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Fermionic Gaussians

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Introduction

The notion of a Gaussian as the exponential of a quadratic is rather familiar. Such functions are of considerable importance in a number of contexts, for example within quantum theory. Thus, in the Schrödinger representation of the canonical commutation relations they alone minimize uncertainty and they appear as ground states for harmonic oscillators. Also in the complex-wave representation of a free boson field they arise as transforms of the Fock vacuum under certain Bogoliubov automorphisms.

Gaussians also feature in fermionic contexts, though these are perhaps less familiar. Here, we carefully construct fermionic Gaussians and present a couple of applications: one to a construction of Pfaffian pairings or relative Pfaffians for operators on arbitrary complex Hilbert spaces; another to a construction of annihilator cyclic vectors in fermionic Fock spaces over separably infinite dimensional complex Hilbert spaces. Of course, this list is incomplete: for instance, when certain Bogoliubov automorphisms are implemented in the Fock representation of a free fermion field they transform the vacuum to a fermionic Gaussian.

We begin with a complex Hilbert space V of arbitrary dimension, fermionic Fock space over which is the Hilbert space direct sum

$$\bigwedge[V] = \bigoplus_{n \geqslant 0} \bigwedge^n[V]$$

of the completed spaces of alternating tensors. The completed space $\bigwedge^2[V]$ of alternating two tensors over V is canonically isometric to the space $\mathbb{S}(V)$ comprising all Hilbert–Schmidt operators Z on V that are antiskew in the sense

$$x, y \in V \Rightarrow \langle Zx \,|\, y\rangle + \langle Zy \,|\, x\rangle = 0$$

and are hence antilinear; the isometry $\bigwedge^2[V] \to \mathbb{S}(V) : \zeta \mapsto \frac{1}{2}Z$ is determined by the rule

$$x, y \in V \Rightarrow \langle \zeta \,|\, x \wedge y\rangle = \langle Zx \,|\, y\rangle.$$

A fermionic Gaussian is now simple to define, modulo questions of convergence: an element $\zeta \in \bigwedge^2[V]$ serves as the quadratic; its exponential

$$\exp\zeta = \sum_{n \geqslant 0} \frac{1}{n!}\zeta^n \in \bigwedge[V]$$

is the corresponding fermionic Gaussian. We write $\epsilon(Z) = \exp\zeta$ when $\zeta \in \bigwedge^2[V]$ and $Z \in \mathbb{S}(V)$ are related as above.

As our first application, we consider Pfaffian pairings or relative Pfaffians. Let X and Y lie in $\mathbb{S}(V)$. The complex-linear operator XY on V is of trace class, whence the

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operator $I-XY$ has a Fredholm determinant. The Pfaffian pairing $Pf(X,Y)$ is a canonical square root of this determinant. Explicitly, we define the Pfaffian pairing as an inner product of fermionic Gaussians, thus

$$Pf(X,Y) = \langle \epsilon(X) \,|\, \epsilon(Y) \rangle;$$

we demonstrate that

$$Pf(X,Y)^2 = \mathrm{Det}\,(I-XY)$$

and show that Pf is a unitary invariant in the sense that

$$Pf(UXU^*, UYU^*) = Pf(X,Y)$$

whenever U is a unitary operator on V.

As our second application, we consider a possibly surprising property of the Fock representation of the canonical anticommutation relations. The Fock vacuum $\Omega = 1 \in \mathbb{C} = \bigwedge^0[V]$ is of course cyclic for creators alone; in fact, creator cyclic vectors are dense in $\bigwedge[V]$. The existence of annihilator cyclic vectors for finite dimensional V is very well known; their non-existence for inseparable V is not difficult to verify. We show that if V is separably infinite dimensional and if $Z \in \mathbb{S}(V)$ is injective then the fermionic Gaussian $\epsilon(Z)$ is cyclic for annihilators alone. In quantum field theoretic language and by analogy with the finite dimensional case, it might have been expected that annihilator cyclic vectors would involve fully occupied states; of course, these do not exist in infinite dimensions.

Some additional remarks are in order. Fermionic Fock space appears under exterior hilbertian powers in [**2**]. Our proof that ϵ is Hölder continuous is patterned after an argument in [**8**]. For Fredholm determinants, see [**1**, **4**, **7**]. Our Pfaffian pairing adapts very easily to yield the relative Pfaffians studied in [**5**]: there, the construction involves Berezin integrals, Pfaffian minors and invertible perturbations. For the Fock representation of the canonical anticommutation relations, see [**1**, **3**, **9**]. Our idea to investigate annihilator cyclic vectors was prompted by a seemingly innocent question concerning creator cyclic vectors that was posed by David Groisser.

1. *Fermionic Gaussians*

In this section, we set forth the construction of fermionic Gaussians and establish a fundamental continuity result.

Let V be a complex Hilbert space of arbitrary dimension. Fermionic Fock space over V is the Hilbert space direct sum

$$\bigwedge[V] = \bigoplus_{n \geqslant 0} \bigwedge^n[V]$$

where $\bigwedge^0[V] = \bigwedge^0(V) = \mathbb{C}$ has unit vector $\Omega = 1$ and where if $n > 0$ then $\bigwedge^n[V]$ is the Hilbert space completion of the n-fold alternating tensor power $\bigwedge^n(V)$ with inner product given on decomposables by the formula

$$\langle x_1 \wedge \ldots \wedge x_n \,|\, y_1 \wedge \ldots \wedge y_n \rangle = \mathrm{Det}\,[\langle x_i \,|\, y_j \rangle]$$

for $x_1, y_1, \ldots, x_n, y_n \in V$. We remark that if $\{v_r : r \in R\}$ is a complete orthonormal system for V with totally ordered index set R then the collection of all vectors of the form $v_{r_1} \wedge \ldots \wedge v_{r_n}$ with $r_1 < \ldots < r_n$ in R constitutes a complete orthonormal system for $\bigwedge^n[V]$ and that the union of these collections as n runs over $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ is a complete orthonormal system for $\bigwedge[V]$ itself; here, the vector $v_\varnothing$ with empty index is conventionally understood to stand for the unit vector $\Omega \in \bigwedge[V]$.

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Our discussion of fermionic Gaussians is facilitated by an alternative description of the space $\bigwedge^2[V]$. Denote by $\mathbb{S}(V)$ the space of all Hilbert–Schmidt real linear operators Z on V that are antiskew in the sense that

$$x, y \in V \Rightarrow \langle Zx \,|\, y\rangle + \langle Zy \,|\, x\rangle = 0.$$

Such an operator Z is automatically conjugate linear (or antilinear) and has real Hilbert–Schmidt norm $\|Z\|_{HS}$ given by

$$\|Z\|_{HS}^2 = 2 \sum_{r \in R} \|Zv_r\|^2$$

since $\{v_r, iv_r : r \in R\}$ is a complete orthonormal system for the real Hilbert space underlying V. Now, let $\zeta \in \bigwedge^2[V]$ and note that the map

$$V \times V \to \mathbb{C} : (x, y) \mapsto \langle x \wedge y \,|\, \zeta\rangle$$

is alternating bilinear and bounded, whence there exists a bounded antiskew operator Z on V such that

$$x, y \in V \Rightarrow \langle \zeta \,|\, x \wedge y\rangle = \langle Zx \,|\, y\rangle.$$

According to the Parseval identity,

$$\begin{aligned} 4\|\zeta\|^2 &= 4 \sum_{s<t} |\langle \zeta \,|\, v_s \wedge v_t\rangle|^2 \\ &= 2 \sum_{s,t} |\langle \zeta \,|\, v_s \wedge v_t\rangle|^2 \\ &= 2 \sum_{s,t} |\langle Zv_s \,|\, v_t\rangle|^2 \\ &= \|Z\|_{HS}^2 \end{aligned}$$

where $\|.\|_{HS}$ denotes real Hilbert–Schmidt norm as before. The assignment $\bigwedge^2[V] \to \mathbb{S}(V) : \zeta \mapsto \frac{1}{2}Z$ is thus an isometry; it is also a surjection, since a given $Z \in \mathbb{S}(V)$ arises from $\zeta \in \bigwedge^2[V]$ where

$$\zeta = \sum_{s<t} \langle Zv_s \,|\, v_t\rangle v_s \wedge v_t.$$

We have now arrived at the following identification.

THEOREM. *The Hilbert spaces* $\mathbb{S}(V)$ *and* $\bigwedge^2[V]$ *are isometrically isomorphic under the correspondence* $\frac{1}{2}Z \leftrightarrow \zeta$ *given by the rule*

$$x, y \in V \Rightarrow \langle \zeta \,|\, x \wedge y\rangle = \langle Zx \,|\, y\rangle.$$

In these terms, if $Z \in \mathbb{S}(V)$ corresponds to $\zeta \in \bigwedge^2[V]$ then we shall define the fermionic Gaussian $\epsilon(Z)$ to be the sum in $\bigwedge[V]$ of the exponential series

$$\exp \zeta = \sum_{j \geq 0} \frac{1}{j!} \zeta^j.$$

Of course, this definition will require some justification: specifically, we must establish convergence of the indicated exponential. For this reason, we next examine more closely the space $\mathbb{S}(V)$ and its relationship to $\bigwedge^2[V]$.

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So, let $Z \in \mathbb{S}(V)$. The bounded complex linear operator Z^2 is self-adjoint on V: indeed, if $x, y \in V$ then

$$\langle Z^2 x \,|\, y \rangle = -\langle Zy \,|\, Zx \rangle = \langle x \,|\, Z^2 y \rangle.$$

Being also compact, Z^2 induces an orthogonal eigendecomposition of V as

$$V = \bigoplus_{\mu \in \mathbb{R}} V(\mu)$$

where if $\mu \in \mathbb{R}$ then

$$V(\mu) = \{v \in V : Z^2 v = \mu v\}.$$

If $\mu \in \mathbb{R}$ and $v \in V(\mu)$ then

$$\|Zv\|^2 = -\langle Z^2 v \,|\, v \rangle = -\mu \|v\|^2$$

so that $\mu > 0$ forces $V(\mu) = 0$. Writing $V(-\lambda^2) = V_\lambda$ when λ is a non-negative real number, it follows that in fact

$$V = V_0 \oplus \sum_{\lambda > 0} V_\lambda$$

where V_0 is actually the kernel of Z itself. Of course, the positive λ for which V_λ is nonzero form a square summable (hence countable) set with 0 as the only possible limit point. Moreover, if $\lambda > 0$ and $V_\lambda \neq 0$ then V_λ is a finite dimensional space on which $(1/\lambda) Z \,|\, V_\lambda$ is an antilinear complex (thus, quaternionic) structure; in consequence, such V_λ are of even complex dimension and have orthonormal bases consisting of pairs v, $(1/\lambda) Zv$ for suitable unit vectors $v \in V_\lambda$.

It is now plain that we may list the (summable) non-zero eigenvalues of $-Z^2$ according to their multiplicities as

$$\lambda_1^2, \lambda_1^2, \ldots, \lambda_n^2, \lambda_n^2, \ldots$$

and that the space

$$(\ker Z)^\perp = \sum_{\lambda > 0} V_\lambda$$

has a corresponding complete orthonormal sequence of the form

$$x_1, y_1, \ldots, x_n, y_n, \ldots$$

with

$$Zx_n = \lambda_n y_n, Zy_n = -\lambda_n x_n$$

for $n > 0$. Fourier expansion now confirms that if $\zeta \in \bigwedge^2[V]$ corresponds to Z as usual then

$$\zeta = \sum_{n > 0} \lambda_n x_n \wedge y_n.$$

For convenience, we record these findings as follows.

THEOREM. *If $Z \in \mathbb{S}(V)$ then the corresponding $\zeta \in \bigwedge^2[V]$ has a (possibly finite) development*

$$\zeta = \sum_{n > 0} \lambda_n x_n \wedge y_n$$

in which $\{\lambda_n : n > 0\}$ are positive scalars and $\{x_n, y_n : n > 0\}$ is a complete orthonormal sequence for $(\ker Z)^\perp$ such that $Zx_n = \lambda_n y_n$ and $Zy_n = -\lambda_n x_n$ whenever $n > 0$.

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Continuing the notation established above, let us also introduce $\zeta_n = x_n \wedge y_n$ for $n > 0$. Being even and decomposable, the vectors $\{\zeta_n : n > 0\}$ mutually commute and self annihilate under wedge product. Accordingly, if $j > 0$ then the j-fold wedge product $\zeta^j \in \bigwedge^{2j}[V]$ of ζ equals $j!$ times the sum of all terms

$$\lambda_{n_1} \dots \lambda_{n_j} \zeta_{n_1} \wedge \dots \wedge \zeta_{n_j}$$

for which $n_1 < \dots < n_j$. We may thus estimate the norm of ζ^j as follows:

$$\begin{aligned}
\|\zeta^j\|^2 &= (j!)^2 \sum \{\lambda_{n_1}^2 \dots \lambda_{n_j}^2 : n_1, \dots, n_j \text{ increasing}\} \\
&= j! \sum \{\lambda_{n_1}^2 \dots \lambda_{n_j}^2 : n_1, \dots, n_j \text{ distinct}\} \\
&\leqslant j! \sum \{\lambda_{n_1}^2 \dots \lambda_{n_j}^2 : n_1, \dots, n_j \text{ arbitrary}\} \\
&= j! \Big(\sum_{n>0} \lambda_n^2 \Big)^j
\end{aligned}$$

whence

$$\|\zeta^j\|^2 \leqslant j! \, \|\zeta\|^{2j}.$$

These inequalities suffice to guarantee convergence of the exponential series

$$\exp \zeta = \sum_{j \geqslant 0} \frac{1}{j!} \zeta^j$$

to an element of $\bigwedge[V]$ satisfying

$$\|\exp \zeta\|^2 \leqslant \exp \|\zeta\|^2.$$

Taking into account the fact that the norm of ζ and the real Hilbert–Schmidt norm of $\frac{1}{2}Z$ coincide, we have established the following result.

THEOREM. *If $Z \in \mathbb{S}(V)$ corresponds to $\zeta \in \bigwedge^2[V]$ as usual, then the fermionic Gaussian*

$$\epsilon(Z) := \exp \zeta \in \bigwedge[V]$$

satisfies the norm estimate

$$\|\epsilon(Z)\|^2 \leqslant \exp \{\tfrac{1}{4} \|Z\|_{HS}^2\}.$$

It is important that the map $\epsilon : \mathbb{S}(V) \to \bigwedge[V]$ is (Hölder) continuous. To see this, let X and Y lie in $\mathbb{S}(V)$ and choose a complex linear functional $\phi \in \bigwedge[V]^*$ of unit norm. The function $f : \mathbb{C} \to \mathbb{C}$ given by

$$z \in \mathbb{C} \Rightarrow f(z) = \phi(\epsilon(\tfrac{1}{2}(X+Y) + z(Y-X)))$$

is certainly entire. From our norm estimate on ϵ it follows that if $|z| \leqslant \frac{1}{2} + \|Y-X\|_{HS}^{-1}$ then

$$\begin{aligned}
|f(z)| &\leqslant \|\epsilon(\tfrac{1}{2}(X+Y) + z(Y-X))\| \\
&\leqslant \exp \tfrac{1}{8} \|\tfrac{1}{2}(X+Y) + z(Y-X)\|_{HS}^2 \\
&\leqslant \exp \tfrac{1}{8} (\|X\|_{HS} + \|Y\|_{HS} + 1)^2
\end{aligned}$$

whence by virtue of the Cauchy inequality it follows that

$$\begin{aligned}
|\phi(\epsilon(Y) - \epsilon(X))| &= |\phi\epsilon(Y) - \phi\epsilon(X)| \\
&= |f(\tfrac{1}{2}) - f(-\tfrac{1}{2})| \\
&\leqslant \sup \{|f'(t)| : -\tfrac{1}{2} \leqslant t \leqslant \tfrac{1}{2}\} \\
&\leqslant \|Y-X\|_{HS} \sup \{|f(z)| : |z| \leqslant \tfrac{1}{2} + \|Y-X\|_{HS}^{-1}\} \\
&\leqslant \|Y-X\|_{HS} \exp \tfrac{1}{8} (\|X\|_{HS} + \|Y\|_{HS} + 1)^2.
\end{aligned}$$

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The unit norm complex linear functional ϕ on $\bigwedge[V]$ being arbitrary, we deduce the following result.

THEOREM. *The map* $\epsilon:\mathbb{S}(V)\to\bigwedge[V]$ *is Hölder continuous: in fact, if* X *and* Y *lie in* $\mathbb{S}(V)$ *then*

$$\|\epsilon(Y)-\epsilon(X)\| \leqslant \|Y-X\|_{HS}\exp\tfrac{1}{8}(\|X\|_{HS}+\|Y\|_{HS}+1)^2.$$

As a particular corollary of this theorem, if the bounded net $(Z_\alpha:\alpha\in\mathbb{A})$ converges to Z in $\mathbb{S}(V)$ then the net $(\epsilon(Z_\alpha):\alpha\in\mathbb{A})$ converges to $\epsilon(Z)$ in $\bigwedge[V]$.

2. *Pfaffian pairings*

Our first application of fermionic Gaussians is to the construction of Pfaffian pairings, or relative Pfaffians as they are called. Let X and Y lie in $\mathbb{S}(V)$: thus, X and Y are Hilbert–Schmidt antiskew operators on V. Of course, the complex linear operator XY on V is of trace class, whence the operator $I-XY$ has a Fredholm determinant. The Pfaffian pairing assigns to (X,Y) a canonical square root $Pf(X,Y)$ of $\mathrm{Det}\,(I-XY)$. Here, we shall realize $Pf(X,Y)$ as the inner product $\langle\epsilon(X)\,|\,\epsilon(Y)\rangle$ between the fermionic Gaussians $\epsilon(X)$ and $\epsilon(Y)$ in fermionic Fock space $\bigwedge[V]$.

Our construction of the Pfaffian pairing proceeds via approximation from finite dimensions. Write $\mathscr{F}(V)$ for the collection of all finite dimensional complex subspaces of V directed by inclusion. When $M\in\mathscr{F}(V)$ we let P_M denote orthogonal projection from V onto M and when also $Z\in\mathbb{S}(V)$ we let Z_M denote the compression $P_M Z P_M$. Notice that Z_M is again Hilbert–Schmidt and antiskew; it may be regarded as an element of either $\mathbb{S}(V)$ or $\mathbb{S}(M)$ as convenience dictates. Our technique of finite dimensional approximation hinges upon the fact that the net $(Z_M:M\in\mathscr{F}(V))$ converges to Z in Hilbert–Schmidt norm, which fact we now verify.

Let $Z\in\mathbb{S}(V)$ and $\zeta\in\bigwedge^2[V]$ be related as usual. Recall that there exist a (possibly finite) set $\{\lambda_n:n>0\}$ of positive scalars and a corresponding complete orthonormal sequence $\{x_n,y_n:n>0\}$ for $(\ker Z)^\perp$ so that

$$\zeta=\sum_{n>0}\lambda_n x_n\wedge y_n$$

where if $n>0$ then $Zx_n=\lambda_n y_n$ and $Zy_n=-\lambda_n x_n$. Here, the real Hilbert–Schmidt norm of Z is of course given by

$$\|Z\|_{HS}^2=4\sum_{n>0}\lambda_n^2.$$

Fourier expansion of $v\in V$ yields

$$v=v_0+\sum_{n>0}\{\langle v\,|\,x_n\rangle x_n+\langle v\,|\,y_n\rangle y_n\}$$

so that

$$Zv=\sum_{n>0}\lambda_n\{\langle v\,|\,x_n\rangle y_n-\langle v\,|\,y_n\rangle x_n\}$$

where v_0 is the component of v in $\ker Z$.

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Now, let $\epsilon > 0$ be given and choose $N \in \mathbb{N}$ so that $16 \sum_{n>N} \lambda_n^2 \leqslant \epsilon^2$. Denote by M_ϵ the linear span of $\{x_1, y_1, \ldots, x_N, y_N\}$ and by P_ϵ orthogonal projection from V onto M_ϵ. If $v \in V$ then

$$(Z - P_\epsilon Z P_\epsilon) v = \sum_{n>N} \lambda_n \{\langle v | x_n \rangle y_n - \langle v | y_n \rangle x_n\}$$

in consequence of which the real Hilbert–Schmidt norm of $Z - P_\epsilon Z P_\epsilon$ is given by

$$\|Z - P_\epsilon Z P_\epsilon\|_{HS} = 2\sqrt{\{\sum_{n>N} \lambda_n^2\}} \leqslant \tfrac{1}{2}\epsilon.$$

Next, let $M \in \mathscr{F}(V)$ contain M_ϵ and in place of P_M write P. Note that

$$PZP - P_\epsilon Z P_\epsilon = PZQ_\epsilon$$

where $Q_\epsilon = P - P_\epsilon$ is orthogonal projection from V onto $M_\epsilon^\perp \cap M$. Thanks to the inclusion $\operatorname{ran} Q_\epsilon \subset M_\epsilon^\perp$ we may restrict Z to $M_\epsilon^\perp$ in computing the real Hilbert–Schmidt norm of PZQ_ϵ and find that

$$\|PZP - P_\epsilon Z P_\epsilon\|_{HS} = \|PZQ_\epsilon\|_{HS} \leqslant \|P\| \, \|Z | M_\epsilon^\perp\|_{HS} \, \|Q_\epsilon\| \leqslant \tfrac{1}{2}\epsilon$$

since $\|Z | M_\epsilon^\perp\|_{HS} = 2\sqrt{\{\sum_{n>N} \lambda_n^2\}}$ and since $\|Q_\epsilon\| \leqslant \|P\| \leqslant 1$.

Reverting to our established notation for compressions, given $Z \in \mathbb{S}(V)$ and $\epsilon > 0$ we have found $M_\epsilon \in \mathscr{F}(V)$ with the property that $\|Z - Z_M\|_{HS} \leqslant \epsilon$ whenever $M \in \mathscr{F}(V)$ contains M_ϵ. Since if $M \in \mathscr{F}(V)$ then the operator norm of P_M is at most unity, we have verified the following.

THEOREM. *If $Z \in \mathbb{S}(V)$ then the net $(Z_M : M \in \mathscr{F}(V))$ is bounded and converges to Z in Hilbert–Schmidt norm.*

This disposes of the machinery that we require to perform our finite dimensional approximation.

Now let V be of finite dimension and let $Z \in \mathbb{S}(V)$ correspond to $\zeta \in \bigwedge^2(V)$ in the usual manner. Place ζ in standard form

$$\zeta = \lambda_1 x_1 \wedge y_1 + \ldots + \lambda_N x_N \wedge y_N$$

for positive scalars $\lambda_1, \ldots, \lambda_N$ and an orthonormal basis $x_1, y_1, \ldots, x_N, y_N$ of $(\ker Z)^\perp$ with $Zx_n = \lambda_n y_n$ and $Zy_n = -\lambda_n x_n$ whenever $1 \leqslant n \leqslant N$. Listed according to multiplicity, the nonzero eigenvalues of $-Z^2$ are

$$\lambda_1^2, \lambda_1^2, \ldots, \lambda_N^2, \lambda_N^2$$

as a consequence of which

$$\operatorname{Det}(I - Z^2) = \prod_{n=1}^{N} (1 + \lambda_n^2)^2.$$

Now

$$\prod_{n=1}^{N} (1 + \lambda_n^2) = \sum_{j=0}^{N} \{\sum_{n_1 < \ldots < n_j} \lambda_{n_1}^2 \ldots \lambda_{n_j}^2\}$$

and

$$\sum_{n_1 < \ldots < n_j} \lambda_{n_1}^2 \ldots \lambda_{n_j}^2 = \frac{\|\zeta^j\|^2}{(j!)^2}$$

so

$$\prod_{n=1}^{N} (1 + \lambda_n^2) = \sum_{j=0}^{N} \frac{\|\zeta^j\|^2}{(j!)^2} = \|\exp \zeta\|^2.$$

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Thus the fermionic Gaussian $\epsilon(Z) = \exp\zeta$ satisfies

$$\|\epsilon(Z)\|^4 = \mathrm{Det}\,(I - Z^2).$$

Again supposing V to be finite dimensional, note that both the inner product $\langle\epsilon(X)\,|\,\epsilon(Y)\rangle$ and the determinant $\mathrm{Det}\,(I-XY)$ are holomorphic (since polynomial) in $X \in \mathbb{S}(V)$ and antiholomorphic (since antipolynomial) in $Y \in \mathbb{S}(V)$. We verified above that if $Z \in \mathbb{S}(V)$ then

$$\langle\epsilon(Z)\,|\,\epsilon(Z)\rangle^2 = \mathrm{Det}\,(I - Z^2).$$

Accordingly, the principle of analytic continuation from the diagonal establishes the following result.

THEOREM. *Let V be finite dimensional. If X and Y lie in $\mathbb{S}(V)$ then*

$$\langle\epsilon(X)\,|\,\epsilon(Y)\rangle^2 = \mathrm{Det}\,(I - XY).$$

Permit the dimension of V to be arbitrary. The Pfaffian pairing

$$Pf : \mathbb{S}(V) \times \mathbb{S}(V) \to \mathbb{C}$$

is defined by

$$Pf(X, Y) = \langle\epsilon(X)\,|\,\epsilon(Y)\rangle$$

for X and Y in $\mathbb{S}(V)$. We proceed to demonstrate that this has the properties expected of a relative Pfaffian.

Recall that if $Z \in \mathbb{S}(V)$ then the net $(Z_M : M \in \mathscr{F}(V))$ is bounded and converges to Z in Hilbert–Schmidt norm. From the close of the previous section, the map $\epsilon : \mathbb{S}(V) \to \bigwedge[V]$ is Hölder continuous; also, it is familiar that the Fredholm determinant is trace norm continuous. Now, let X and Y lie in $\mathbb{S}(V)$. Allow M to run over the directed set $\mathscr{F}(V)$: as $X_M \to X$ and $Y_M \to Y$ in Hilbert–Schmidt norm, it follows that $X_M\, Y_M \to XY$ in trace norm. Thus

$$\begin{aligned} Pf(X, Y)^2 &= \lim\,(Pf(X_M, Y_M)^2 : M \to V) \\ &= \lim\,(\mathrm{Det}\,(I - X_M\, Y_M) : M \to V) \\ &= \mathrm{Det}\,(I - XY) \end{aligned}$$

thanks to our finite dimensional analysis. Note that here, the inner product $\langle\epsilon(X_M)\,|\,\epsilon(Y_M)\rangle$ is calculated equally in both $\bigwedge[V]$ and $\bigwedge[M]$. Note also that the determinant $\mathrm{Det}\,(I - X_M\, Y_M)$ is calculated equally on both M and V.

THEOREM. *If V is arbitrary then the Pfaffian pairing Pf satisfies*

$$Pf(X, Y)^2 = \mathrm{Det}\,(I - XY)$$

for all X and Y in $\mathbb{S}(V)$.

In particular, for $Z \in \mathbb{S}(V)$ we have the precise formula

$$\|\epsilon(Z)\|^4 = \mathrm{Det}\,(I - Z^2)$$

which refines our estimate

$$\|\epsilon(Z)\|^4 \leqslant \exp\{\tfrac{1}{2}\|Z\|_{HS}^2\}$$

based on the real Hilbert–Schmidt norm.

The Pfaffian pairing has a number of other fundamental properties. Aside from its Hölder continuity, perhaps the most important of these is a transformation rule. Let G be a bounded linear operator on V. If $Z \in \mathbb{S}(V)$ corresponds to $\zeta \in \bigwedge^2[V]$ as usual,

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then $G.Z := GZG^*$ corresponds to $\zeta' = G.\zeta$ in the functorial action on Fock space: indeed, if $x, y \in V$ then

$$\begin{aligned}\langle \zeta' \,|\, x \wedge y\rangle &= \langle GZG^*x \,|\, y\rangle \\ &= \langle ZG^*x \,|\, G^*y\rangle \\ &= \langle \zeta \,|\, G^*(x \wedge y)\rangle \\ &= \langle G.\zeta \,|\, x \wedge y\rangle.\end{aligned}$$

As a consequence, it follows from this that

$$\epsilon(G.Z) = \exp G.\zeta = G.\exp\zeta = G.\epsilon(Z).$$

Now, let E and F be bounded linear operators on V: if X and Y lie in $\mathbb{S}(V)$ then

$$\begin{aligned}Pf(E.X, F.Y) &= \langle \epsilon(E.X) \,|\, \epsilon(F.Y)\rangle \\ &= \langle E.\epsilon(X) \,|\, F.\epsilon(Y)\rangle.\end{aligned}$$

Taking $E = F$ to be an isometry yields the following transformation rule for the Pfaffian pairing.

THEOREM. *If X and Y lie in $\mathbb{S}(V)$ and if U is an isometry on V then*

$$Pf(UXU^*, UYU^*) = Pf(X, Y).$$

In particular, the Pfaffian pairing is a unitary invariant. We remark that, similarly, if U is instead an anti-isometry (especially, an antiunitary operator) on V then

$$Pf(UXU^*, UYU^*) = \overline{Pf(X, Y)}$$

for all X and Y in $\mathbb{S}(V)$. Moreover, if X and Y lie in $\mathbb{S}(V)$ then

$$\overline{Pf(X, Y)} = Pf(Y, X)$$

since the Pfaffian pairing arises from an (Hermitian) inner product.

3. *Annihilator cyclicity*

Our second application of fermionic Gaussians is to the Fock representation of the canonical anticommutation relations or CAR. As we shall recall briefly below, the Fock representation of the CAR over V is defined on fermionic Fock space $\bigwedge[V]$ in terms of creators and annihilators. Almost by definition, the standard unit vector $\Omega \in \bigwedge[V]$ is cyclic for creators alone; in fact, the set of vectors that are cyclic for creators alone is dense in $\bigwedge[V]$. It is perhaps natural to ask whether $\bigwedge[V]$ contains any vectors that are cyclic for annihilators alone. We show here that the answer to this question is negative if and only if V is inseparable.

For convenience, we briefly describe the Fock representation of the CAR over V. Creators and annihilators are defined initially on the Grassmann algebra

$$\bigwedge(V) = \bigoplus_{n \geq 0} \bigwedge^n(V).$$

Let $v \in V$. The creator (or creation operator) $c(v)$ is defined by

$$\zeta \in \bigwedge(V) \Rightarrow c(v)\,\zeta = v \wedge \zeta$$

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and depends on v in complex linear fashion. The annihilator (or annihilation operator) $a(v)$ depends on v in antilinear fashion, being defined by $a(v)\,\Omega = 0$ and

$$a(v)\,(v_0 \wedge \ldots \wedge v_n) = \sum_{j=0}^{n} (-1)^j \langle v_j \,|\, v\rangle\, v_0 \wedge \ldots \wedge \hat{v}_j \wedge \ldots \wedge v_n$$

where $v_0, \ldots, v_n \in V$ and where $\hat{\ }$ signifies omission as usual. In fact, $c(v)$ and $a(v)$ are bounded linear operators of norm $\|v\|$ on $\bigwedge(V)$; they therefore extend uniquely to $\bigwedge[V]$ by continuity, preserving their norm. The extended operators $c(v)$ and $a(v)$ are mutually adjoint: if $\xi, \eta \in \bigwedge[V]$ then

$$\langle c(v)\,\xi \,|\, \eta\rangle = \langle \xi \,|\, a(v)\,\eta\rangle.$$

Further, creators and annihilators satisfy the canonical anticommutation relations on Fock space: if $x, y \in V$ then

$$\begin{aligned} c(x)\,c(y) + c(y)\,c(x) &= 0 \\ a(x)\,a(y) + a(y)\,a(x) &= 0 \\ c(x)\,a(y) + a(y)\,c(x) &= \langle x \,|\, y\rangle\, I. \end{aligned}$$

When taken together, the creators and annihilators act irreducibly on Fock space: the bounded linear operators on $\bigwedge[V]$ commuting with every creator and annihilator are precisely the scalars; equivalently, each non-zero vector in $\bigwedge[V]$ is cyclic for creators and annihilators.

The standard unit vector Ω is plainly cyclic for creators alone: if $v_1, \ldots, v_n \in V$ then

$$c(v_n) \ldots c(v_1)\,\Omega = v_n \wedge \ldots \wedge v_1$$

and the decomposables span $\bigwedge(V)$. More generally, each $\zeta \in \bigwedge(V)$ with $\omega := \langle \zeta \,|\, \Omega\rangle$ non-zero is cyclic for creators alone. To see this, note first that $\zeta - \omega\Omega$ is nilpotent in the Grassmann algebra, whence $\zeta - \omega\Omega = -\omega p\,.\,\Omega$ for some nilpotent polynomial p in creators. Now, if $N \in \mathbb{N}$ is such that $p^N = 0$ then

$$\begin{aligned} (I + p + \ldots + p^{N-1})\,\zeta &= (I + p + \ldots + p^{N-1})\,(I - p)\,\omega\Omega \\ &= \omega\Omega. \end{aligned}$$

All that remains is to invoke the cyclic nature of Ω itself. Thus, $\bigwedge[V]$ actually contains a dense set of vectors that are cyclic for creators alone. Notice that this result does not depend on the dimension of V.

We are now prompted to ask whether $\bigwedge[V]$ contains vectors that are cyclic for annihilators alone. It turns out that the dimension of V influences the answer to this question: as we shall see, the answer is negative if and only if V is inseparable.

Let us first suppose that V is inseparable: the complete orthonormal system $\{v_r : r \in R\}$ for V has uncountable index set. If $\zeta \in \bigwedge[V]$ then at most countably many vectors $v_{r_1} \wedge \ldots \wedge v_{r_n}$ with $r_1 < \ldots < r_n$ in R can have $\langle \zeta \,|\, v_{r_1} \wedge \ldots \wedge v_{r_n}\rangle$ non-zero. As a consequence, there exists $r \in R$ such that $\operatorname{ran} c(v_r)$ and ζ are perpendicular. We claim more generally that if q is a polynomial in annihilators and if $\xi \in \bigwedge[V]$ then $q\,.\,\zeta$ and $c(v_r)\,\xi$ are perpendicular: since the adjoint q^* is a polynomial in creators, if we suppose without essential loss that q is even or odd then

$$\begin{aligned} \langle q\zeta \,|\, c(v_r)\,\xi\rangle &= \langle \zeta \,|\, q^* c(v_r)\,\xi\rangle \\ &= \pm \langle \zeta \,|\, c(v_r)\,q^*\xi\rangle \\ &= 0 \end{aligned}$$

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by virtue of the CAR among creators. Of course, it is now plain that ζ is not cyclic for annihilators.

THEOREM. *If V is inseparable then $\bigwedge[V]$ contains no vectors that are cyclic for annihilators alone.*

Next, let V be separable. If the dimension of V is finite, then $\bigwedge[V] = \bigwedge(V)$ surely contains vectors that are cyclic for annihilators alone: for example, the wedge product of vectors in a basis for V: in complete generality, any vector not orthogonal to such a wedge product. We are thus reduced to a consideration of the case in which V is separably infinite dimensional; in this case, we shall exhibit fermionic Gaussians that are cyclic for annihilators alone.

To begin, let $Z \in \mathbb{S}(V)$ correspond to $\zeta \in \bigwedge^2[V]$ as usual: thus, if $x, y \in V$ then

$$\langle \zeta \,|\, x \wedge y \rangle = \langle Zx \,|\, y \rangle.$$

As an immediate consequence, if $v \in V$ then

$$a(v)\,\zeta = Zv$$

so that if $n \geqslant 0$ then

$$a(v)\,\zeta^n = nc\,(Zv)\,\zeta^{n-1}$$

or

$$a(v)\frac{\zeta^n}{n!} = c(Zv)\frac{\zeta^{n-1}}{(n-1)!}.$$

By continuity, it follows from this that individual annihilators act as creators on the fermionic Gaussian $\epsilon(Z) = \exp \zeta$.

THEOREM. *If $Z \in \mathbb{S}(V)$ and $v \in V$ then $a(v)\,\epsilon(Z) = c(Zv)\,\epsilon(Z)$.*

Of course, here the dimension of V is immaterial. Proceeding a little further, if $x, y \in V$ then

$$\begin{aligned} a(y)\,a(x)\,\epsilon(Z) &= a(y)\,c(Zx)\,\epsilon(Z) \\ &= [\langle Zx \,|\, y \rangle - c(Zx)\,a(y)]\,\epsilon(Z) \\ &= \langle Zx \,|\, y \rangle\,\epsilon(Z) - c(Zx)\,c(Zy)\,\epsilon(Z) \\ &= \langle Zx \,|\, y \rangle\,\epsilon(Z) + c(Zy)\,c(Zx)\,\epsilon(Z) \end{aligned}$$

by virtue of the CAR. In general and in a similar manner, the CAR imply by induction that if $v_1, \ldots, v_n \in V$ then

$$a(v_n) \ldots a(v_1)\,\epsilon(Z) = c(Zv_n) \ldots c(Zv_1)\,\epsilon(Z) + p\,.\,\epsilon(Z)$$

where p is a polynomial of degree at most $n-2$ in creators of the form $c(Zv)$ for $v \in V$.

At this point, some notation will simplify matters. If W is any subspace of V, then we denote by $\mathscr{A}(W)$ the algebra of all polynomials in annihilators $a(w)$ for $w \in W$ and by $\mathscr{C}(W)$ the algebra of all polynomials in creators $c(w)$ for $w \in W$. In these terms, it follows from above that $\mathscr{A}(V)\,.\,\epsilon(Z) \subset \mathscr{C}(\operatorname{ran} Z)\,.\,\epsilon(Z)$ and (by another induction) that $\mathscr{A}(V)\,.\,\epsilon(Z) \supset \mathscr{C}(\operatorname{ran} Z)\,.\,\epsilon(Z)$. Thus we have the equality

$$\mathscr{A}(V)\,.\,\epsilon(Z) = \mathscr{C}(\operatorname{ran} Z)\,.\,\epsilon(Z).$$

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By virtue of the fact that if $u \in V$ then $\|c(u)\| = \|u\|$ we deduce that the closures of $\mathscr{A}(V).\epsilon(Z)$ and $\mathscr{C}(\overline{\operatorname{ran} Z}).\epsilon(Z)$ coincide: thus,

$$\overline{\mathscr{A}(V).\epsilon(Z)} = \overline{\mathscr{C}(\overline{\operatorname{ran} Z}).\epsilon(Z)}.$$

Once again, this result is valid for all V regardless of dimension.

Now, let V be separably infinite dimensional. We may choose $Z \in \mathbb{S}(V)$ so that ran Z is dense in V; equivalently Z is injective, since $(\operatorname{ran} Z)^{\perp} = \ker Z$. Let us agree to simplify notation a little by writing $\mathscr{A}$ in place of $\mathscr{A}(V)$ and $\mathscr{C}$ in place of $\mathscr{C}(V)$. This done, we have now arrived at the following result.

THEOREM. *If V is separably infinite dimensional and if $Z \in \mathbb{S}(V)$ is injective then*

$$\overline{\mathscr{A}.\epsilon(Z)} = \overline{\mathscr{C}.\epsilon(Z)}.$$

We can now demonstrate that under the hypotheses of this theorem, the fermionic Gaussian $\epsilon(Z)$ is cyclic for annihilators alone. Let π be any polynomial in creators and annihilators. The CAR allow us to express π as a sum of terms pq with $p \in \mathscr{C}$ and $q \in \mathscr{A}$; this is simply Wick ordering. From above, it follows at once that

$$\pi.\epsilon(Z) \in \mathscr{C}.\mathscr{A}.\epsilon(Z) \subset \mathscr{C}.\epsilon(Z) \subset \overline{\mathscr{A}.\epsilon(Z)}.$$

Now we invoke irreducibility of the Fock representation: the non-zero vector $\epsilon(Z)$ is cyclic for creators and annihilators together; as a result, $\epsilon(Z)$ is cyclic for annihilators alone.

THEOREM. *If V is separably infinite dimensional and $Z \in \mathbb{S}(V)$ is injective then the fermionic Gaussian $\epsilon(Z) \in \bigwedge[V]$ is cyclic for annihilators alone.*

Of course, we now have the following complement to the inseparable case.

THEOREM. *If V is separable then $\bigwedge[V]$ contains vectors that are cyclic for annihilators alone.*

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