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INVERSE BERNSTEIN INEQUALITIES AND MIN–MAX–MIN PROBLEMS ON THE UNIT CIRCLE

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Abstract. We give a short and elementary proof of an inverse Bernstein-type inequality found by S. Khrushchev for the derivative of a polynomial having all its zeros on the unit circle. The inequality is used to show that equally-spaced points solve a min–max–min problem for the logarithmic potential of such polynomials. Using techniques recently developed for polarization (Chebyshev-type) problems, we show that this optimality also holds for a large class of potentials, including the Riesz potentials $1/r^s$ with $s > 0$.

§1. *Inverse Bernstein-type inequality.* Inequalities involving the derivatives of polynomials often occur in approximation theory (see, e.g., [3, 4, 6]). One of the most familiar of these inequalities, which is due to Bernstein, provides an upper bound for the derivative of a polynomial on the unit circle $\mathbb{T}$ of the complex plane. In [9], Khrushchev derived a rather striking inverse Bernstein-type inequality, a slight improvement of which may be stated as follows.

THEOREM 1.1. *If*

$$\begin{gathered} P(z) = \prod_{j=1}^{n} (z - z_j), \quad z_j \in \mathbb{T}, \\ z_j = \mathrm{e}^{\mathrm{i}t_j}, \quad 0 \leqslant t_1 < \cdots < t_n < 2\pi, \quad t_{n+1} := t_1 + 2\pi, \\ m := \min_{1 \leqslant j \leqslant n} \Big(\max_{t \in [t_j, t_{j+1}]} |P(\mathrm{e}^{\mathrm{i}t})| \Big), \end{gathered} \tag{1.1}$$

then

$$|P'(z)|^2 \geqslant \left(\frac{n}{2}\right)^2 (|P(z)|^2 + (m^2 - |P(z)|^2)_+) \geqslant \left(\frac{nm}{2}\right)^2, \quad z \in \mathbb{T}, \tag{1.2}$$

where $(x)_+ := \max\{x, 0\}$.

Khrushchev used a potential theoretic method to prove his inequality. Here we offer a simple and short proof based on an elementary zero counting argument.

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Proof. Write

$$P(\mathrm{e}^{\mathrm{i}t}) = R(t)\mathrm{e}^{\mathrm{i}\varphi(t)}, \qquad R(t) := |P(\mathrm{e}^{\mathrm{i}t})|,$$

where R and φ are differentiable functions on

$$[0, 2\pi) \setminus \{t_1, \dots, t_n\}.$$

Since $P'(\mathrm{e}^{\mathrm{i}t})$ is a continuous function of $t \in \mathbb{R}$, in the rest of the proof we may assume that

$$t \in [0, 2\pi) \setminus \{t_1, \dots, t_n\}.$$

We have

$$P'(\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{\mathrm{i}t}\mathrm{i} = R'(t)\mathrm{e}^{\mathrm{i}\varphi(t)} + \mathrm{i}R(t)\mathrm{e}^{\mathrm{i}\varphi(t)}\varphi'(t),$$

and it follows that

$$|P'(\mathrm{e}^{\mathrm{i}t})|^2 = (R'(t))^2 + R(t)^2(\varphi'(t))^2. \tag{1.3}$$

Using the fact that $w = z/(z - z_j)$ maps $\mathbb{T}$ onto the vertical line $\operatorname{Re} w = 1/2$, we have

$$\varphi'(t) = \operatorname{Re}\left(\varphi'(t) - \frac{R'(t)}{R(t)}\mathrm{i}\right) = \operatorname{Re}\left(\frac{P'(\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{\mathrm{i}t}}{P(\mathrm{e}^{\mathrm{i}t})}\right) = \operatorname{Re}\left(\sum_{j=1}^{n}\frac{\mathrm{e}^{\mathrm{i}t}}{\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{\mathrm{i}t_j}}\right) = \frac{n}{2}.$$

Thus from (1.3) we get

$$|P'(\mathrm{e}^{\mathrm{i}t})|^2 = (R'(t))^2 + \left(\frac{n}{2}\right)^2 R(t)^2, \tag{1.4}$$

and so, if $R(t) \geqslant m$, then (1.2) follows immediately. Observe from (1.4) that on each subarc determined by the successive zeros of $P(z)$, the ratio $|P/P'|$ varies from zero to $2/n$, with the latter value occurring at the unique maximum point of $|P|$ on that subarc†.

Assume now that $R(t) < m$. Observe that Q defined by $Q(t) := R(t)^2 = |P(\mathrm{e}^{\mathrm{i}t})|^2$ is a real trigonometric polynomial of degree n; that is, $Q \in \mathcal{T}_n$. Now let

$$t_0 \in [0, 2\pi) \setminus \{t_1, \dots, t_n\}$$

be fixed, and let $T \in \mathcal{T}_n$ be defined by

$$T(t) := m^2\cos^2(n(t - \alpha)/2) = \frac{m^2}{2}(1 + \cos(n(t - \alpha)),$$

where $\alpha \in \mathbb{R}$ is chosen so that

$$T(t_0) = Q(t_0) \tag{1.5}$$

and

$$\operatorname{sign}(T'(t_0)) = \operatorname{sign}(Q'(t_0)). \tag{1.6}$$

We claim that

$$|Q'(t_0)| \geqslant |T'(t_0)|. \tag{1.7}$$

† We thank both Gergely Ambrus and the referee for pointing out this *equi-oscillation* property of $|P/P'|$.

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Indeed, $|Q'(t_0)| < |T'(t_0)|$ together with (1.5) and (1.6) would imply that the not identically zero trigonometric polynomial $T - Q \in \mathcal{T}_n$ had at least $2n + 2$ zeros in the period $[0, 2\pi)$ (at least two zeros on each of the intervals $[t_1, t_2), \ldots, [t_n, t_{n+1})$, and at least four zeros on the interval (t_j, t_{j+1}) containing t_0) by counting multiplicities, a contradiction. Thus (1.7) holds and implies that

$$|Q'(t_0)| \geqslant |m^2 n \cos(n(t_0 - \alpha)/2) \sin(n(t_0 - \alpha)/2)|,$$

which, together with (1.5), yields

$$\begin{aligned} |Q'(t_0)|^2 &\geqslant n^2(m^2 \cos^2(n(t_0 - \alpha)/2))(m^2 \sin^2(n(t_0 - \alpha)/2)) \\ &= n^2 |Q(t_0)|(m^2 - |Q(t_0)|). \end{aligned}$$

Substituting $Q(t_0) = R(t_0)^2$ and $Q'(t_0) = 2R(t_0)R'(t_0)$ in the above inequality, we conclude that

$$(R'(t_0))^2 \geqslant \frac{n^2}{4}(m^2 - R(t_0)^2).$$

Finally, combining this last inequality with (1.4) and recalling that $R(t_0) = |P(e^{it_0})|$ yields (1.2). □

A natural question that arises is that of finding the maximal value $m^*(n)$ of the quantity m given in (1.1) or, equivalently (using the notation of Theorem 1.1), determining

$$m^*(n) := \max_{\omega_n \in \Omega_n} \min_{1 \leqslant j \leqslant n} \max_{t \in [t_j, t_{j+1}]} \left| \prod_{k=1}^{n} (e^{it} - z_k) \right|, \quad z_k = e^{it_k}, \tag{1.8}$$

where Ω_n is the collection of all n-tuples $\omega_n \in [0, 2\pi)^n$ of the form

$$\omega_n = (t_1, \ldots, t_n), \quad 0 \leqslant t_1 \leqslant \cdots \leqslant t_n < 2\pi.$$

In Corollary 6.9 of [**9**], Khrushchev proved that $m^*(n) = 2$, the value of m corresponding to $P(z) = z^n - 1$ for which equality holds throughout in (1.2). Here we deduce this fact as a simple consequence of Theorem 1.1.

COROLLARY 1.2. *Let $m^*(n)$ be as in (1.8). Then $m^*(n) = 2$ and this maximum is attained only for n distinct equally-spaced points $\{z_1, \ldots, z_n\}$ on the unit circle.*

In other words, for any monic polynomial of degree n with n distinct zeros on the unit circle, there must be some subarc formed from consecutive zeros on which the modulus of the polynomial is at most 2.

Proof of Corollary 1.2. Assume $m^*(n) > 2$. According to Theorem 1.1, $|P_o'(z)| > n$ for all z on $\mathbb{T}$, where P_o is a monic polynomial of degree n for which the maximum value $m^*(n)$ is attained. By the Gauss–Lucas theorem, P_o' has all its zeros in the open unit disk (clearly it can't have any on $\mathbb{T}$). So now consider $f(z) := P_o'(z)/z^{n-1}$, which is analytic on and outside $\mathbb{T}$, even at infinity where it equals n. Since f does not vanish outside or on $\mathbb{T}$, its modulus must attain its minimum on $\mathbb{T}$. But $|f(z)| > n$ on $\mathbb{T}$, while $f(\infty) = n$, which gives the desired

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contradiction. Thus $m^*(n) = 2$ and the argument above also shows that if this maximum is attained by a polynomial P_o, then $|f(z)| = 2$ for all z on or outside $\mathbb{T}$, which implies that f is constant and so P_o has equally-spaced zeros on $\mathbb{T}$. □

Observe that the determination of $m^*(n)$ can alternatively be viewed as a min–max–min problem on the unit circle for the logarithmic potential $\log(1/r)$ with r denoting Euclidean distance between points on $\mathbb{T}$. In the next section we consider such problems for a general class of potentials.

§2. *Min–max–min problems on* $\mathbb{T}$. Let g be a positive, extended real-valued, even function defined on $\mathbb{R}$ that is periodic with period 2π and satisfies $g(0) = \lim_{t\to 0} g(t)$. Further suppose that g is non-increasing and strictly convex on $(0, \pi]$. For $\omega_n = (t_1, \dots, t_n) \in \Omega_n$, we set

$$P_{\omega_n}(t) := \sum_{j=1}^{n} g(t - t_j). \tag{2.1}$$

Here and in the following we assume that t_j is extended so that

$$t_{j+n} = t_j + 2\pi \quad (j \in \mathbb{Z});$$

in particular, we have $t_0 = t_n - 2\pi$ and $t_{n+1} = t_1 + 2\pi$. For $\omega_n = (t_1, \dots, t_n) \in \Omega_n$ and $\gamma \in [0, 2\pi)$, let $\omega_n + \gamma$ denote the element in Ω_n corresponding to the set $\{e^{i(t_k+\gamma)}\}_{k=1}^n$. Then $P_{\omega_n+\gamma}(t) = P_{\omega_n}(t - \gamma)$. We further let $\widetilde{\omega}_n := (\widetilde{t}_1, \dots, \widetilde{t}_n)$ denote the equally-spaced configuration given by

$$\widetilde{t}_j := 2(j-1)\pi/n, \quad j = 1, \dots, n.$$

By the convexity of g, it follows that

$$\min_{t\in[0,2\pi)} P_{\widetilde{\omega}_n}(t) = P_{\widetilde{\omega}_n}(\pi/n).$$

Motivated by recent articles on polarization of discrete potentials on the unit circle (cf. [**1**, **2**, **7**, **8**]) we shall prove the following generalization of Corollary 1.2.

THEOREM 2.1. *Let g be a positive, extended real-valued, even function defined on $\mathbb{R}$ that is periodic with period 2π and satisfies $g(0) = \lim_{t\to 0} g(t)$. Suppose further that g is non-increasing and strictly convex on $(0, \pi]$. Then we have*

$$\min_{\omega_n\in\Omega_n} \left\{ \max_{1\leqslant j\leqslant n} \left\{ \min_{t\in[t_j,t_{j+1}]} P_{\omega_n}(t) \right\} \right\} = P_{\widetilde{\omega}_n}(\pi/n); \tag{2.2}$$

that is, the solution to the min–max–min problem on $\mathbb{T}$ is given by n distinct equally-spaced points on $\mathbb{T}$ and, moreover, these are the only solutions.

2.1. *Logarithmic and Riesz kernels.* It is straightforward to verify that

$$g(t) = g_{\log}(t) := \log(1/|e^{it} - 1|) = -\log(2\sin|t/2|)$$

satisfies the hypotheses of Theorem 2.1 providing an alternate proof of Corollary 1.2. Furthermore, for the case (relating to Euclidean distance)

$$g(t) = g_s(t) := |e^{it} - 1|^{-s} = (2\sin|t/2|)^{-s}, \quad s > 0, \tag{2.3}$$

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we obtain the *Riesz s-potential* and it is again easily verified that g_s satisfies the hypotheses of Theorem 2.1. Consequently, with $z_k = e^{it_k}$,

$$\min_{\omega_n \in \Omega_n} \max_{1 \leqslant j \leqslant n} \min_{t \in [t_j, t_{j+1}]} \sum_{k=1}^{n} |e^{it} - z_k|^{-s} = \sum_{k=1}^{n} |e^{i\pi/n} - e^{2k\pi i/n}|^{-s} = M_n^s(\mathbb{T}), \tag{2.4}$$

where $M_n^s(\mathbb{T})$ is the *Riesz s-polarization constant* for n points on the unit circle (cf. [8]). We remark that for s an even integer, say $s = 2m$, the precise value of $M_n^s(\mathbb{T})$ can be expressed as a polynomial in n; namely, as a consequence of the formulas derived in [5],

$$M_n^{2m}(\mathbb{T}) = \frac{2}{(2\pi)^{2m}} \sum_{k=1}^{m} n^{2k} \zeta(2k) \alpha_{m-k}(2m)(2^{2k} - 1), \quad m \in \mathbb{N}, \tag{2.5}$$

where $\zeta(s)$ is the classical Riemann zeta function and $\alpha_j(s)$ is defined via the power series for $\operatorname{sinc} z = (\sin \pi z)/(\pi z)$:

$$(\operatorname{sinc} z)^{-s} = \sum_{j=0}^{\infty} \alpha_j(s) z^{2j}, \quad \alpha_0(s) = 1;$$

see Corollary 3 from [8]. In particular,

$$M_n^2(\mathbb{T}) = \frac{n^2}{4}, \qquad M_n^4(\mathbb{T}) = \frac{n^2}{24} + \frac{n^4}{48}, \qquad M_n^6(\mathbb{T}) = \frac{n^2}{120} + \frac{n^4}{192} + \frac{n^6}{480}.$$

2.2. *Proof of Theorem 2.1.* Theorem 2.1 is a consequence of the following lemma, which is the basis of the proof of the polarization theorem established by Hardin *et al* in [8]. (In §3 we state a slightly stronger version of this polarization result as Theorem 3.1 and present some related results.)

LEMMA 2.2. *Let g be as in Theorem 2.1 and suppose $\omega_n = (t_1, \ldots, t_n)$ and $\omega_n' = (t_1', \ldots, t_n')$ are in Ω_n. Then there is some $\ell \in \{0, \ldots, n\}$ and some $\gamma \in [0, 2\pi)$ (where ℓ and γ depend on ω_n and ω_n' but not on g) such that*

$$P_{\omega_n'}(t - \gamma) \leqslant P_{\omega_n}(t), \quad t \in [t_\ell, t_{\ell+1}], \tag{2.6}$$

and $[t_\ell, t_{\ell+1}] \subset [t_\ell' + \gamma, t_{\ell+1}' + \gamma]$.

The inequality is strict for $t \in (t_\ell, t_{\ell+1})$ unless $t_{j+1} - t_j = t_{j+1}' - t_j'$ for all $j = 1, \ldots, n$.

Sketch of proof. This lemma follows from techniques developed in [8], specifically from Lemmas 5 and 6 in that paper. For the convenience of the reader, we provide here an outline of its proof. First, the convexity of g implies that, for $n = 2$, the inequality

$$P_{(t_1 - \Delta, t_2 + \Delta)}(t) < P_{(t_1, t_2)}(t), \quad t \in (t_1, t_2), \tag{2.7}$$

holds for sufficiently small $\Delta > 0$ (this observation was first presented in [1] and also utilized in [2]). That is, the potential due to two points decreases on an

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interval when the points are moved symmetrically *away* from the interval. For simplicity, we consider the case that

$$\operatorname{sep}(\omega_n) := \min_j (t_{j+1} - t_j) > 0,$$

(see [8] for the case of coincident points where $\operatorname{sep}(\omega_n) = 0$).

Next, using elementary linear algebra, we find a vector $\mathbf{\Delta} = (\Delta_1, \dots, \Delta_n)$ such that (a) $\Delta_k \geqslant 0$ for all k, (b) $\Delta_\ell = 0$ for some ℓ and (c) $\mathbf{\Delta}$ solves the equations

$$(t'_{j+1} - t'_j) = (t_{j+1} - t_j) - \Delta_{j+1} + 2\Delta_j - \Delta_{j-1} \quad (j = 1, \dots, n) \tag{2.8}$$

where we take $\Delta_0 := \Delta_n$ and $\Delta_{n+1} := \Delta_1$. For $j = 1, \dots, n$, consider the transformation

$$\tau_{j,\Delta}(\omega_n) := (t_1, \dots, t_{j-2}, t_{j-1} - \Delta, t_j + \Delta, t_{j+1}, \dots, t_n).$$

Then (2.8) implies that $\omega''_n := \tau_{1,\Delta_1} \circ \tau_{2,\Delta_2} \circ \cdots \circ \tau_{n,\Delta_n}(\omega_n)$ equals $\omega'_n + \gamma$ for some $\gamma \in [0, 2\pi)$. If $\max_j \Delta_j \leqslant (1/2)\operatorname{sep}(\omega_n)$ then, since $\Delta_\ell = 0$ and $\Delta_k \geqslant 0$, we may apply the inequality (2.7) n times to obtain (2.6). Moreover, unless $\Delta_k = 0$ for all k, inequality (2.8) is strict. If $\max_j \Delta_j > (1/2)\operatorname{sep}(\omega_n)$, then we may choose m such that $(1/m)\max_j \Delta_j < (1/2)\operatorname{sep}(\omega_n)$ and then, recursively applying $\tau_{(1/m)\mathbf{\Delta}}$ to ω_n the number m times, we again obtain (2.6).

Finally, since $\Delta_\ell = 0$ and $\Delta_{\ell-1}, \Delta_{\ell+1} \geqslant 0$, we have $[t_\ell, t_{\ell+1}] \subset [t''_\ell, t''_{\ell+1}] = [t'_\ell + \gamma, t'_{\ell+1} + \gamma]$. □

Proof of Theorem 2.1. Let $\omega_n \in \Omega_n$ be fixed but arbitrary and recall that $\widetilde{\omega}_n$ denotes an equally-spaced configuration. By Lemma 2.2, there is some $\ell \in \{0, \dots, n\}$ and some $\gamma \in [0, 2\pi)$ such that

$$P_{\widetilde{\omega}_n}(t - \gamma) \leqslant P_{\omega_n}(t), \quad t \in [t_\ell, t_{\ell+1}].$$

Hence,

$$\begin{aligned} P_{\widetilde{\omega}_n}(\pi/n) &= \min_{t\in[0,2\pi)} P_{\widetilde{\omega}_n}(t) \leqslant \min_{t\in[t_\ell,t_{\ell+1}]} P_{\widetilde{\omega}_n}(t - \gamma) \\ &\leqslant \min_{t\in[t_\ell,t_{\ell+1}]} P_{\omega_n}(t) \leqslant \max_j \min_{t\in[t_j,t_{j+1}]} P_{\omega_n}(t). \end{aligned}$$ □

2.3. *Derivatives of logarithmic potentials.* We next consider a class of kernels g derived from $g_{\log}$ that were considered in [7]. For an even positive integer m, we define the kernel

$$g_m(t) := g^{(m)}_{\log}(t) = \frac{d^m}{dt^m} g_{\log}(t).$$

Then, for $t \in [0, 2\pi)$,

$$g_2(t) = \frac{d}{dt}\left(-\frac{1}{2}\cot\left(\frac{t}{2}\right)\right) = \frac{1}{4}\csc^2\left(\frac{t}{2}\right)$$

and hence

$$g_m(t) = \frac{1}{4} f^{(m-2)}(t),$$

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where $f(t) := \csc^2(t/2)$. Following [7], we next verify that g_m satisfies the hypotheses of Theorem 2.1. It is well known and elementary to check that

$$\tan t = \sum_{j=1}^{\infty} a_j t^j, \quad t \in (-\pi/2, \pi/2),$$

with each $a_j \geqslant 0$, $j = 1, 2, \ldots$. Hence, if $h(t) := \tan(t/2)$, then

$$h^{(k)}(t) > 0, \quad t \in (0, \pi), k = 0, 1, \ldots.$$

Now observe that

$$f(t) = \csc^2\left(\frac{t}{2}\right) = \sec^2\left(\frac{\pi - t}{2}\right) = 2h'(\pi - t),$$

and hence,

$$(-1)^k f^{(k)}(t) = 2h^{(k+1)}(\pi - t) > 0, \quad t \in (0, \pi).$$

This implies that if m is even, then $g_m(t) = \frac{1}{4} f^{(m-2)}(t)$ is a positive, decreasing, strictly convex function on $(0, \pi)$. It is also clear that if m is even, then g_m is even, since f is even. Thus, $g = g_m$ satisfies the hypotheses of Theorem 2.1.

We note that, for an even positive integer m, an induction argument implies that

$$g_m(t) = p_m(r^{-2}), \qquad r = 2\sin(t/2),$$

for some polynomial p_m of degree $m/2$. The induction follows from the recursive relation

$$p_{m+2}(x) = (6x^2 - x)p'_m(x) + (4x^3 - x^2)p''_m(x), \tag{2.9}$$

which is easily derived using $(r')^2 = 1 - (r/2)^2$ and $r'' = -r/4$. Thus, g_m can be expressed as a linear combination of Riesz s-potentials with $s = 2, 4, \ldots, m$ with coefficients corresponding to the polynomial p_m. Table 1 displays p_m for $m = 2, 4, 6$, and 8.

For $\omega_n \in \Omega_n$, we let

$$Q_{\omega_n}(t) := \prod_{j=1}^{n} \sin\left|\frac{t - t_j}{2}\right|$$

and set

$$T_n(t) := Q_{\widetilde{\omega}_n}(t) = 2^{1-n} \sin\left|\frac{nt}{2}\right|.$$

Our next two results are consequences of Lemma 2.2 and Theorem 2.1, respectively.

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Table 1: The polynomials $p_m(x)$ and the values $-(\log|T_n|)^{(m)}(\pi/n)$ from (2.10) (see Theorem 2.4 and Corollary 3.3) for $m = 2, 4, 6, 8$, and for $n \in \mathbb{N}$.

m	$p_m(x)$	$-(\log\|T_n\|)^{(m)}(\pi/n)$
2	x	$n^2/4$
4	$6x^2 - x$	$n^4/8$
6	$120x^3 - 30x^2 + x$	$n^6/4$
8	$5040x^4 - 1680x^3 + 126x^2 - x$	$17n^8/16$

THEOREM 2.3. *Let m be a positive even integer and $\omega_n \in \Omega_n$. Then there is some $\gamma \in [0, 2\pi)$ and some $j \in \{1, \dots, n\}$ (with γ and j depending on ω_n) such that*

$$-(\log|Q_{\omega_n}|)^{(m)}(t) \geqslant -(\log|T_n|)^{(m)}(t-\gamma), \quad t \in (t_j, t_{j+1}).$$

Proof. This is an immediate consequence of Lemma 2.2 with $g = g_m$ and $\omega_n' = \widetilde{\omega}_n$, and so $P_{\omega_n}(t) = -(\log|Q_{\omega_n}|)^{(m)}(t)$ and $P_{\widetilde{\omega}_n}(t) = -(\log|T_n|)^{(m)}(t)$). □

Since g_m satisfies the hypotheses of Theorem 2.1, we obtain the following theorem.

THEOREM 2.4. *We have*

$$\min_{\omega_n \in \Omega_n} \left\{ \max_{1 \leqslant j \leqslant n} \left\{ \min_{t \in [t_j, t_{j+1}]} -(\log|Q_{\omega_n}|)^{(m)}(t) \right\} \right\} = -(\log|T_n|)^{(m)}(\pi/n),$$

for every even positive integer m.

From (2.9), one can show that the leading coefficient of p_m is $(m-1)!$. A somewhat more detailed computation using (2.5) and (2.9) yields

$$-(\log|T_n|)^{(m)}(\pi/n) = \frac{2}{(2\pi)^m}\zeta(m)(m-1)!(2^m - 1). \tag{2.10}$$

Table 1 gives the values $-(\log|T_n|)^{(m)}(\pi/n)$ for $m = 2, 4, 6, 8$, and for $n \in \mathbb{N}$.

§3. *Comments on polarization.* The main part of the following "polarization" theorem was proved in [**8**]. Here we extend this theorem using the observation from [**7**] that, for each $\omega_n \in \Omega_n$, we may restrict the set over which we search for a minimum to

$$E(\omega_n) := [0, 2\pi) \setminus \bigcup_{j=1}^{n} (t_j - \pi/n, t_j + \pi/n) \pmod{2\pi}.$$

THEOREM 3.1. *Let g be as in Theorem 2.1. Then*

$$\max_{\omega_n \in \Omega_n} \left\{ \min_{t \in [0, 2\pi)} P_{\omega_n}(t) \right\} = \max_{\omega_n \in \Omega_n} \left\{ \min_{t \in E(\omega_n)} P_{\omega_n}(t) \right\} = P_{\widetilde{\omega}_n}(\pi/n). \tag{3.1}$$

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Proof. Let $\omega_n \in \Omega_n$ be arbitrary. The proof follows from Lemma 2.2 and is similar to the proof of Theorem 2.1, except that the roles of $\widetilde{\omega}_n$ and ω_n are switched. By Lemma 2.2, there is some $\ell \in \{0, \dots, n\}$ and some $\gamma \in [0, 2\pi)$ such that

$$P_{\omega_n}(t-\gamma) \leqslant P_{\widetilde{\omega}_n}(t), \quad t \in [\widetilde{t}_\ell, \widetilde{t}_{\ell+1}],$$

and $[\widetilde{t}_\ell, \widetilde{t}_{\ell+1}] \subset [t_\ell + \gamma, t_{\ell+1} + \gamma]$. Then

$$t_\ell + \pi/n \leqslant \pi(2\ell-1)/n - \gamma \leqslant t_{\ell+1} - \pi/n,$$

and so $\pi(2\ell+1)/n - \gamma \in E(\omega_n)$. We then obtain

$$\begin{aligned}\min_{t\in[0,2\pi)} P_{\omega_n}(t) &\leqslant \min_{t\in E(\omega_n)} P_{\omega_n}(t) \leqslant P_{\omega_n}(\pi(2\ell+1)/n - \gamma) \\ &\leqslant P_{\widetilde{\omega}_n}(\pi(2\ell+1)/n) = P_{\widetilde{\omega}_n}(\pi/n),\end{aligned}$$

which completes the proof. □

THEOREM 3.2. *Let $\omega_n \in \Omega_n$. Then there is a number $\theta \in [0, 2\pi)$ (depending on ω_n) such that*

$$-(\log|Q_{\omega_n}|)^{(m)}(t) \leqslant -(\log|T_n|)^{(m)}(t-\theta), \quad t \in (\theta, \theta + 2\pi/n),$$

for every non-negative even integer m.

Proof. Let m be a non-negative even integer. We apply Lemma 2.2 with $g = g_m$, $\omega_n' = \omega_n$ and $\omega_n = \widetilde{\omega}_n$ (in which case, $P_{\omega_n}(t) = -(\log|Q_{\omega_n}|)^{(m)}(t)$ and $P_{\widetilde{\omega}_n}(t) = -(\log|T_n|)^{(m)}(t)$) to deduce that there is an $\ell \in \{1, \dots, n\}$ and a number $\gamma \in [0, 2\pi)$ (depending on ω_n) such that

$$-(\log|Q_{\omega_n}|)^{(m)}(t-\gamma) \leqslant -(\log|T_n|)^{(m)}(t), \quad t \in [\widetilde{t}_\ell, \widetilde{t}_{\ell+1}),$$

which can be rewritten using $\theta := \widetilde{t}_\ell - \gamma$, $u := t - \gamma$, and the fact that T_n is $2\pi/n$ periodic as

$$-(\log|Q_{\omega_n}|)^{(m)}(u) \leqslant -(\log|T_n|)^{(m)}(u-\theta), \quad u \in [\theta, \theta + 2\pi/n).$$ □

COROLLARY 3.3. *We have*

$$\begin{aligned}\max_{\omega_n\in\Omega_n}\left\{\min_{t\in[0,2\pi)} -(\log|Q_{\omega_n}|)^{(m)}(t)\right\} &= \max_{\omega_n\in\Omega_n}\left\{\min_{t\in E(\omega_n)} -(\log|Q_{\omega_n}|)^{(m)}(t)\right\} \\ &= -(\log|T_n|)^{(m)}(\pi/n)\end{aligned}$$

for every even integer m.

Proof. This is an immediate corollary of Theorem 3.1. □

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