A CONSTRUCTIVE MINIMAL INTEGRAL WHICH INCLUDES LEBESGUE INTEGRABLE FUNCTIONS AND DERIVATIVES

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1. *Introduction*

In this paper we provide a minimal constructive integration process of Riemann type which includes the Lebesgue integral and also integrates the derivatives of differentiable functions. We provide a new solution to the classical problem of recovering a function from its derivative by integration, which, unlike the solution provided by Denjoy, Perron and many others, does not possess the generality which is not needed for this purpose.

The descriptive version of the problem was treated by A. M. Bruckner, R. J. Fleissner and J. Foran in [**2**]. Their approach was based on the trivial observation that for the required minimal integral, a function F is the indefinite integral of f if and only if $F' = f$ almost everywhere and there exists a differentiable function H such that $F-H$ is absolutely continuous. They strengthen this definition by proving that $F-H$ can have arbitrary small variation. Nevertheless, their definition still needs a choice of a differentiable function.

Our approach is based on a Riemann-type integral and so does not need any choice of differentiable functions. It has been proposed, under the name of 'C-integral', by Bongiorno [**1**]. The basic idea behind the notion of the C-integral is similar to the one used by Zahorski in his fundamental study of derivatives [**8**].

DEFINITION 1. Given a function $f:[a,b]\longrightarrow \mathbf{R}$ we say that f is *C-integrable* on $[a,b]$ if there exists a constant A such that for each $\varepsilon > 0$ there is a gage δ so that

$$\left|\sum_{i=1}^{p} f(x_i)\,|I_i| - A\right| < \varepsilon$$

for each δ-fine (McShane) partition $\{(I_1,x_1),\dots,(I_p,x_p)\}$ satisfying the condition $\sum_{i=1}^{p} \operatorname{dist}(x_i,I_i) < 1/\varepsilon$.

We recall that a gage δ on $[a,b]$ is a positive function defined on $[a,b]$ and a δ-fine partition is a collection $\{(I_1,x_1),\dots,(I_p,x_p)\}$ of pairwise non-overlapping intervals $I_i \subset [a,b]$ such that $I_i \subset (x_i-\delta(x_i), x_i+\delta(x_i))$ and $\sum_{i=1}^{p} |I_i| = b-a$. If $\sum_{i=1}^{p} |I_i| \leqslant b-a$, then we say that the collection $\{(I_1,x_1),\dots,(I_p,x_p)\}$ is a partial partition. (We consider only so-called McShane partitions.)

We remark that Lebesgue integrable functions are C-integrable. The easiest way to see this is to recall that the Lebesgue integral is equivalent to the McShane one

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[**4**, **5**] and note that the McShane integral is contained in the C-integral. We remark also that C-integrability implies Denjoy integrability (with the same value of the integral). In order to avoid the use of many integrals, every integral will be understood as a Denjoy integral, unless specified otherwise.

Other simple facts about the C-integral include the following:

(1) The C-integral is linear; in particular the space of C-integrable functions is a vector space.

(2) C-integrability on an interval implies C-integrability on each subinterval.

(3) f is C-integrable if and only if, given $\varepsilon > 0$, there exists a gage δ so that

$$\sum_{i=1}^{p} \left| f(x_i)\,|I_i| - \int_{I_i} f \right| < \varepsilon$$

for each δ-fine partial partition $\{(I_1, x_1), \dots, (I_p, x_p)\}$ satisfying the condition $\sum_{i=1}^{p} \operatorname{dist}(x_i, I_i) < 1/\varepsilon$ (Henstock's lemma).

For additional properties of the C-integral we refer to [**1**].

THEOREM 1. *The C-integral is the minimal integral which includes Lebesgue integrable functions and derivatives.*

As usual, we use the term 'derivative' to denote a function h for which there is an everywhere differentiable function H such that $h = H'$.

In a more precise form our result may be stated as follows.

MAIN THEOREM. *A function $f\colon [a, b] \longrightarrow \mathbf{R}$ is C-integrable if and only if there exists a derivative h such that $f-h$ is Lebesgue integrable.*

The 'if' part was proved in [**1**, Theorem 1]. For completeness we will reproduce the simple argument at the end of the next section.

2. *Proof of Main Theorem*

For the proof of the 'only if' part we need the following notation and lemmas. By $\Delta[\alpha, \beta]$ we denote the family of all derivatives on the interval $[\alpha, \beta]$.

Given a real function f, an interval $[\alpha, \beta]$ will be called *f-regular* if there is $h \in \Delta[\alpha, \beta]$ such that $f-h$ is Lebesgue integrable in $[\alpha, \beta]$.

By $\mathscr{C}([\alpha, \beta])$ and by $L^1([\alpha, \beta])$ we denote respectively the family of all continuous functions and the family of all Lebesgue integrable functions in $[\alpha, \beta]$.

LEMMA 1. *If $[\alpha, \beta]$ is f-regular, then for each $\varepsilon > 0$ there exists $h_\varepsilon \in \Delta[\alpha, \beta]$ such that $h_\varepsilon(\alpha) = h_\varepsilon(\beta) = 0$ and*

$$\int_\alpha^\beta |f-h_\varepsilon| < \varepsilon, \qquad \int_\alpha^\beta h_\varepsilon = \int_\alpha^\beta f.$$

Proof. Let $h \in \Delta[\alpha, \beta]$ be such that $f-h \in L^1([\alpha, \beta])$. By the absolute continuity of the Lebesgue integral and by Lusin's theorem there exists $f_1 \in \mathscr{C}([\alpha, \beta])$ such that

$$\int_\alpha^\beta |f-h-f_1| < \frac{\varepsilon}{4}.$$

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Let $f_2 \in \mathscr{C}([\alpha,\beta])$ be such that $f_2(\alpha) = h(\alpha)+f_1(\alpha)$, $f_2(\beta) = h(\beta)+f_1(\beta)$ and $\int_\alpha^\beta |f_2| < \varepsilon/4$.

Define now $f_3 \in \mathscr{C}([\alpha,\beta])$ such that $f_3(\alpha) = f_3(\beta) = 0$ and

$$\int_\alpha^\beta f_3 = \int_\alpha^\beta (h+f_1-f_2) - \int_\alpha^\beta f.$$

We can assume that $f_3 \geqslant 0$ or $f_3 \leqslant 0$. Then

$$\int_\alpha^\beta |f_3| = \left| \int_\alpha^\beta f_3 \right| \leqslant \int_\alpha^\beta |f-h-f_1| + \int_\alpha^\beta |f_2|$$

$$< \frac{\varepsilon}{4}+\frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

Finally define $h_\varepsilon = h+f_1-f_2-f_3$. Then $h_\varepsilon \in \Delta[\alpha,\beta]$ and

$$\int_\alpha^\beta h_\varepsilon = \int_\alpha^\beta f.$$

Moreover

$$\int_\alpha^\beta |f-h_\varepsilon| \leqslant \int_\alpha^\beta |f-h-f_1| + \int_\alpha^\beta |f_2| + \int_\alpha^\beta |f_3|$$

$$< \frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

LEMMA 2. (a) *A subinterval of an f-regular interval is f-regular.*

(b) *If an interval is the union of two f-regular intervals, then it is f-regular.*

Proof. Statement (a) is obvious. To prove (b) let $[\alpha,\beta] = [\alpha_1,\beta_1] \cup [\alpha_2,\beta_2]$, with $[\alpha_i,\beta_i]$ f-regular, $i = 1,2$. By (a) we may assume that $\alpha_1 = \alpha$, $\alpha_2 = \beta_1$ and $\beta_2 = \beta$. By Lemma 1 there are $h_i \in \Delta[\alpha_i,\beta_i]$, $i = 1,2$, such that $h_i(\alpha_i) = h_i(\beta_i) = 0$, $\int_{\alpha_i}^{\beta_i} |f-h_i| < +\infty$. We define $h_i(x) = 0$ for $x \notin [\alpha_i,\beta_i]$, $i = 1,2$ and let $h = h_1+h_2$. Clearly h is the desired derivative. $\square$

LEMMA 3. *Let f be C-integrable on* $[\alpha,\beta]$ *and let F be a closed subset of* $[\alpha,\beta]$ *such that* $\alpha,\beta \in F$ *and each closed interval contained in* $[\alpha,\beta]\backslash F$ *is f-regular. Suppose also that* $\int_F |f| < +\infty$ *and*

$$\sum_{p\in\mathbf{N}} \sup_{\substack{J\subset I_p \\ J \text{ interval}}} \left| \int_J f \right| < +\infty,$$

where $I_p = (\alpha_p,\beta_p)$ *is the sequence of all connected components of* $[\alpha,\beta]\backslash F$. *Then* $[\alpha,\beta]$ *is f-regular.*

Since the proof of this lemma is more involved than the previous ones, we defer it to the last section of this paper.

2.1. *Proof of the 'only if' part of the Main Theorem*

Let f be C-integrable on $[a,b]$. We have to prove that $[a,b]$ is f-regular. Let G be the union of the interiors of all f-regular intervals contained in $[a,b]$ and let $F = [a,b]\backslash G$. By Lemma 2, every closed interval $J \subset [a,b]\backslash F$ is f-regular, since it can

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be covered by finitely many f-regular intervals. We first observe that the closure of any connected component (α,β) of $[a,b]\setminus F$ is f-regular. Indeed, by the continuity of the indefinite Denjoy integral of f, we have

$$\sup_{\substack{J\subset(\alpha,\beta)\\ J \text{ interval}}}\left|\int_J f\right| < +\infty.$$

Consequently, the assumptions of Lemma 3 hold with F replaced by $F\cap[\alpha,\beta]$, and we conclude that $[\alpha,\beta]$ is f-regular. In particular, if $F=\{a,b\}$, then $[a,b]$ is f-regular and we are done.

To finish the proof, we assume that $F\cap(a,b)\neq\varnothing$ and find a contradiction. We note that $F\cap(a,b)$ does not have any isolated point. In fact for such a point γ we could find $\alpha<\gamma<\beta$, $\alpha,\beta\in F$ such that $(\alpha,\gamma)\cap F=\varnothing=(\gamma,\beta)\cap F$. By the above observation, $[\alpha,\gamma]$ and $[\gamma,\beta]$ are f-regular, so by Lemma 2, $[\alpha,\beta]$ is f-regular and $\gamma\notin F$.

Since f is Denjoy-integrable there exists [7, Chapter VIII, Theorem 1.4] an interval $[\alpha,\beta]\subset[a,b]$ such that $F\cap(\alpha,\beta)\neq\varnothing$, $\int_{F\cap[\alpha,\beta]}|f|<+\infty$ and

$$\sum_{p\in\mathbf{N}}\sup_{\substack{J\subset I_p\\ J \text{ interval}}}\left|\int_J f\right| < +\infty,$$

where $\{I_p\}$ is the sequence of connected components of $[\alpha,\beta]\setminus F$. Since $F\cap(a,b)$ has no isolated points, we may achieve $\alpha,\beta\in F$. Then all assumptions of Lemma 3 hold with F replaced by $F\cap[\alpha,\beta]$, *so* $[\alpha,\beta]$ is f-regular, which, by definition of F means that (α,β) cannot meet F. This is the final contradiction proving the 'only if' part of the theorem.

2.2. *Proof of the 'if' part of the Main Theorem*

Let h be a derivative such that $f-h$ is Lebesgue integrable. Then $f-h$ is C-integrable. To complete the proof it is enough to show that h is C-integrable. Let $\varepsilon>0$ and let $h=H'$. By the definition of derivative, given $x\in[a,b]$ there exists $\delta(x)>0$ such that

$$\left|\frac{H(y)-H(x)}{y-x}-h(x)\right|<\frac{\varepsilon^2}{2}, \tag{1}$$

for each $y\in[a,b]$ with $|y-x|<\delta(x)$.

Let $I=(\alpha,\beta)$ be an interval such that $\operatorname{dist}(x,I)<\delta(x)$. By (1) we get

$$\begin{aligned}|H(I)-h(x)\,|I|\,| &\leqslant |H(\beta)-H(x)-h(x)\,(\beta-x)|+|H(\alpha)-H(x)-h(x)\,(\alpha-x)|\\ &<\frac{\varepsilon^2}{2}|\beta-x|+\frac{\varepsilon^2}{2}|\alpha-x|\leqslant\varepsilon^2\operatorname{dist}(x,I).\end{aligned} \tag{2}$$

Then, if $\{(I_1,x_1),\dots,(I_p,x_p)\}$ is a δ-fine partition satisfying the condition $\sum_{i=1}^p\operatorname{dist}(x_i,I_i)<1/\varepsilon$, we get by (2)

$$\left|\sum_{i=1}^p h(x_i)\,|I_i|-(H(b)-H(a))\right|\leqslant\sum_{i=1}^p|h(x_i)\,|I_i|-H(I_i)|<\varepsilon^2\sum_{i=1}^p\operatorname{dist}(x_i,I_i)<\varepsilon^2\frac{1}{\varepsilon}=\varepsilon.\quad\square$$

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3. *Proof of Lemma* 3

In this section we assume that f is a function C-integrable on $[\alpha,\beta]$, F is a closed subset of $[\alpha,\beta]$ such that $\alpha,\ \beta\in F$ and each closed interval contained in $[\alpha,\beta]\backslash F$ is f-regular. We also assume that $\int_F |f| < +\infty$ and

$$\sum_{p\in\mathbf{N}} \sup_{\substack{J\subset I_p\\ J \text{ interval}}} \left|\int_J f\right| < +\infty,$$

where $I_p = (\alpha_p,\beta_p)$ is the sequence of all connected components of $[\alpha,\beta]\backslash F$.

If $F=[\alpha,\beta]$ then $[\alpha,\beta]$ is f-regular, since $0\in\Delta[\alpha,\beta]$, $f\in L^1([\alpha,\beta])$ and $f-0=f\in L^1([\alpha,\beta])$. We will therefore assume that $F\neq[\alpha,\beta]$ and even that $\{I_p\}$ is infinite (indeed, if it is finite we can reduce the proof to the infinite case by adding a suitable sequence of points to F).

Let

$$\tilde{f}=f-f\chi_F.$$

Note that $\tilde{f}=0$ on F and $\tilde{f}$ is C-integrable on $[\alpha,\beta]$, by the assumption $\int_F|f|<+\infty$, so if we prove that $[\alpha,\beta]$ is $\tilde{f}$-regular, then $[\alpha,\beta]$ is f-regular. Thus, from now, we assume $f=0$ on F.

Now let $\{\varepsilon_n\}$ be a sequence of positive real numbers such that

$$\sum_{n\in\mathbf{N}} \varepsilon_n < +\infty, \tag{3}$$

and

$$\beta-\alpha+2 < \frac{1}{\varepsilon_n}. \tag{4}$$

Moreover, let $\delta_n(x)$ be the gage corresponding to $\varepsilon=\varepsilon_n$ in the definition of the C-integral of f. Hence, if $\{(x_1,I_1),\dots,(x_p,I_p)\}$ is a δ_n-fine partial partition in $[\alpha,\beta]$ such that $x_i\in F$ and $\sum_{i=1}^p \operatorname{dist}(x_i,I_i)<\beta-\alpha+2$, we have by Henstock's lemma

$$\sum_{i=1}^{p}\left|\int_{I_i} f\right| < \varepsilon_n. \tag{5}$$

Fix an increasing sequence of compact sets H_k such that $(\alpha,\beta)\backslash F=\bigcup_{k=1}^{\infty}H_k$, $H_k\cap I_p$ is a proper interval for finitely many p and $H_k\cap I_p=\varnothing$ for the remaining indices p, and

$$\sum_{p\in\mathbf{N}} \sup_{\substack{J\subset I_p\backslash H_k\\ J \text{ interval}}} \left|\int_J f\right| < +\infty. \tag{6}$$

For natural n,l set

$$F_{n,l}=\left\{x\in F:\min(\delta_1(x),\dots,\delta_n(x))>\frac{3}{l}\right\},$$

$$G_{n,l}=\left\{x\in[\alpha,\beta]:\operatorname{dist}(x,F_{n,l})<\frac{1}{l}\right\}.$$

CLAIM 1. *There exists a family* $\{\gamma_{p,q};p\in\mathbf{N},\ q\in\mathbf{Z}\}$ *of real numbers such that*
(c_1) $\gamma_{p,0}=(\alpha_p+\beta_p)/2$;
(c_2) $\gamma_{p,q}\leqslant\gamma_{p,q+1}$;
(c_3) $\alpha_p<\gamma_{p,q}<\beta_p$;
(c_4) $\inf_{q\in\mathbf{Z}}\{\gamma_{p,q}\}=\alpha_p$ *and* $\sup_{q\in\mathbf{Z}}\{\gamma_{p,q}\}=\beta_p$;

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and such that the intervals $I_{p,q} = [\gamma_{p,q}, \gamma_{p,q+1}]$ *satisfy*

(c_5) *for each* $x \in F$ *and each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\left| \int_J f \right| \leqslant \varepsilon \operatorname{dist}(x, I_{p,q}),$$

for any interval J *contained in* $I_{p,q}$, *provided that* $I_{p,q} \subset (x-\delta, x+\delta)$;

(c_6) $\sum_{p\in\mathbf{N}, q\in\mathbf{Z}} |\int_{I_{p,q}} f| < +\infty$.

Proof. We construct only the family $\{\gamma_{p,q}; p \in \mathbf{N}, q \geqslant 0\}$ and prove the corresponding version of conditions (c_1)–(c_6). The family $\{\gamma_{p,q}; p \in \mathbf{N}, q \leqslant 0\}$ can be constructed in a similar way.

DEFINITION 2. According to condition (c_1) we have to define $\gamma_{p,0} = (\alpha_p + \beta_p)/2$ and we will continue by induction. Suppose that $\gamma_{p,q}$ has been defined. To define $\gamma_{p,q+1}$ we set

$$\mathscr{F}_{p,q} = \{(k,l) : [\gamma_{p,q}, \beta_p) \subset G_{k,l} \text{ and } [\gamma_{p,q}, \beta_p) \cap H_k = \varnothing\}.$$

For the case when $\mathscr{F}_{p,q}$ is empty we define $\gamma_{p,q+1} = (\gamma_{p,q} + \beta_p)/2$. For the opposite case we remark that $\mathscr{F}_{p,q}$ is finite. Indeed $[\gamma_{p,q}, \beta_p) \cap H_k = \varnothing$ only for finitely many k, the sequence $\{H_k\}$ being increasing. Moreover the inclusion $[\gamma_{p,q}, \beta_p) \subset G_{k,l}$ can hold only if $l < 1/(\beta_p - \gamma_{p,q})$.

Set

$$t_{p,q} = \sup\{t \in [\gamma_{p,q}, \beta_p) : |\textstyle\int_J f| \leqslant \varepsilon_k \operatorname{dist}([\gamma_{p,q}, t], F_{k,l}), \text{ for each interval } J \subset [\gamma_{p,q}, t] \text{ and for each } (k,l) \in \mathscr{F}_{p,q}\}.$$

In the case $t_{p,q} < \beta_p$ we define $\gamma_{p,q+1} = t_{p,q}$; in the case $t_{p,q} = \beta_p$ we define $\beta_p > \gamma_{p,q+1} > (\gamma_{p,q} + \beta_p)/2$ such that $[\gamma_{p,q+1}, \beta_p) \cap H_{k+1} = \varnothing$, for every k for which $[\gamma_{p,q}, \beta_p) \cap H_k = \varnothing$. It is clear that $\gamma_{p,q} \leqslant \gamma_{p,q+1}$ and that conditions (c_2) and (c_3) are satisfied, for any p and q.

Condition (c_4). Suppose to the contrary that $\gamma = \sup_{q\in\mathbf{N}}\{\gamma_{p,q}\} < \beta_p$. Take $0 < \delta < \beta_p - \gamma$ such that $|\int_J f| < \varepsilon_k(\beta_p - \gamma)/2$ for each k for which $\gamma \notin H_k$ and for each interval $J \subset (\gamma - \delta, \gamma + \delta)$. Choose $\gamma_{p,q} > \gamma - \delta$. We claim that $\gamma_{p,q+1} > \gamma$. If $\mathscr{F}_{p,q} = \varnothing$ then $\gamma_{p,q+1} = (\gamma_{p,q} + \beta_p)/2 > (\gamma - \delta + \beta_p)/2 > \gamma$. If $\mathscr{F}_{p,q} \neq \varnothing$ and $t_{p,q} = \beta_p$, then $\gamma_{p,q+1} > (\gamma_{p,q} + \beta_p)/2 > \gamma$. If $\mathscr{F}_{p,q} \neq \varnothing$ and $t_{p,q} < \beta_p$, we prove that $t_{p,q} \geqslant \gamma + \delta/2$. Let J be a subinterval of $[\gamma_{p,q}, \gamma + (\delta/2)]$ and let $(k,l) \in \mathscr{F}_{p,q}$. By the definition of $\mathscr{F}_{p,q}$ we have $H_k \cap [\gamma_{p,q}, \gamma + (\delta/2)] = \varnothing$, so $\gamma \notin H_k$. Moreover, $J \subset (\gamma - \delta, \gamma + \delta)$. Hence

$$\begin{aligned}
\left| \int_J f \right| &< \varepsilon_k \frac{\beta_p - \gamma}{2} = \varepsilon_k \left(\beta_p - \gamma - \frac{\beta_p - \gamma}{2} \right) \\
&< \varepsilon_k \left(\beta_p - \left(\gamma + \frac{\delta}{2} \right) \right) \\
&= \varepsilon_k \operatorname{dist}\left(\left[\gamma_{p,q}, \gamma + \frac{\delta}{2} \right], F \right) \\
&\leqslant \varepsilon_k \operatorname{dist}\left(\left[\gamma_{p,q}, \gamma + \frac{\delta}{2} \right], F_{k,l} \right).
\end{aligned}$$

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Then, by the definition of $\gamma_{p,q+1}$, we have

$$\gamma_{p,q+1} = t_{p,q} \geqslant \gamma + \frac{\delta}{2} > \gamma.$$

Hence in all three cases we get a contradiction with the definition of γ.

Condition (c_5). Let $x \in F$ and $\varepsilon > 0$. Take $k \in \mathbf{N}$ such that $\varepsilon_k < \varepsilon$ and take $l \in \mathbf{N}$ such that $x \in F_{k,l}$. We choose $0 < \delta < 1/2l$ such that $(x-2\delta, x+2\delta) \cap H_k = \varnothing$. If $I_{p,q} \subset (x-\delta, x+\delta)$ then $(k,l) \in \mathscr{F}_{p,q}$. To see this we have to prove that $[\gamma_{p,q}, \beta_p) \subset G_{k,l}$ and $[\gamma_{p,q}, \beta_p) \cap H_k = \varnothing$. If $x \geqslant \beta_p$ then $[\gamma_{p,q}, \beta_p) \subset (x-\delta, x)$, so both statements hold since $(x-\delta, x) \subset G_{k,l}$ and $(x-\delta, x) \cap H_k = \varnothing$. If $x \leqslant \alpha_p$ then $(\alpha_p, \gamma_{p,q}) \subset (x, x+\delta)$. Since $\gamma_{p,q} \geqslant \gamma_{p,0}$ (which is the centre of I_p), we get $[\gamma_{p,q}, \beta_p) \subset I_p \subset (x, x+2\delta)$. Then the required statement follows from $(x, x+2\delta) \subset G_{k,l}$ and $(x, x+2\delta) \cap H_k = \varnothing$. Now, from $(k,l) \in \mathscr{F}_{p,q}$, we see that $t_{p,q}$ is defined and, by continuity of the indefinite integral, we have

$$\left| \int_J f \right| \leqslant \varepsilon_k \operatorname{dist}([\gamma_{p,q}, t_{p,q}], F_{k,l}),$$

for all $J \subset [\gamma_{p,q}, t_{p,q}]$. Since $I_{p,q} \subset [\gamma_{p,q}, t_{p,q}]$, $x \in F_{k,l}$ and $\varepsilon_k < \varepsilon$, we conclude that

$$\left| \int_J f \right| \leqslant \varepsilon \operatorname{dist}(x, I_{p,q}),$$

for all $J \subset I_{p,q}$.

Condition (c_6). Decompose

$$\sum_{p,q} = \sum_{(p,q) \in \mathscr{I}_1} + \sum_{(p,q) \in \mathscr{I}_2} + \sum_{(p,q) \in \mathscr{I}_3},$$

where $\mathscr{I}_1$ is the set of those p, q for which $\mathscr{F}_{p,q} = \varnothing$, $\mathscr{I}_2$ is the set of those p, q for which $\mathscr{F}_{p,q} \neq \varnothing$ and $t_{p,q} = \beta_p$, $\mathscr{I}_3$ is the set of those p, q for which $\mathscr{F}_{p,q} \neq \varnothing$ and $t_{p,q} < \beta_p$.

Concerning the sum $\sum_{(p,q) \in \mathscr{I}_1}$ we claim that it is finite since $\mathscr{I}_1$ is a finite set. Indeed, assume $\{(p,q) : \mathscr{F}_{p,q} = \varnothing\}$ is infinite. Then there exists an infinite sequence of different pairs (p_i, q_i) such that $\mathscr{F}_{p_i,q_i} = \varnothing$. Hence there exists $x \in F$ such that for every positive δ the inclusion $I_{p_i,q_i} \subset (x-\delta, x+\delta)$ holds for infinitely many indices i. However $x \in F_{k,l}$ for some l and some k, so there exists $\delta^* > 0$ such that $(x-\delta^*, x+\delta^*) \subset G_{k,l}$ and $(x-\delta^*, x+\delta^*) \cap H_k = \varnothing$. For those indices i for which $I_{p_i,q_i} \subset (x-\delta^*, x+\delta^*)$ we have $\mathscr{F}_{p_i,q_i} \neq \varnothing$, hence a contradiction.

We now estimate the second sum. For every $(p,q) \in \mathscr{I}_2$ we choose $k_{p,q}$ such that $[\gamma_{p,q}, \beta_p] \cap H_{k_{p,q}} = \varnothing$ and $[\gamma_{p,q}, \beta_p] \cap H_{k_{p,q}+1} \neq \varnothing$. We first show that, given k and p, there is at most one $q \geqslant 0$ such that $k_{p,q} = k$. To see this, suppose that $q_1 < q_2$ with $(p,q_1), (p,q_2) \in \mathscr{I}_2$ and $k_{p,q_1} = k_{p,q_2} = k$. By definition of k_{p,q_1} we have $[\gamma_{p,q_1}, \beta_p) \cap H_k = \varnothing$. Since $t_{p,q_1} = \beta_p$, the point γ_{p,q_1+1} was chosen so that $[\gamma_{p,q_1+1}, \beta_p) \cap H_{k+1} = \varnothing$. However $q_2 \geqslant q_1 + 1$, so $[\gamma_{p,q_2}, \beta_p) \cap H_{k+1} = \varnothing$, against the assumption that $k_{p,q_2} = k$. Noting that $I_{p,q} \subset I_p \backslash H_{k_{p,q}}$ we therefore infer from (3) and (6) that

$$\sum_{p,q \in \mathscr{I}_2} \left| \int_{I_{p,q}} f \right| \leqslant \sum_k \sum_{k_{p,q}=k} \left| \int_{I_{p,q}} f \right|$$

$$\leqslant \sum_k \sum_p \sup_{J \subset I_p \backslash H_k} \left| \int_J f \right| < \sum_k \varepsilon_k < +\infty.$$

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Finally we estimate the third sum. Assume that $(p,q)\in\mathscr{I}_3$; then for every $(k,l)\in\mathscr{F}_{p,q}$ we have

$$\left|\int_J f\right| \leqslant \varepsilon_k \operatorname{dist}(I_{p,q}, F_{k,l}),$$

for all intervals $J\subset I_{p,q}$. Since $t_{p,q}<\beta_p$, we infer from the continuity of the indefinite integral of f that there exist an interval $J_{p,q}\subset I_{p,q}$ and indices $k_{p,q}$ and $l_{p,q}$ such that $(k_{p,q},l_{p,q})\in\mathscr{F}_{p,q}$ and

$$\left|\int_{J_{p,q}} f\right| = \varepsilon_{k_{p,q}} \operatorname{dist}(I_{p,q}, F_{k_{p,q},l_{p,q}}). \tag{7}$$

Now we estimate

$$\begin{aligned}\sum_{(p,q)\in\mathscr{I}_3}\left|\int_{I_{p,q}} f\right| &\leqslant \sum_k \sum_{k_{p,q}=k}\left|\int_{I_{p,q}} f\right| \\ &\leqslant \sum_k \varepsilon_k \sum_{k_{p,q}=k} \operatorname{dist}(I_{p,q}, F_{k_{p,q},l_{p,q}}).\end{aligned}$$

Because of (3) we finish the proof by showing that, for each k,

$$\sum_{k_{p,q}=k} \operatorname{dist}(I_{p,q}, F_{k_{p,q},l_{p,q}}) \leqslant 2.$$

Suppose to the contrary that for some k this is false. Then there are a finite set $\mathscr{I}\subset\mathscr{I}_3$ and $(p_0,q_0)\in\mathscr{I}_3\setminus\mathscr{I}$ such that $k_{p_0,q_0}=k=k_{p,q}$ for each $(p,q)\in\mathscr{I}$,

$$\sum_{(p,q)\in\mathscr{I}} \operatorname{dist}(I_{p,q}, F_{k_{p,q},l_{p,q}}) < 2$$

and

$$\sum_{(p,q)\in\mathscr{I}\cup\{(p_0,q_0)\}} \operatorname{dist}(I_{p,q}, F_{k_{p,q},l_{p,q}}) \geqslant 2. \tag{8}$$

If $0<\varepsilon<1/l$ and $x\in F_{k_{p,q},l_{p,q}}$ is such that $\operatorname{dist}(x,I_{p,q})<\operatorname{dist}(I_{p,q},F_{k_{p,q},l_{p,q}})+\varepsilon$, we infer from the fact that the centre of $I_{p,q}$ belongs to $G_{k_{p,q},l_{p,q}}$ that $\operatorname{dist}(I_{p,q},F_{k_{p,q},l_{p,q}})+|I_{p,q}|/2<1/l_{p,q}$ and so, by definition of $F_{k_{p,q},l_{p,q}}$, that $I_{p,q}\subset(x-3/l_{p,q},x+3/l_{p,q})\subset(x-\delta_k(x),x+\delta_k(x))$. Consequently, for each $(p,q)\in\mathscr{I}$ we may choose $x_{p,q}\in F_{k_{p,q},l_{p,q}}$ such that $I_{p,q}\subset(x_{p,q}-\delta_k(x_{p,q}),x_{p,q}+\delta_k(x_{p,q}))$ and $\sum_{(p,q)\in\mathscr{I}}\operatorname{dist}(x_{p,q},I_{p,q})<2$. Hence

$$\sum_{(p,q)\in\mathscr{I}} \operatorname{dist}(x_{p,q},J_{p,q}) < 2+\sum_{(p,q)\in\mathscr{I}}|I_{p,q}| \leqslant b-a+2.$$

Since $J_{p,q}\subset I_{p,q}\subset(x_{p,q}-\delta_k(x_{p,q}),x_{p,q}+\delta_k(x_{p,q}))$, by (5) we have

$$\sum_{(p,q)\in\mathscr{I}}\left|\int_{J_{p,q}} f\right| < \varepsilon_k.$$

Thus, using (7) we have

$$\varepsilon_k \sum_{(p,q)\in\mathscr{I}} \operatorname{dist}(I_{p,q}, F_{k_{p,q},l_{p,q}}) = \sum_{(p,q)\in\mathscr{I}}\left|\int_{J_{p,q}} f\right| < \varepsilon_k;$$

hence

$$\sum_{(p,q)\in\mathscr{I}\cup\{(p_0,q_0)\}} \operatorname{dist}(I_{p,q}, F_{k_{p,q},l_{p,q}}) < 1+\operatorname{dist}(I_{p_0,q_0}, F_{k_{p_0,q_0}}) \leqslant 1+\frac{1}{l_{p_0,q_0}} \leqslant 2,$$

which contradicts (8). □

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Proof of the regularity of $[\alpha,\beta]$. For each $p\in\mathbf{N}$ and $q\in\mathbf{Z}$ we put

$$c_{p,q}=\frac{1}{|I_{p,q}|}\int_{I_{p,q}} f.$$

Let

$$0<\eta_{p,q}<\operatorname{dist}^2(I_{p,q},F)\quad\text{and}\quad\sum_{p\in\mathbf{N},q\in\mathbf{Z}}\eta_{p,q}<+\infty. \tag{9}$$

By Lemma 1 there exists $h_{p,q}\in\Delta[\alpha,\beta]$ such that $h_{p,q}=0$ outside $I_{p,q}$,

$$\int_{I_{p,q}}|f-c_{p,q}-h_{p,q}|<\eta_{p,q}$$

and

$$\int_{I_{p,q}}h_{p,q}=0.$$

Define

$$h=\sum_{p\in\mathbf{N},q\in\mathbf{Z}}h_{p,q}.$$

We will prove the f-regularity of $[\alpha,\beta]$ by showing that $f-h\in L^1([\alpha,\beta])$ and $h\in\Delta[\alpha,\beta]$. Since $f=0$ on F, we have

$$\begin{aligned}\int_\alpha^\beta|f-h|&\leqslant\sum_{p\in\mathbf{N},q\in\mathbf{Z}}\int_{I_{p,q}}(|c_{p,q}|+|f-c_{p,q}-h_{p,q}|)\\&\leqslant\sum_{p\in\mathbf{N},q\in\mathbf{Z}}\left|\int_{I_{p,q}}f\right|+\sum_{p\in\mathbf{N},q\in\mathbf{Z}}\eta_{p,q}.\end{aligned}$$

Then, by condition (c_6) and by (9) we conclude that $f-h\in L^1(\alpha,\beta)$. Note that $h=(h-f)+f$ is Denjoy integrable. Let

$$H(x)=\int_\alpha^x h,\quad x\in[\alpha,\beta].$$

Clearly $H'(x)=h(x)$ if $x\notin F$. We prove that $H'(x)=0=h(x)$, for each $x\in F$. First let $\tilde{F}=F\cup\{\gamma_{p,q};p\in\mathbf{N},q\in\mathbf{Z}\}$ and observe that $H(x)=0$ for every $x\in\tilde{F}$. Also note that $h(x)=0$ for $x\in F$.

Now fix $x\in F$ and $\tau>0$. By (c_5) and by the absolute continuity of the Lebesgue primitive of $h-f$, there exists $0<\delta<\tau$ such that

$$\left|\int_J f\right|\leqslant\tau\operatorname{dist}(x,I_{p,q})\quad\text{and}\quad\int_J|h-f|<\eta_{p,q}, \tag{10}$$

whenever J is an interval and $J\subset I_{p,q}\subset(x-\delta,x+\delta)$. By (9) and (10), for the same intervals J we obtain

$$\begin{aligned}\left|\int_J h\right|&\leqslant\left|\int_J f\right|+\eta_{p,q}\\&\leqslant\tau\operatorname{dist}(x,I_{p,q})+\operatorname{dist}^2(I_{p,q},F)\\&\leqslant\tau\operatorname{dist}(x,I_{p,q})+\operatorname{dist}^2(x,I_{p,q})\\&\leqslant\tau\operatorname{dist}(x,I_{p,q})+\delta\operatorname{dist}(x,I_{p,q})\\&\leqslant 2\tau\operatorname{dist}(x,I_{p,q}).\end{aligned}$$

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We finish the proof by showing that

$$|H(y)-H(x)| \leqslant 2\tau|y-x| \tag{11}$$

for $y\in(u_1,u_2)$, where $u_1,u_2\in\tilde{F}$ are chosen such that

$$x-\delta < u_1 < x < u_2 < x+\delta.$$

Indeed (11) is obvious if $y\in F$. If $y\in(u_1,u_2)\backslash F$ there are p,q such that $y\in I_{p,q}$ (see (c_4)). Then

$$I_{p,q}\subset(u_1,u_2)\subset(x-\delta,x+\delta).$$

Hence

$$|H(y)-H(x)| = \left|\int_{\gamma_{p,q}}^{y} h\right| \leqslant 2\tau\,\mathrm{dist}(x,I_{p,q}) \leqslant 2\tau|y-x|. \qquad \square$$

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