\dots,K\}^*}$ is a filtration for μ and

$$\mu(A_\sigma) = p_\sigma. \tag{4·2}$$

(see [**6**]). As for the choice of maximal finite antichains let

$$q_j := \left(p_j s_j^{bp}\right)^{D_{bp}/(D_{bp}+bp)} \text{ for } j \in \{1, \dots, K\}.$$

Then by (2·6), $\sum_{j=1}^K q_j = 1$. Let $q = q_{\max} := \max q_j$ and set

$$\Gamma_n := \Gamma(q^{n-1}) = \{\sigma \in \{1, \dots, K\}^* : q_{\sigma^-} \geqslant q^{n-1} > q_\sigma\} \text{ if } n \geqslant 1, \Gamma_0 := \{\varnothing\}. \tag{4·3}$$

Then Γ_n is a maximal finite antichain and Γ_n is a refinement of Γ_{n-1}. By Lemma 1 we know that

$$\operatorname{card}(\Gamma_n) \approx q^{-n} \text{ as } n \to \infty$$

so that condition (3·9) is satisfied with $\gamma = 1/q$. For $\sigma \in \Gamma_1$ we have $v_1(\sigma) = \varnothing$ and for $\sigma \in \Gamma_n$ with $n \geqslant 2$ we have

$$v_n(\sigma) = \begin{cases} \sigma^-, & q_{\sigma^-} < q^{n-2} \\ \sigma, & q_{\sigma^-} \geqslant q^{n-2}. \end{cases}$$

Since $\operatorname{diam}(A_\sigma) = s_\sigma \operatorname{diam}(A)$ for all $\sigma \in \{1, \dots, K\}^*$ we deduce that for every $n \geqslant 1$ and every $\sigma \in \Gamma_n$,

$$\operatorname{diam}(A_{v_n(\sigma)}) = \begin{cases} \operatorname{diam}(A_\sigma), & v_n(\sigma) = \sigma \\ s_{\sigma_{|\sigma|}}^{-1} \operatorname{diam}(A_\sigma), & v_n(\sigma) = \sigma^- \end{cases}$$

This implies

$$\operatorname{diam}(A_{v_n(\sigma)}) \leqslant s_{\min}^{-1} \operatorname{diam}(A_\sigma).$$

where $s_{\min} := \min s_j$. Thus condition (3·8), is satisfied with $f(\sigma) := \operatorname{diam}(A_\sigma)$. It follows from Proposition 1 using regular variation of φ that for every $N \in \mathbb{N}$,

$$\begin{aligned} \forall p \in (0, \rho), \forall r \in (0, p], e_{N,r}(X, L^p(\mu)) &\leqslant C \left(\sum_{\sigma \in \Gamma_{n(N)}} \mu(A_\sigma) \varphi(\operatorname{diam}(A_\sigma))^p \right)^{1/p} \\ &\leqslant C \left(\sum_{\sigma \in \Gamma_{n(N)}} p_\sigma \varphi(s_\sigma)^p \right)^{1/p} \end{aligned} \tag{4·4}$$

where

$$n(N) = \left\lfloor \frac{\log \log N}{-\log q} + C_1 \right\rfloor^+, C_1 \in \mathbb{R}. \tag{4·5}$$

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Next we provide an estimate for

$$B_n := \sum_{\sigma\in\Gamma_n} p_\sigma \varphi(s_\sigma)^p.$$

Note that $B_0 = \varphi(1)^p$.

(a) For $n \geqslant 1$ and $p > 0$ we have

$$B_n = \sum_{\sigma\in\Gamma_n} p_\sigma s_\sigma^{bp} \psi(s_\sigma)^p = \sum_{\sigma\in\Gamma_n} q_\sigma q_\sigma^{bp/D_{bp}} \psi(s_\sigma)^p \tag{4·6}$$

where ψ is the slowly varying part of φ. For $\sigma \in \Gamma_n$ we have $q_\sigma < q^{n-1}$, hence

$$q_\sigma^{bp/D_{bp}} \leqslant q^{(n-1)bp/D_{bp}}.$$

Assume that ψ is decreasing on (0,1). For $\sigma \in \Gamma_n$ we have

$$q^{|\sigma|-1} = q^{|\sigma^-|} \geqslant q_{\sigma^-} \geqslant q^{n-1}$$

so that $|\sigma| \leqslant n$. This implies

$$s_\sigma \geqslant s_{\min}^{|\sigma|} \geqslant s_{\min}^n$$

where $s_{\min} := \min s_j$. Thus using regular variation of $\varphi \circ \exp(-\cdot)$ at infinity we obtain

$$\begin{aligned}
\psi(s_\sigma) &\leqslant \psi(s_{\min}^n) \\
&= \psi(\exp(-n(\log s_{\min}/\log q^{1/D_{bp}})(-\log q^{1/D_{bp}}))) \\
&\leqslant C\psi(\exp(n \log q^{1/D_{bp}})) \\
&= C\psi(q^{n/D_{bp}}).
\end{aligned}$$

Assume that ψ is increasing on (0,1). For $\sigma \in \Gamma_n$ we have

$$q_{\min}^{|\sigma|} \leqslant q_\sigma < q^{n-1}$$

where $q_{\min} := \min q_j$. This implies

$$s_\sigma \leqslant s_{\max}^{|\sigma|} \leqslant s_{\max}^{(n-1)\log q/\log q_{\min}}$$

where $s_{\max} := \max s_j$ and therefore

$$\psi(s_\sigma) \leqslant \psi\left(s_{\max}^{(n-1)\log q/\log q_{\min}}\right).$$

As above one shows that

$$\psi(s_\sigma) \leqslant C\psi(q^{(n-1)/D_{bp}}).$$

In view of (3·2) these inequalities lead to

$$B_n \leqslant C\varphi(q^{n/D_{bp}})^p. \tag{4·7}$$

Consequently,

$$B_{n(N)} \leqslant C\varphi((\log N)^{-1/D_{bp}})^p$$

and the assertion follows from (4·4).

(b) Under the assumption $p_i = 1/K$ and $s_i = s$ for some $s \in (0,1)$ and for every $i \in \{1,\ldots,K\}$, the numbers $p_\sigma\varphi(s_\sigma)^p$ only depend on the length of σ,

$$p_\sigma \varphi(s_\sigma)^p = K^{-n}\varphi(s^n)^p \text{ if } |\sigma| = n.$$

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Moreover, $q_i = 1/K = q$, $D_{bp} = D_\infty = -\log K/\log s$ and $\Gamma_n = \{1, \dots, K\}^n$, $n \geqslant 0$. Therefore,

$$B_n = \varphi(s^n)^p = \varphi(K^{-n/D_\infty})^p.$$

Consequently, the assertion follows from (4·4) and (4·5).

5. *Proof of Theorem* 2 *for* $p < \rho$

For $n \geqslant 0$ and $\sigma \in \{1, 2\}^n$ define inductively

$$I_\sigma = [0, 1] \text{ if } \sigma = \varnothing \text{ that is, } n = 0$$

and, if I_{σ^-} is an already defined closed interval $[a, b] \subset [0, 1]$ then

$$I_\sigma = \begin{cases} [a, a + a_n(b-a)], & \sigma_{|\sigma|} = 1 \\ [b - a_n(b-a), b], & \sigma_{|\sigma|} = 2. \end{cases}$$

Then the generalized Cantor set $\mathbf{C} = \mathbf{C}((a_n))$ satisfies

$$\mathbf{C} = \bigcap_{n \geqslant 0} \bigcup_{\sigma \in \{1,2\}^n} I_\sigma.$$

For $A := \mathbf{C}$ and $A_\sigma := \mathbf{C} \cap I_\sigma$, the family $(A_\sigma)_{\sigma \in \{1,2\}^*}$ is a filtration for the homogeneous Cantor measure $\mu = \mu((a_n))$ with

$$\text{diam } (A_\sigma) = \prod_{i=1}^{|\sigma|} a_i$$

and

$$\mu(A_\sigma) = 2^{-|\sigma|}$$

for all $\sigma \in \{1, 2\}^*$ where $\prod_{i=1}^0 a_i := 1$ (see (2·14)). For $n \geqslant 0$ set

$$\Gamma_n := \{1, 2\}^n.$$

Then $(\Gamma_n)_{n \geqslant 0}$ is a sequence of maximal finite antichains such that $\Gamma_0 = \{\varnothing\}$, Γ_n is a refinement of Γ_{n-1} and $\text{card}(\Gamma_n) = 2^n$. Thus condition (3·9) is satisfied with $\gamma = 2$. By (2·17), we have

$$\text{diam } (A_\sigma) \leqslant C_1 2^{-|\sigma|/D} =: f(\sigma)$$

for every $\sigma \in \{1, 2\}^*$. Since for $\sigma \in \Gamma_n$ with $n \geqslant 1$ we have $v_n(\sigma) = \sigma^-$, condition (3.8) is satisfied. Consequently by Proposition 1 using regular variation of φ,

$$\begin{aligned} \forall p \in (0, \rho), \forall r \in (0, p], e_{N,r}(X, L^p(\mu)) &\leqslant C \left(\sum_{\sigma \in \Gamma_{n(N)}} 2^{-|\sigma|} \varphi(C_1 2^{-|\sigma|/D})^p \right)^{1/p} \\ &\leqslant C\varphi(2^{-n(N)/D}) \\ &\leqslant C\varphi((\log N)^{-1/D}). \end{aligned}$$

6. *A nonconstructive approach and the case* $p = \rho$

In this section we derive a precise link between the quantization errors of the Borel probability measure μ on $\mathbb{R}^d$ with respect to suitable loss functions and the functional quantization errors $e_{N,r}(X, L^p(\mu))$. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function. Beside the usual

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quantization errors

$$e_{N,r}(\mu, I\!R^d) = \inf\left\{\left(\int \min_{t\in\alpha} |x-t|^r d\mu(x)\right)^{1/r} : \alpha \subset I\!R^d,\ \ \mathrm{card}(\alpha) \leqslant N\right\}$$

provided $\int |x|^r d\mu(x) < \infty$ (see (1·2)) we consider for $T \subset I\!R^d$

$$e_{N,\varphi,p}(\mu, T) := \inf\left\{\left(\int \min_{t\in\alpha} \varphi(|x-t|)^p d\mu(x)\right)^{1/p} : \alpha \subset T,\ \mathrm{card}(\alpha) \leqslant N\right\} \qquad (6{\cdot}1)$$

provided $\int \varphi(|x-t|)^p d\mu(x) < \infty$ for some $t \in T$. Note that if $\varphi(0+) = 0$ and $\mathrm{supp}(\mu) \subset T$ we have $\lim_{N\to\infty} e_{N,\varphi,p}(\mu, T) = 0$ (see [**2**, proposition 2·5]).

THEOREM 3. *Assume that the measurable process $X = (X_t)_{t\in T}$, $T \subset I\!R^d$ satisfies* (2·1) *and* (2·2) *for some $\rho \in [1, \infty)$ and some function φ. Let μ be a Borel probability measure on $I\!R^d$ satisfying supp$(\mu) \subset T$ and*

$$\int \varphi(|x-t|)^\rho d\mu(x) < \infty \text{ for some } t \in T. \qquad (6{\cdot}2)$$

Let $p \in [1, \rho]$.

(a) *If*

$$e_{N,\varphi,p}(\mu, T) = O(g(N)) \text{ as } N \to \infty$$

where $g : (0, \infty) \to (0, \infty)$ is a regularly varying function at infinity with index < 0, then

$$\forall r \in (0, p), e_{N,r}(X, L^p(\mu)) = O(g(\log N)) \text{ as } N \to \infty.$$

(b) *Assume $\varphi(u) = cu^b$, $b > 0$. If*

$$e_{N,bp}(\mu, I\!R^d) = O(g(N)) \text{ as } N \to \infty$$

where $g : (0, \infty) \to (0, \infty)$ is a regularly varying function at infinity with index < 0, then

$$\forall r \in (0, p), e_{N,r}(X, L^p(\mu)) = O(g(\log N)^b) \text{ as } N \to \infty.$$

Remarks. (a) Condition (6·2) is satisfied if supp(μ) is compact.
(b) The proof of Theorem 3 (b) shows that one can replace $e_{N,\varphi,p}(\mu, T)$ in part (a) by $e_{N,\varphi,p}(\mu, I\!R^d)$ if φ is C-subadditiv in the sense that ther is a constant $C \in (0, \infty)$ such that $\varphi(s+t) \leqslant C(\varphi(s) + \varphi(t))$ for every $s, t > 0$. One also easily checks that $e_{N,\varphi,p}(\mu, T) = e_{N,\varphi,p}(\mu, I\!R^d)$ if T contains the closed convex hull of supp(μ).
(c) A drawback of Theorem 3 consists in the fact that the case $r = p$ is excluded in contrast to Theorem 1.

Proof. First observe that, for any $t \in T$

$$\begin{aligned} I\!E||X||^\rho_{L^\rho(\mu)} &\leqslant 2^{\rho-1}\left(\int I\!E|X_s - X_t|^\rho d\mu(s) + I\!E|X_t|^\rho\right) \\ &\leqslant 2^{\rho-1}\left(\int \varphi(|s-t|)^\rho d\mu(s) + I\!E|X_t|^\rho\right) < \infty \end{aligned}$$

so that the sample paths of X a.s. lie in $L^\rho(\mu)$. We exploit the relationship between the n-th

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linear width of X defined by

$$d_{n,s}(X, L^p(\mu)) := \inf\{(I\!E||X - V(X)||^s_{L^p(\mu)})^{1/s} : V : L^p(\mu) \longrightarrow L^p(\mu) \text{ linear}, rkV < n\} \tag{6·3}$$

for $s \in (0, \infty)$, $n \in I\!N$ and the quantization errors $e_{2^n,r}(X, L^p(\mu))$ with $r < s$.

For $n \in I\!N$ and $\alpha = \{t_1, \ldots, t_n\} \subset I\!R^d$ consider a Voronoi partition $\{B_1, \ldots, B_n\}$ of $I\!R^d$ with respect to α, that is a partition consisting of Borel sets such that

$$B_j \subset \{x \in I\!R^d : |x - t_j| = \min_{1 \leqslant i \leqslant n} |x - t_i|\}$$

for every j. Then

$$X^n := \sum_{j=1}^n \int_{B_j} X_s d\mu(s) \mu(B_j)^{-1} 1_{B_j}$$

satisfies

$$\begin{aligned} ||X - X^n||^p_{L^p(\mu)} &= \sum_{j=1}^n \int_{B_j} |\int_{B_j} (X_t - X_s) d\mu(s) \mu(B_j)^{-1}|^p d\mu(t) \\ &\leqslant \sum_{j=1}^n \int_{B_j} \int_{B_j} |X_t - X_s|^p d\mu(s) \mu(B_j)^{-1} d\mu(t). \end{aligned}$$

(a) Now assume $\alpha \subset T$. We obtain

$$\begin{aligned} ||X - X^n||^p_{L^p(\mu)} &\leqslant 2^{p-1} \sum_{j=1}^n \mu(B_j)^{-1} \int_{B_j} \int_{B_j} (|X_t - X_{t_j}|^p + |X_s - X_{t_j}|^p) d\mu(s) d\mu(t) \\ &= 2^p \sum_{j=1}^n \int_{B_j} |X_s - X_{t_j}|^p d\mu(s) \end{aligned}$$

so that

$$\begin{aligned} I\!E||X - X^n||^p_{L^p(\mu)} &\leqslant 2^p \sum_{j=1}^n \int_{B_j} \varphi(|s - t_j|)^p d\mu(s) \\ &= 2^p \sum_{j=1}^n \int_{B_j} \min_{1 \leqslant i \leqslant n} \varphi(|s - t_i|)^p d\mu(s) \\ &= 2^p \int \min_{1 \leqslant i \leqslant n} \varphi(|s - t_i|)^p d\mu(s). \end{aligned}$$

Consequently, for every $n \in I\!N$

$$\forall s \in (0, p], d_{n+1,s}(X, L^p(\mu)) \leqslant 2 e_{n,\varphi,p}(\mu, T). \tag{6·4}$$

Now we use a Carl type inequality (see [**1**, theorem 4·6·1]) that connects linear widths and quantizaton errors of X (in the Banach space case $p \geqslant 1$). From this inequality, (6·4) and the assumptions we deduce

$$\forall r \in (0, p), e_{2^n,r}(X, L^p(\mu)) = O(g(n))$$

which yields the assertion.

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(b) Here we have

$$\begin{aligned}
I\!E||X - X^n||^p_{L^p(\mu)} &\leqslant \sum_{j=1}^n \mu(B_j)^{-1} \int_{B_j}\int_{B_j} I\!E|X_s - X_t|^p d\mu(s)d\mu(t)\\
&\leqslant c\sum_{j=1}^n \mu(B_j)^{-1} \int_{B_j}\int_{B_j} |s-t|^{bp} d\mu(s)d\mu(t)\\
&\leqslant c2^{bp}\sum_{j=1}^n \mu(B_j)^{-1} \int_{B_j}\int_{B_j} (|s-t_j|^{bp} + |t-t_j|^{bp}) d\mu(s)d\mu(t)\\
&= c2^{bp+1}\sum_{j=1}^n \int_{B_j} |s-t_j|^{bp} d\mu(s)\\
&= c2^{bp+1}\int \min_{1\leqslant i\leqslant n} |s-t_i|^{bp} d\mu(s)
\end{aligned}$$

and hence

$$\forall s \in (0, p], d_{n+1,s}(X, L^p(\mu)) \leqslant (c2^{bp+1})^{1/p} e_{n,bp}(\mu, I\!R^d)^b. \tag{6·5}$$

The assertion follows as in (a).

The remaining parts of Theorems 1 and 2 are immediate consequences of Theorem 3.

Proof of Theorem 1. *for* $p = \rho \geqslant 1$. Let $\Gamma_k = \Gamma(q^{k-1})$ if $k \geqslant 1$ and $\Gamma_0 = \{\varnothing\}$ as in the proof of Theorem 1, $N = N(k) :=$ card (Γ_k) and $t^\sigma \in A_\sigma$. Then $(N(k))_{k\geqslant 0}$ is strictly increasing to infinity, $N(k) \approx q^{-k}$ as $k \mapsto \infty$ and

$$\begin{aligned}
e_{N,\varphi,\rho}(\mu, T)^\rho &\leqslant \int \min_{\sigma\in\Gamma_k} \varphi(|x - t^\sigma|)^\rho d\mu(x)\\
&\leqslant \sum_{\sigma\in\Gamma_k} \int_{A_\sigma} \varphi(|x - t^\sigma|)^\rho d\mu(x)\\
&\leqslant \sum_{\sigma\in\Gamma_k} \mu(A_\sigma)\varphi(s_\sigma \text{ diam } (A))^\rho\\
&\leqslant C\sum_{\sigma\in\Gamma_k} p_\sigma \varphi(s_\sigma)^\rho\\
&\leqslant C\varphi(N^{-1/D_{b\rho}})^\rho\\
&= Cg(N)^\rho
\end{aligned}$$

using (4·7) in part (a) where $g(u) := \varphi(u^{-1/D_{b\rho}})$. Since g is regularly varying at infinity with index $-b/D_{b\rho}$, the assertion follows from Theorem 3(a).

Proof of Theorem 2. *for* $p = \rho \geqslant 1$. Choose $t^I \in I \bigcap T$ for $I \in \mathcal{D}_n$. Then

$$\begin{aligned}
e_{2^n,\varphi,\rho}(\mu, T)^\rho &\leqslant \int \min_{I\in\mathcal{D}_n} \varphi(|x - t^I|)^\rho d\mu(x)\\
&\leqslant \sum_{I\in\mathcal{D}_n} \int_I \varphi(|x - t^I|)^\rho d\mu(x) \leqslant 2^{-n} \sum_{I\in\mathcal{D}_n} \varphi\left(\prod_{i=1}^n a_i\right)^\rho\\
&= \varphi\left(\prod_{i=1}^n a_i\right)^\rho \leqslant Cg(2^n)^\rho
\end{aligned}$$

where $g(u) := \varphi(u^{-1/D})$. The assertion follows from Theorem 3(a).

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Remark. Further solutions of the $e_{N,\varphi,p}(\mu)$- and $e_{N,bp}(\mu)$-problem can be found e.g. in [**2**, **7**, **13**, **19**, **24**].

An even more general approach is possible based on the intrinsic semimetric of the process X defined by

$$\delta_{X,p}(s,t) := (I\!E|X_s - X_t|^p)^{1/p}$$

and the quantization errors

$$e_{N,\delta,p}(\mu, T) := \inf\left\{\left(\int \min_{t\in\alpha} \delta(s,t)^p d\mu(s)\right)^{1/p} : \alpha \subset T,\ \mathrm{card}(\alpha) \leqslant N\right\} \tag{6·6}$$

where $\delta = \delta_{X,p}$.

Just as Theorem 3(a) one proves the following result.

THEOREM 4. *Let $p \in [1,\infty)$, let $(T, \mathcal{B})$ be a measurable space and let $X = (X_t)_{t\in T}$ be a measurable process such that $X_t \in L^p(I\!P)$ for every $t \in T$. Let μ be a probability measure on $(T, \mathcal{B})$ such that $L^p(\mu)$ is separable satisfying*

$$\int \delta(s,t)^p d\mu(s) < \infty \text{ for some } t \in T$$

where $\delta = \delta_{X,p}$. If

$$e_{N,\delta,p}(\mu, T) = O(g(N))$$

where $g\colon (0,\infty) \to (0,\infty)$ is regularly varying at infinity with index < 0, then

$$\forall r \in (0, p),\ e_{N,r}(X, L^p(\mu)) = O(g(\log N)).$$

REFERENCES

[**1**] J. CREUTZIG. Approximation of Gaussian random vectors in Banach spaces. Ph.D. thesis (University of Jena, 2002).

[**2**] S. DELATTRE, S. GRAF, H. LUSCHGY and G. PAGÈS. Quantization of probability distributions under norm-based distortion measures. *Statist. Decisions* **22** (2004), 261–282.

[**3**] S. DEREICH. High resolution coding of stochastic processes and small ball probabilities. Ph.D. thesis (TU Berlin, 2003).

[**4**] S. DEREICH, F. FEHRINGER, A. MATOUSSI and M. SCHEUTZOW. On the link between small ball probabilities and the quantization problem for Gaussian measures on Banach spaces. *J. Theoret. Probab*. **16** (2003), 249–265.

[**5**] K. FALCONER. *Techniques in Fractal Geometry* (Wiley, 1997).

[**6**] S. GRAF. On Bandt's tangential distribution for self-similar measures. *Monatsh. Math*. **120** (1995), 223–246.

[**7**] S. GRAF and H. LUSCHGY. *Foundations of Quantization for Probability Distributions* (Springer, 2000).

[**8**] S. GRAF and H. LUSCHGY. Asymptotics of the quantization errors for self-similar probabilities. *Real Anal. Exchange* **26** (2000) 01, 795–810.

[**9**] S. GRAF and H. LUSCHGY. The quantization dimension of self-similar probabilities. *Math. Nachri.* **241** (2002), 103–109.

[**10**] S. GRAF and H. LUSCHGY. The point density measure in the quantization of self-similar probabilities. *Math. Proc. Camb. Phil. Soc*. **138** (2005), 513–531.

[**11**] S. GRAF, H. LUSCHGY and G. PAGÈS. Functional quantization and small ball probabilities for Gaussian processes. *J. Theoret. Probab*. **16** (2003), 1047–1062.

[**12**] S. GRAF, H. LUSCHGY and G. PAGÈS. Optimal quantizers for Radon random vectors in a Banach space. *J. Approx. Theory* **144** (2007), 27–53.

[**13**] P. M. GRUBER. Optimum quantization and its applications. *Adv. Math*. **186** (2004), 456–497.

[**14**] J. HUTCHINSON. Fractals and self-similarity. *Indiana Univ. J*. **30** (1981), 713–747.

https://doi.org/10.1017/S0305004110000344 Published online by Cambridge University Press

[**15**] M. KESSEBÖHMER and S. ZHU. Stability of quantization dimension and quantization for homogeneous Cantor measures. *Math. Nachr.* **280** (2007), 1–16.
[**16**] W. KREITMEIER. Optimal quantization for dyadic homogeneous Cantor distributions. *Math. Nachr.* **281** (2008), 1307–1327.
[**17**] M. A. LIFSHITS, W. LINDE and Z. SHI. Small deviations of Riemann–Liouville processes in L_q-spaces with respect to fractal measures. *Proc. London Math. Soc*. **92** (2006), 224–250.
[**18**] M. A. LIFSHITS, W. LINDE and Z. SHI. Small deviations of Gaussian random fields in L_q-spaces. *Electron. J. Probab*. **11** (2006), 1204–1233.
[**19**] L. LINDSAY and R. D. MAULDIN. Quantization dimension for conformal iterated function systems. *Nonlinearity* **15** (2002), 189–199.
[**20**] H. LUSCHGY and G. PAGÈS. Sharp asymptotics of the functional quantization problem for Gaussian processes. *Ann. Probab*. **32** (2004), 1574–1599.
[**21**] H. LUSCHGY and G. PAGÈS. Functional quantization rate and mean pathwise regularity of processes with an application to Lévy processes. *Ann. Appl. Probab*. **18** (2008), 427–469.
[**22**] H. LUSCHGY and G. PAGÈS. Moment estimates for Lévy processes. *Electron. Comm. Probab*. **13** (2008), 422–434.
[**23**] A. I. NAZAROV. Logarithmic L_2-small ball asymptotics with respect to self-similar measures for some Gaussian random processes. *J. Math. Sci*. **133** (2006), 1314–1327.
[**24**] K. PÖTZELBERGER. The quantization error of self-similar distributions. *Math. Proc. Camb. Phil. Soc*. **137** (2004), 725–740.

https://doi.org/10.1017/S0305004110000344 Published online by Cambridge University Press