*Ergod. Th. & Dynam. Sys.* (2019), **39**, 3207–3223 © *Cambridge University Press, 2018*
doi:10.1017/etds.2018.18

# Recurrence on affine Grassmannians

YVES BENOIST and CAROLINE BRUÈRE

*Université Paris-Sud, Orsay 91405, France*
*(e-mail: yves.benoist@math.u-psud.fr, caroline.arvis@math.u-psud.fr)*

(*Received* 31 *March* 2017 *and accepted in revised form* 4 *January* 2018)

*Abstract.* We study the action of the affine group $G$ of $\mathbb{R}^d$ on the space $X_{k,d}$ of $k$-dimensional affine subspaces. Given a compactly supported Zariski dense probability measure $\mu$ on $G$, we show that $X_{k,d}$ supports a $\mu$-stationary measure $\nu$ if and only if the $(k+1)$th Lyapunov exponent of $\mu$ is strictly negative. In particular, when $\mu$ is symmetric, $\nu$ exists if and only if $2k \geq d$.

## 1. *Introduction*

1.1. *Recurrence and Lyapunov exponents.* Consider a locally compact group $G$ acting continuously on a locally compact second-countable space $X$ and $\mu$ a probability measure on $G$. The *associated random walk on* $X$ is the Markov chain over $X$ defined by the transition probabilities $P_x = \mu * \delta_x$ for all $x \in X$. Our aim is to study the recurrence properties of such a random walk. We will not focus here on the *almost sure recurrence* as in [**5**] and [**9**] but on the *recurrence in law* as in [**3**, **6**, **10**].

*Definition 1.1.* The random walk on $X$ is *recurrent in law* at a point $x \in X$ if, for all $\varepsilon > 0$, there exist a compact set $C \subset X$ and $n_0 \in \mathbb{N}^*$ such that for all $n \geq n_0$,

$$\mu^{*n} * \delta_x(C) \geq 1 - \varepsilon.$$

The random walk on $X$ is *uniformly recurrent in law* if the same compact set $C$ can be chosen for all the starting points $x$. A probability measure $\nu$ on $X$ is said to be *$\mu$-stationary* or *$\mu$-invariant* if one has $\mu * \nu = \nu$.

Those definitions are tightly linked. Indeed, there exists a $\mu$-stationary probability measure on $X$ if and only if the random walk on $X$ is recurrent in law at some point $x \in X$ (see Lemma 3.3 for one implication).

In this paper, $G$ will always be a real algebraic group acting algebraically on a real algebraic variety $X$; the measure $\mu$ will be compactly supported and *Zariski dense*, which means that its support spans a Zariski dense subgroup in $G$.

When $G$ is a reductive group and $X = G/H$ is an algebraic homogeneous space, it is proven in [**3**] that there exists a $\mu$-stationary probability measure on $X$ if and only if $X$

https://doi.org/10.1017/etds.2018.18 Published online by Cambridge University Press

is compact. The aim of our article is to focus on situations where the algebraic group $G$ is not reductive. In particular, in Corollary 1.5, we will exhibit examples of non-compact homogeneous spaces on which there always exists a $\mu$-stationary probability measure.

The key tool in our analysis will be to link the recurrence properties of these random walks to the Lyapunov exponents of $\mu$. The definition of these Lyapunov exponents depends on the choice of a linear action of $G$ on $\mathbb{R}^d$.

*Definition 1.2.* Given a linear action of $G$ on $\mathbb{R}^d$, the *Lyapunov exponents* of $\mu$ are the real numbers $\lambda_1, \ldots, \lambda_d$ such that, for all $1 \leq p \leq d$, we have

$$\lambda_1 + \cdots + \lambda_p = \lim_{n\to\infty} \frac{1}{n} \int_G \log \|\Lambda^p g\| \, d\mu^{*n}(g). \tag{1.1}$$

The sequence of Lyapunov exponents is always decreasing: $\lambda_1 \geq \cdots \geq \lambda_d$ (see [**14**, Proposition 1.2]). More properties of these exponents are given in [**14**, **16**]; their use in the context of reductive groups is detailed in [**4**, **11**–**13**].

1.2. *Action on the affine Grassmannians.* We assume now that $G$ is either the affine group $G = \mathrm{GL}(d, \mathbb{R}) \ltimes \mathbb{R}^d$ or the special affine group $G = \mathrm{SL}(d, \mathbb{R}) \ltimes \mathbb{R}^d$. For $1 \leq p \leq d$, we denote by $\lambda_p$ the $p$th Lyapunov exponent corresponding to the linear action of $G$ on $\mathbb{R}^d$. For instance, in dimension $d = 1$, one has

$$\lambda_1 = \int_{\mathbb{R}^* \ltimes \mathbb{R}} \log |a| \, d\mu(a, u),$$

where $g = (a, u) \in \mathbb{R}^* \ltimes \mathbb{R}$. For any $d \geq 1$, Bougerol and Picard have shown in [**6**] that there exists a $\mu$-stationary probability measure on $\mathbb{R}^d$ if and only if the first Lyapunov exponent of $\mu$ is strictly negative: $\lambda_1 < 0$.

The main result of this paper is the following Theorem 1.3, which extends this equivalence to the affine Grassmannians $X_{k,d}$, where $0 \leq k < d$. By definition, the affine Grassmannian $X_{k,d}$ is the space of $k$-dimensional affine subspaces of $\mathbb{R}^d$. The group $G$ acts transitively on $X_{k,d}$.

THEOREM 1.3. *Let $G$ be the affine group or the special affine group of $\mathbb{R}^d$, let $\mu$ be a Zariski dense probability measure with compact support on $G$ and let $0 \leq k < d$.*

(a) *If $\lambda_{k+1} \geq 0$, then the random walk on $X_{k,d}$ is nowhere recurrent in law, there exists no $\mu$-stationary probability measure on $X_{k,d}$ and, for all $x$ in $X_{k,d}$, the sequence of means of transition probabilities weakly converges to 0:*

$$\frac{1}{n} \sum_{j=1}^{n} \mu^{*j} * \delta_x \xrightarrow[n\to\infty]{} 0.$$

(b) *If $\lambda_{k+1} < 0$, then the random walk on $X_{k,d}$ is uniformly recurrent in law, there exists a unique $\mu$-stationary probability measure $\nu$ on $X_{k,d}$ and, for all $x$ in $X_{k,d}$, the sequence of means of transition probabilities weakly converges to $\nu$:*

$$\frac{1}{n} \sum_{j=1}^{n} \mu^{*j} * \delta_x \xrightarrow[n\to\infty]{} \nu.$$

https://doi.org/10.1017/etds.2018.18 Published online by Cambridge University Press

The result of Bougerol and Picard in [**6**] covers the $k = 0$ case. In fact, their proof uses only the weaker assumption that $\mu$ has a finite first moment and that its support does not preserve any proper affine subspace of $\mathbb{R}^d$. Assuming compact support allows for the construction of a contracted function in §3.2 and the non-degeneracy hypothesis needed to apply the law of the iterated logarithm in §4.4.

The following corollary, which is particularly noteworthy insofar as it does not mention Lyapunov exponents, is deduced from Theorem 1.3.

COROLLARY 1.4. *Assume that $\mu$ is symmetric. Then there exists a $\mu$-stationary probability measure $\nu$ on $X_{k,d}$ if and only if $2k \geq d$. In this case, $\nu$ is unique.*

*Proof.* Since $\mu$ is symmetric, for all $1 \leq p \leq d$, the Lyapunov exponents satisfy the equalities $\lambda_p = -\lambda_{d+1-p}$. Moreover, since $\mu$ is Zariski dense in $G$, it follows from the Guivarc'h–Raugi simplicity theorem that the sequence of Lyapunov exponents is strictly decreasing: $\lambda_1 > \cdots > \lambda_d$ (see [**4**, Corollary 10.15]). Therefore, one has the equivalence $\lambda_{k+1} < 0 \Longleftrightarrow 2k \geq d$. □

COROLLARY 1.5. *Let $d \geq 2$. When $G$ is the special affine group and $k = d - 1$, there exists a unique $\mu$-stationary probability measure on $X_{k,d}$.*

*Proof.* In this case, the sum of the Lyapunov exponents is zero. Hence, the simplicity of the Lyapunov exponents implies that $\lambda_d < 0$. □

For instance, when $G$ is the special affine group of $\mathbb{R}^2$, the random walk on the space of affine lines of $\mathbb{R}^2$ is always uniformly recurrent in law, while the random walk on the space of points of $\mathbb{R}^2$ is nowhere recurrent in law.

1.3. *Action on $X_{V,W}$.* By embedding the affine Grassmannian $X_{k,d}$ of $\mathbb{R}^d$ in the projective space of a suitable exterior power $V$ of $\mathbb{R}^{d+1}$, we will deduce Theorem 1.3 from the following Theorem 1.6.

We first need two definitions. An algebraic group $G$ is *Zariski connected* if it is connected for the Zariski topology. A linear action of $G$ on a vector space $W$ is *proximal* if there exists a rank-1 linear endomorphism $\pi$ of $W$ which is a limit of a sequence $\lambda_n \gamma_n$ with $\lambda_n > 0$ and $\gamma_n$ in $\Gamma$.

THEOREM 1.6. *Let $V$ be a finite-dimensional real vector space, $G$ a Zariski connected algebraic subgroup of* $\mathrm{GL}(V)$ *and $W$ a $G$-invariant subspace of $V$ such that:*

(H1) *$G$ acts irreducibly and proximally on $W$ and on $W' := V/W$;*

(H2) *the representations of $G$ in $W$ and $W'$ are not equivalent;*

(H3) *$W$ has no $G$-invariant complementary subspace in $V$.*

*Let $X_{V,W} := \mathbb{P}(V) \smallsetminus \mathbb{P}(W)$, let $\mu$ be a Zariski dense probability measure with compact support on $G$ and let $\lambda_1 = \lambda_{1,W}$ and $\lambda'_1 = \lambda_{1,W'}$ be the first Lyapunov exponents of $\mu$ in $W$ and $W'$, respectively.*

(a) *If $\lambda_1 \geq \lambda'_1$, then the random walk on $X_{V,W}$ is nowhere recurrent in law, there exists no $\mu$-stationary probability measure on $X_{V,W}$ and, for all $x$ in $X_{V,W}$, one has the weak convergence $(1/n) \sum_{j=1}^{n} \mu^{*j} * \delta_x \xrightarrow[n\to\infty]{} 0$.*

https://doi.org/10.1017/etds.2018.18 Published online by Cambridge University Press

(b) *If $\lambda_1 < \lambda_1'$, then the random walk on $X_{V,W}$ is uniformly recurrent in law, there exists a unique $\mu$-stationary probability measure $\nu$ on $X_{V,W}$ and, for all $x$ in $X_{V,W}$, one has the weak convergence* $(1/n)\sum_{j=1}^{n} \mu^{*j} * \delta_x \xrightarrow[n\to\infty]{} \nu$.

1.4. *Strategy of the proof.* In §2, we explain how to embed the affine Grassmannian $X_{k,d}$ in the variety $\mathbb{P}(\Lambda^{k+1}\mathbb{R}^{d+1}) \smallsetminus \mathbb{P}(\Lambda^{k+1}\mathbb{R}^d)$ and we deduce Theorem 1.3 from Theorem 1.6.

The last three sections will deal with the proof of Theorem 1.6.

In §3, we prove the uniform recurrence in law when $\lambda_1 < \lambda_1'$ (Corollary 3.6). The crux of the proof is the construction of a proper function on $X_{V,W}$, which is contracted by the averaging operator (Proposition 3.5).

In §4, we prove the non-recurrence in law when $\lambda_1 \geq \lambda_1'$ (Proposition 4.4). The key point is the study of the ratio of the norms in $W$ and in $W'$ of a random product $b_1 \cdots b_n$. On the one hand, the existence of a $\mu$-stationary probability measure on $X_{V,W}$ would imply that these ratios are bounded (Lemma 4.5). On the other hand, when $\lambda_1 \geq \lambda_1'$, the law of large numbers and the law of the iterated logarithm for these products prevent these ratios from being bounded (Lemma 4.6).

In §5, we prove the uniqueness of the $\mu$-stationary measure on $X_{V,W}$ (Proposition 5.3). Indeed, using the joining measure (Corollary 4.2) of two distinct $\mu$-stationary probability measures on $X_{V,W}$, we construct (Lemma 5.1) a $\mu$-stationary measure $\overline{\nu}$ on the space $\mathbb{P}(W \oplus W') \smallsetminus (\mathbb{P}(W) \cup \mathbb{P}(W'))$. This contradicts the classification of stationary measures in [**3**] since this space does not contain compact $G$-orbits (Lemma 5.2). The weak convergence of the sequence of means of transition probabilities follows easily (Corollary 5.4).

In Appendix A, we collect known facts on random walks on reductive groups.

In this paper, all the vector spaces will be finite-dimensional real vector spaces, all the measures will be Borel measures and we will not distinguish between a real algebraic group and its group of real points.

## 2. *Recurrence on affine Grassmannians*

We explain first how to deduce Theorem 1.6 from Theorem 1.3.

We use the notation of Theorem 1.6. The group $G$ is the affine group or the special affine group of $\mathbb{R}^d$, the space $X_{k,d}$ is the affine Grassmannian of $\mathbb{R}^d$ and the probability measure $\mu$ on $G$ is Zariski dense and compactly supported.

Let us construct $G$-vector spaces $W \subset V$ to which we will apply Theorem 1.6. We identify the affine space $\mathbb{R}^d$ with the affine hyperplane of $\mathbb{R}^{d+1} = \mathbb{R}^d \oplus \mathbb{R}$:

$$\mathcal{A} = \{(w,\ 1) \mid w \in \mathbb{R}^d\}.$$

The group $G$ is then a subgroup of $\mathrm{GL}(d+1,\ \mathbb{R})$, which stabilizes $\mathcal{A}$, and we have

$$X_{k,d} = \mathrm{Gr}_{k+1}(d+1) \smallsetminus \mathrm{Gr}_{k+1}(d),$$

where $\mathrm{Gr}_{k+1}(d+1)$ and $\mathrm{Gr}_{k+1}(d)$ are the Grassmannians of $(k+1)$-dimensional vector subspaces of $\mathbb{R}^{d+1}$ and of $\mathbb{R}^d$, respectively. Now let

$$V := \Lambda^{k+1}\mathbb{R}^{d+1} \quad \text{and} \quad W := \Lambda^{k+1}\mathbb{R}^d.$$

https://doi.org/10.1017/etds.2018.18 Published online by Cambridge University Press

The group $G$ acts linearly on the vector space $V$ and leaves invariant its vector subspace $W$. The Plücker map

$$\varphi : \mathrm{Gr}_{k+1}(d+1) \longrightarrow \mathbb{P}(V); \quad U \longmapsto \Lambda^{k+1}U$$

is an embedding of the Grassmannian variety in the projective space of $V$. It induces a $G$-equivariant injection

$$\varphi : X_{k,\,d} \hookrightarrow X_{V,W} := \mathbb{P}(V) \smallsetminus \mathbb{P}(W).$$

PROPOSITION 2.1. *With the above notation:*

(a) *Hypotheses* (H1), (H2) *and* (H3) *hold for these* $V$, $W$ *and* $W' = V/W$;

(b) *the* $G$*-equivariant inclusion* $X_{k,\,d} \hookrightarrow X_{V,W}$ *has closed image;*

(c) *we have the equality* $\lambda_{k+1} = \lambda_{1,\,W} - \lambda_{1,\,W'}$.

*Proof of Proposition 2.1.* (a) (H1): The representation of $\mathrm{SL}(d, \mathbb{R})$ in $W = \Lambda^{k+1}\mathbb{R}^d$ is irreducible by [**7**, Ch. 8.13.1.4]. This representation is proximal since the image in $\mathrm{GL}(W)$ of a diagonal element of $G$ with positive distinct eigenvalues is a proximal element of $\mathrm{GL}(W)$. The same is true for the representation in $W' \simeq \Lambda^k\mathbb{R}^d$.

(H2): The fact that the representations of $\mathrm{SL}(d, \mathbb{R})$ in $W$ and $W'$ are not equivalent is also proven in [**7**, Ch. 8.13.1.4].

(H3): Since the representations of $\mathrm{SL}(d, \mathbb{R})$ in $W$ and $W'$ are irreducible and are not equivalent, by Schur's lemma, the only $\mathrm{SL}(d, \mathbb{R})$-invariant complementary subspace of $W$ in $V \simeq W \oplus W'$ is $W'$. But $W'$ is not invariant by the translations of $G$.

(b) The image $\varphi(X_{k,d})$ is closed in $X_{V,W}$ since $\varphi^{-1}(\mathbb{P}(W)) = \mathrm{Gr}_{k+1}(d)$.

(c) This equality is the difference of the equalities

$$\lambda_{1,\,W} = \lambda_1 + \cdots + \lambda_{k+1} \quad \text{and} \quad \lambda_{1,\,W'} = \lambda_1 + \cdots + \lambda_k,$$

which follow from the very definition (1.1) of the Lyapunov exponents. □

*Proof of Theorem 1.6 $\Longrightarrow$ Theorem 1.3.* We use Proposition 2.1.

If $\lambda_{k+1} \geq 0$, then we can apply Theorem 1.6 in the case where $\lambda_{1,\,W} \geq \lambda_{1,\,W'}$ and there can be no $\mu$-stationary probability measure on $X_{k,\,d}$.

Conversely, if $\lambda_{k+1} < 0$, we are in the case where $\lambda_{1,\,W} < \lambda_{1,\,W'}$. Since $X_{k,\,d}$ is a $G$-invariant closed subset of $X_{V,W}$, we obtain uniform recurrence in law on $X_{k,\,d}$. Lemma 3.3 then ensures the existence of a $\mu$-stationary probability measure on $X_{k,\,d}$, which is thus the unique $\mu$-stationary probability measure on $X_{V,W}$. □

## 3. *Uniform recurrence when* $\lambda_1 < \lambda_1'$

The goal of this section is to show that the random walk on $X_{V,W}$ is uniformly recurrent in law when $\lambda_1 < \lambda_1'$ (Corollary 3.6).

### 3.1. *The contraction hypothesis.*

We recall in this section the uniform contraction hypothesis and why this condition implies the uniform recurrence in law.

The setting is very general (see [**15**], [**10**] or [**2**] for more details). Let $X$ be a locally compact second-countable space and $P$ a Markov–Feller operator on $X$.

https://doi.org/10.1017/etds.2018.18 Published online by Cambridge University Press

*Definition 3.1.* The operator $P$ satisfies the *uniform contraction hypothesis (UCH)* if there exist a proper map $u : X \to [0, \infty[$ and two constants $0 < a < 1$ and $b > 0$ such that, over $X$,

$$Pu \leq au + b. \tag{3.1}$$

We recall that a map is *proper* if the inverse image of every compact set is relatively compact. The definition of recurrence in law extends to Markov chains on $X$. Uniform recurrence in law is fundamentally linked with (UCH), as follows.

PROPOSITION 3.2. *If $P$ satisfies* (UCH)*, then the associated Markov chain on $X$ is uniformly recurrent in law.*

*Proof.* See [**15**, Theorem 15.0.1], [**10**, Lemma 3.1] or [**2**, Lemma 2.1]. □

LEMMA 3.3. *If $P$ is recurrent in law at a point $x \in X$, then there exists a $P$-invariant probability measure on $X$.*

*Proof.* By the Banach–Alaoglu theorem, the sequence of means of transition probabilities $\nu_n = (1/n) \sum_{j=1}^{n} P_x^j$ has at least one accumulation point $\nu_\infty$ for the weak-$*$ topology. This finite measure $\nu_\infty$ is $P$-invariant. Since $P$ is recurrent in law at $x$, there is no escape of mass and $\nu_\infty$ is a probability measure. □

The following lemma is a useful tool to check (UCH).

LEMMA 3.4. *Let $n \geq 1$. If $P^n$ satisfies* (UCH)*, then $P$ satisfies* (UCH) *too.*

*Proof.* Let $u$ be the proper map and $a$,$b$ the constants such that $P^n u \leq au + b$ over $X$. Let $\alpha_k = a^{-k/n}$ for $0 \leq k \leq n-1$, $a' = a^{1/n}$, $b' = (a'/a)b$. Then the proper map $u' : X \to \mathbb{R}_+$ defined by $u' = \sum_{k=0}^{n-1} \alpha_k P^k u$ satisfies the inequality $Pu' \leq a'u' + b'$ on $X$ and thus $P$ satisfies (UCH). □

3.2. *Finding a contracted function.* In this section, we use again the notation and assumptions of Theorem 1.6. We will prove that the averaging operator satisfies the uniform contraction hypothesis.

We recall that $W \subset V$ are real vector spaces and $G$ is a Zariski connected algebraic subgroup of $\mathrm{GL}(V)$ preserving $W$ and satisfying (H1), (H2) and (H3). We identify the quotient $W' = V/W$ with a complementary subspace $W_s$ of $W$ in $V$. Note that this subspace $W_s$ is not $G$-invariant. We recall also that $\mu$ is a Zariski dense probability measure on $G$ with compact support and that $\lambda_1$ and $\lambda'_1$ are the first Lyapunov exponents of $\mu$ in $W$ and $W'$, and that we are studying the associated random walk on the $G$-space $X_{V,W} := \mathbb{P}(V) \smallsetminus \mathbb{P}(W)$.

The corresponding Markov operator $P_\mu : \mathcal{C}^0(X_{V,W}) \longrightarrow \mathcal{C}^0(X_{V,W})$ is given by

$$P_\mu f(x) = \int_G f(gx)\, d\mu(g).$$

PROPOSITION 3.5. *Same notation and assumptions as in Theorem 1.6.*
*If $\lambda_1 < \lambda'_1$, then the Markov operator $P_\mu$ satisfies* (UCH).

https://doi.org/10.1017/etds.2018.18 Published online by Cambridge University Press

*Proof.* The space $X_{V,W}$ can be seen as the set

$$X_{V,W} = \{[w,\ w'] \mid w \in W,\ w' \in W_s \smallsetminus \{0\}\}.$$

Choose a norm on $V$ and, for $\delta > 0$, consider the functions

$$u_\delta : X_{V,W} \longrightarrow \mathbb{R}_+; \quad [w,\ w'] \longmapsto \frac{\|w\|^\delta}{\|w'\|^\delta}.$$

These functions are proper and well defined. We want to find $\delta > 0$, $a \in ]0,\ 1[$, $b > 0$ and $n_0 \in \mathbb{N}^*$ such that, over $X_{V,W}$, one has the inequality

$$P_\mu^{n_0} u_\delta \leq a u_\delta + b. \tag{3.2}$$

Since $W$ is $G$-invariant, we can write $g \in G$ as

$$g = \begin{pmatrix} a_g & c_g \\ 0 & d_g \end{pmatrix} \text{ with } a_g \in \mathrm{GL}(W),\ d_g \in \mathrm{GL}(W_s),\ c_g \in \mathcal{L}(W_s,\ W). \tag{3.3}$$

Let $0 < \varepsilon < (\lambda_1' - \lambda_1)/8$. Then, by a lemma due to Furstenberg (cf. [**4**, Theorem 4.28], [**11**]), since $G$ acts irreducibly on $W$ and $W'$, there exists $n_0 \in \mathbb{N}^*$ such that for all $n \geq n_0$ and for all non-zero $w \in W$, $w' \in W_s$, the following inequalities hold:

$$\lambda_1 - \varepsilon \leq \frac{1}{n} \int_G \log \frac{\|a_g w\|}{\|w\|}\, d\mu^{*n}(g) \leq \lambda_1 + \varepsilon, \tag{3.4}$$

$$\lambda_1' - \varepsilon \leq \frac{1}{n} \int_G \log \frac{\|d_g w'\|}{\|w'\|}\, d\mu^{*n}(g) \leq \lambda_1' + \varepsilon. \tag{3.5}$$

For $\delta > 0$ and $x = [w,\ w'] \in X_{V,W}$, one computes

$$P_\mu^{n_0} u_\delta(x) = u_1(x)^\delta \int_G \frac{u_1(gx)^\delta}{u_1(x)^\delta}\, d\mu^{*n_0}(g).$$

We will give an upper bound for the right-hand integral for all $x$ in the complementary set of some compact $K$ in $X$; since the map $P_\mu^{n_0} u_\delta$ is bounded on the compact set $K$, this will give inequality (3.2). Let $c > 0$ be the constant defined by

$$c^{-1} = \frac{4}{n_0(\lambda_1' - \lambda_1)} \int_G \|c_g\|\ \|a_g^{-1}\|\, d\mu^{*n_0}(g).$$

Let $K$ be the compact subset of $X_{V,W}$ given by

$$K = \{[w,\ w'] \mid w \in W,\ w' \in W_s,\ \|w'\| \geq c\,\|w\|\}.$$

For $\mu^{*n_0}$-almost every $g \in G$, for all $x \in X \smallsetminus K$, the following ratio is bounded:

$$\frac{u_1(gx)}{u_1(x)} = \frac{\|a_g w + c_g w'\|}{\|w\|}\, \frac{\|w'\|}{\|d_g w'\|} \leq \sup_{g \in \mathrm{Supp}\,\mu^{*n_0}} \|d_g^{-1}\|(\|a_g\| + c\|c_g\|).$$

Therefore, we can find some constant $M_{n_0} > 0$ such that for all $\delta > 0$, for all $x \in X \smallsetminus K$ and for $\mu^{*n_0}$-almost every $g \in G$, we can write

$$\frac{u_1(gx)^\delta}{u_1(x)^\delta} = e^{\delta \log (u_1(gx))/(u_1(x))} \leq 1 + \delta \log \frac{u_1(gx)}{u_1(x)} + \delta^2 M_{n_0}.$$

https://doi.org/10.1017/etds.2018.18 Published online by Cambridge University Press

For all $x \in X \smallsetminus K$, for $\mu^{*n_0}$-almost every $g \in G$, the following upper bound holds:

$$\begin{aligned}\log \frac{u_1(gx)}{u_1(x)} &= \log \frac{\|a_g w\|}{\|w\|} - \log \frac{\|d_g w'\|}{\|w'\|} + \log \frac{\|a_g w + c_g w'\|}{\|a_g w\|} \\ &\le \log \frac{\|a_g w\|}{\|w\|} - \log \frac{\|d_g w'\|}{\|w'\|} + \frac{\|c_g w'\|}{\|a_g w\|} \\ &\le \log \frac{\|a_g w\|}{\|w\|} - \log \frac{\|d_g w'\|}{\|w'\|} + \|c_g\|\|a_g^{-1}\| c.\end{aligned}$$

Using inequalities (3.4), (3.5) and the definition of $c$, we get the inequality

$$\int_G \log \frac{u_1(gx)}{u_1(x)}\, d\mu^{*n_0}(g) \le n_0(\lambda_1 - \lambda_1' + 2\varepsilon) + \frac{n_0(\lambda_1' - \lambda_1)}{4} \le \frac{n_0(\lambda_1 - \lambda_1')}{2}.$$

Let $\kappa = (n_0(\lambda_1' - \lambda_1))/2 > 0$. We then get the upper bound, for all $x \in X \smallsetminus K$,

$$\int_G \frac{u_1(gx)^\delta}{u_1(x)^\delta}\, d\mu^{*n_0}(g) \le 1 - \delta\kappa + \delta^2 M_{n_0}.$$

Choose $\delta > 0$ such that $a_{n_0,\,\delta} := 1 - \delta\kappa + \delta^2 M_{n_0}$ is strictly between 0 and 1. Therefore, since $K$ is compact, there exists a constant $b_{n_0,\,\delta}$ such that for all $x \in X$,

$$P_\mu^{n_0} u_\delta(x) \le a_{n_0,\,\delta}\, u_\delta(x) + b_{n_0,\,\delta}$$

and, by Lemma 3.4, the operator $P_\mu$ satisfies (UCH). □

COROLLARY 3.6. *Same notation and assumptions as in Theorem 1.6.*
*If $\lambda_1 < \lambda_1'$, then the random walk on $X$ is uniformly recurrent in law.*

*Proof.* This is a direct consequence of Proposition 3.5: since $P_\mu$ satisfies (UCH), we only need to apply Proposition 3.2. □

## 4. *Non-recurrence in law when $\lambda_1 \ge \lambda_1'$*

The goal of this section is to show that the random walk on $X_{V,W}$ is nowhere recurrent in law when $\lambda_1 \ge \lambda_1'$ (Proposition 4.4).

### 4.1. *The limit measures.*

We recall in this section the definition and the properties of the limit probability measures associated to a stationary measure.

The setting is very general. Let $G$ be a locally compact group acting on a second-countable locally compact space $X$ and $\mu$ be a probability measure on $G$. Let $B$ be the product space $B = G^{\mathbb{N}^*}$ and $\beta$ be the product measure $\beta = \mu^{\otimes \mathbb{N}^*}$. The following lemma is due to Furstenberg. See [**1**, Lemma 3.2] or [**4**, Lemma 2.17].

LEMMA 4.1. *Let $\nu$ be a $\mu$-stationary probability measure on $X$. For $\beta$-almost every $b \in B$, the sequence $(b_1 \cdots b_n)_*\nu$ of probability measures on $X$ has a limit $\nu_b$, which we will call the* limit probability. *Moreover, we have $\nu = \int_B \nu_b\, d\beta(b)$.*

The following construction will be useful in §5. See [**1**, Corollary 3.5] for a proof.

https://doi.org/10.1017/etds.2018.18 Published online by Cambridge University Press

COROLLARY 4.2. *Let $\nu_1$ and $\nu_2$ be two $\mu$-stationary probability measures on $X$. Then the probability measure on $X \times X$,*

$$\nu_1 \boxtimes \nu_2 := \int_B \nu_{1,\,b} \otimes \nu_{2,\,b}\, d\beta(b), \tag{4.1}$$

*is $\mu$-stationary. It is called the* joining measure *of $\nu_1$ and $\nu_2$.*

This corollary will be used in combination with the following basic lemma.

LEMMA 4.3. *Let $m_1$ and $m_2$ be probability measures on a topological space $X$ and let $\Delta_X := \{(x, x) \mid x \in X\}$ be the diagonal of $X$.*

*If $m_1 \otimes m_2(\Delta_X) = 1$, then $m_1$ and $m_2$ are identical Dirac measures.*

*Proof.* By assumption, we have $m_1 \otimes m_2(\Delta_X) = \int_X m_1(\{x\})\, dm_2(x) = 1$. Hence, for $m_2$-almost every $x \in X$, we have $m_1(\{x\}) = 1$, which implies that the measures $m_1$ and $m_2$ are identical Dirac measures. □

4.2. *No stationary measures on $X_{V,W}$.* In this section, we again use the same notation and assumptions as in Theorem 1.6. We will prove that the space $X_{V,W}$ supports no $\mu$-stationary measures.

Recall that $W \subset V$ are real vector spaces and $G$ is a Zariski connected algebraic subgroup of $\mathrm{GL}(V)$ preserving $W$ and satisfying (H1), (H2) and (H3). Also recall that $\mu$ is a Zariski dense probability measure on $G$ with compact support, that $\lambda_1$ and $\lambda'_1$ are the first Lyapunov exponents of $\mu$ in $W$ and in $W' := V/W$ and that we are studying the associated random walk on the $G$-space $X_{V,W} := \mathbb{P}(V) \smallsetminus \mathbb{P}(W)$.

PROPOSITION 4.4. *Same notation and assumptions as in Theorem 1.6.*

*If $\lambda_1 \geq \lambda'_1$, then the random walk on $X_{V,W}$ is nowhere recurrent in law and there exists no $\mu$-stationary probability measure on $X_{V,W}$.*

*Proof.* By Lemma 3.3, the first assertion follows from the second one. This second assertion is a consequence of the following Lemmas 4.5 and 4.6. □

Let $B = G^{\mathbb{N}^*}$ and $\beta = \mu^{\otimes \mathbb{N}^*}$. For $b = (b_1, b_2, \ldots)$ in $B$, we write as in (3.3),

$$b_1 \cdots b_n = \begin{pmatrix} a_n & c_n \\ 0 & d_n \end{pmatrix}. \tag{4.2}$$

LEMMA 4.5. *Same notation and assumptions as in Theorem 1.6. If there exists a $\mu$-stationary probability measure on $X_{V,W}$, then, for $\beta$-almost every $b \in B$, we have*

$$\sup_{n \geq 1} \|a_n\|/\|d_n\| < \infty. \tag{4.3}$$

The proof of Lemma 4.5 will be given in §4.3. It relies on the properties of the limit probability measures $\nu_b$.

LEMMA 4.6. *Same notation and assumptions as in Theorem 1.6. If $\lambda_1 \geq \lambda'_1$, then, for $\beta$-almost every $b \in B$, one has*

$$\sup_{n \geq 1} \|a_n\|/\|d_n\| = \infty. \tag{4.4}$$

The proof of Lemma 4.6 will be given in §4.4. It relies on the law of large numbers and on the law of the iterated logarithm for the random variables $\log \|a_n\| - \log \|d_n\|$.

https://doi.org/10.1017/etds.2018.18 Published online by Cambridge University Press

4.3. *Using the limit measures.* The aim of this section is to prove Lemma 4.5.

We will need the following analog of [**4**, Proposition 4.6], originally found in [**11**], for a non-irreducible action.

LEMMA 4.7. *Same notation and assumptions as in Theorem 1.6. Let $\nu$ be a $\mu$-stationary probability measure on $\mathbb{P}(V)$ such that $\nu(\mathbb{P}(W)) = 0$. Then, for every proper subspace $U$ of $V$, we have $\nu(\mathbb{P}(U)) = 0$.*

*Proof.* Assume that there exists a proper subspace $U$ of $V$ such that $\nu(\mathbb{P}(U)) > 0$. Let $r_0$ be the minimal dimension of such a subspace $U$. If $U_1$ and $U_2$ are two distinct vector subspaces of dimension $r_0$, one has the equality

$$\nu(\mathbb{P}(U_1) \cup \mathbb{P}(U_2)) = \nu(\mathbb{P}(U_1)) + \nu(\mathbb{P}(U_2)).$$

Let $\alpha := \sup\{\nu(\mathbb{P}(U)) \mid U \subset V, \dim U = r_0\} > 0$ and consider the set

$$F = \{U \subset V \mid \nu(\mathbb{P}(U)) = \alpha,\ \dim U = r_0\}.$$

This set is finite and non-empty. By $\mu$-stationarity of $\nu$, for $\mu$-almost every $g \in G$, we have $g^{-1}F = F$. Therefore, since $\mu$ is Zariski dense in $G$, this set $F$ is $G$-invariant. Since $G$ is Zariski connected, all the subspaces $U$ belonging to $F$ are $G$-invariant. But by (H1), (H2) and (H3), the only proper $G$-invariant subspace of $V$ is $W$. This is contradictory since, by assumption, we have $\nu(\mathbb{P}(W)) = 0$. □

*Proof of Lemma 4.5.* We assume also that there exists a $\mu$-stationary probability measure $\nu$ on $X_{V,W}$. In order to prove (4.3), it is enough to check that for $\beta$-almost every $b \in B$, for all accumulation points $\pi$ in $\mathrm{End}(V)$ of the sequence $p_n := (b_1 \cdots b_n)/(\|b_1 \cdots b_n\|)$, the image of $\pi$ is not included in $W$:

$$\mathrm{Im}\,\pi \not\subset W. \tag{4.5}$$

Lemma 4.7 shows that $\nu(\mathbb{P}(\mathrm{Ker}\,\pi)) = 0$; hence, the image probability measure $\pi_*\nu$ is well defined and the sequence $p_{n*}\nu$ weakly converges to $\pi_*\nu$. By Lemma 4.1, this sequence $p_{n*}\nu$ also weakly converges to $\nu_b$ and therefore we have

$$\pi_*\nu = \nu_b.$$

Therefore, for $\beta$-almost all $b$ in $B$, one has, for all accumulation points $\pi$,

$$\nu_b(\mathbb{P}(\mathrm{Im}\,\pi)) = 1.$$

Since $\nu(\mathbb{P}(W)) = 0$, one also has, for $\beta$-almost all $b$ in $B$,

$$\nu_b(\mathbb{P}(W)) = 0$$

and hence the images $\mathrm{Im}\,\pi$ are not contained in $W$. This proves (4.5). □

https://doi.org/10.1017/etds.2018.18 Published online by Cambridge University Press

4.4. *Using the Cartan projection.* The aim of this section is to prove Lemma 4.6.

Let $\rho$ be the natural projection

$$\rho : G \longrightarrow \mathrm{GL}(W) \times \mathrm{GL}(W'); \quad \begin{pmatrix} a & c \\ 0 & d \end{pmatrix} \longmapsto (a, d). \tag{4.6}$$

The image group $\overline{G} := \rho(G)$ is a reductive subgroup of $\mathrm{GL}(W) \times \mathrm{GL}(W')$. The image measure $\overline{\mu} := \rho_* \mu$ is a Zariski dense probability measure on $\overline{G}$.

The proof of Lemma 4.6 will use the notation of Appendix A with the reductive group $\overline{G}$ and its probability measure $\overline{\mu}$. In particular, $\mathfrak{g}$ is the Lie algebra of $\overline{G}$, $\mathfrak{a}$ is the Lie algebra of a maximal split torus of $\overline{G}$, $\kappa$ is the Cartan projection, $\sigma_{\overline{\mu}}$ is the Lyapunov vector, $\Phi_{\overline{\mu}}$ is the covariance 2-tensor, $\mathfrak{a}_{\overline{\mu}}$ is its linear span and $K_{\overline{\mu}}$ is the unit ball of $\mathfrak{a}_{\overline{\mu}}$.

We will also use the following two lemmas. We set $r = \dim W$ and $r' = \dim W'$.

LEMMA 4.8. *The highest weights $\chi$ and $\chi'$ of the representations of $\overline{G}$ in $W$ and $W'$ are distinct.*

*Proof.* Since $\overline{G}$ is Zariski connected, Condition (H1) tells us that $W$ and $W'$ are irreducible representations of $\mathfrak{g}$ and that their highest-weight spaces are one dimensional. Condition (H2) tells us that these representations of $\mathfrak{g}$ are not equivalent. Therefore, as in [**7**, Ch. 8.6.3], the highest weights $\chi$ and $\chi'$ must be distinct. □

LEMMA 4.9. *The center $Z$ of $\overline{G}$ is equal to $Z = \{(\alpha I_r, \beta I_{r'}) \in \overline{G} \mid \alpha,\ \beta \in \mathbb{R}^*\}$.*

*Proof.* By Schur's lemma, the commutant of $\overline{G}$ in $\mathrm{End}(W)$ is a division algebra. Since the representation of $\overline{G}$ in $W$ is proximal, this commutant is the field $\mathbb{R}$ of scalar matrices. Therefore, $Z$ acts on $W$ (and also on $W'$) by scalar matrices. □

*Proof of Lemma 4.6.* Fix norms on $W$ and $W'$ as in Lemma A.1, so that, for any element $g = (a, d)$ in $\overline{G}$ with $a \in \mathrm{GL}(W)$, $d \in \mathrm{GL}(W')$, one has

$$\log \|a\| = \chi(\kappa(g)) \quad \text{and} \quad \log \|d\| = \chi'(\kappa(g)). \tag{4.7}$$

In particular, the first Lyapunov exponents in $W$ and $W'$ are given by

$$\lambda_1 = \chi(\sigma_{\overline{\mu}}) \quad \text{and} \quad \lambda_1' = \chi'(\sigma_{\overline{\mu}}).$$

Let $\overline{B} = \overline{G}^{\mathbb{N}^*}$ and $\overline{\beta} = \overline{\mu}^{\otimes \mathbb{N}^*}$. For $b = (b_1, b_2, \ldots) \in \overline{B}$, we write $b_1 \cdots b_n = (a_n,\ d_n)$. We distinguish three cases.

*First case:* $\lambda_1 > \lambda_1'$. In this case one has $(\chi - \chi')(\sigma_{\overline{\mu}}) > 0$. According to (4.7) and the law of large numbers A.4, for $\overline{\beta}$-almost every $b \in \overline{B}$, we have

$$\lim_{n\to\infty} \log(\|a_n\|/\|d_n\|) = \lim_{n\to\infty} (\chi - \chi')(\kappa(b_1 \cdots b_n)) = \infty.$$

*Second case:* $\lambda_1 = \lambda_1'$ and $(\chi - \chi')(\mathfrak{a}_{\overline{\mu}}) \neq 0$. In this case, one has $(\chi - \chi')(\sigma_{\overline{\mu}}) = 0$ and there exists $x$ in the unit ball $K_{\overline{\mu}}$ of $\mathfrak{a}_{\overline{\mu}}$ such that $(\chi - \chi')(x) > 0$. According to the law of the iterated logarithm A.4, for $\overline{\beta}$-almost every $b \in \overline{B}$, there exists an increasing sequence of integers $n_i$ such that

$$\lim_{i\to\infty} \frac{\kappa(b_1 \cdots b_{n_i}) - n_i\, \sigma_{\widetilde{\mu}}}{\sqrt{2n_i \log\log n_i}} = x$$

https://doi.org/10.1017/etds.2018.18 Published online by Cambridge University Press

and therefore such that

$$\lim_{i\to\infty} \log(\|a_{n_i}\|/\|d_{n_i}\|) = \lim_{i\to\infty} (\chi - \chi')(\kappa(b_1 \cdots b_{n_i})) = \infty.$$

*Third case:* $\lambda_1 = \lambda_1'$ and $(\chi - \chi')(\mathfrak{a}_{\overline{\mu}}) = 0$. Let

$$S := \{(a, d) \in \overline{G} \mid |\det a| = |\det d| = 1\}.$$

Since the group $\overline{G}$ is reductive, by Lemma 4.9, the subgroup $S$ is semisimple. Let $\mathfrak{s}$ be the Lie algebra of $S$. By [**4**, Theorem 13.19], we have $\mathfrak{a} \cap \mathfrak{s} \subset \mathfrak{a}_{\overline{\mu}}$ and thus also

$$(\chi - \chi')(\mathfrak{a} \cap \mathfrak{s}) = 0. \tag{4.8}$$

We introduce the group morphism $\delta$ defined by

$$\delta : \overline{G} \longrightarrow \mathbb{R}; \quad (a, d) \longmapsto \frac{1}{r} \log|\det a| - \frac{1}{r'} \log|\det d|.$$

For every $g = (a, d)$ in $\overline{G}$, we can write $g = sz$ with $s \in S$ and $z \in Z$. Using Equations (4.7), (4.8) and the equality $\kappa(g) = \kappa(s) + \kappa(z)$, we compute

$$\log(\|a\|/\|d\|) = (\chi - \chi')(\kappa(g)) = (\chi - \chi')(\kappa(z)) = \delta(z) = \delta(g). \tag{4.9}$$

We want to describe the behavior of the random variable $T_n = \log(\|a_n\|/\|d_n\|)$ on $\overline{B}$, where as above $(a_n,\ d_n) = b_1 \cdots b_n$. Using Equation (4.9), we see that

$$T_n = \delta(b_1 \cdots b_n) = \delta(b_1) + \cdots + \delta(b_n)$$

is the sum of $n$ real-valued independent and identically distributed random variables $\delta(b_i)$. Note that the law of the variable $\delta(b_1)$ has compact support. Since $\lambda_1 = \lambda_1'$, we have $\mathbb{E}(\delta(b_1)) = (1/n)\mathbb{E}(T_n) \xrightarrow[n\to\infty]{} 0$. Thus, the variable $\delta(b_1)$ is centered.

If this random variable $\delta(b_1)$ were almost surely 0, it would mean that for $\overline{\mu}$-almost every $g = (a,\ d) \in \overline{G}$, we have $\delta(g) = 0$. Since $\overline{\mu}$ is Zariski dense in $\overline{G}$, this would imply that $\delta(G) = 0$ or, equivalently,

$$(\chi - \chi')(\mathfrak{z}) = 0, \tag{4.10}$$

where $\mathfrak{z} \subset \mathfrak{a}$ is the Lie algebra of $Z$. Equalities (4.8) and (4.10) would tell us that the highest weights $\chi$ and $\chi'$ were equal. This would contradict Lemma 4.8.

Therefore, this centered variable $\delta(b_1)$ is not almost surely 0. Thus, the classical recurrence properties of real random walks (cf., e.g., [**8**, Theorem 3.38]) tell us that $\sup_{n\geq 1} T_n = \infty$ almost surely.

In each of these three cases, we have checked (4.4). □

## 5. *Uniqueness of the stationary measure*

The main aim of this section is to prove the uniqueness of the stationary measure on $X_{V,W}$ (Proposition 5.3).

https://doi.org/10.1017/etds.2018.18 Published online by Cambridge University Press

5.1. *No stationary measures on $Y_{V,W}$.* The proof of uniqueness will rely on the following Lemma 5.1.

We keep the notation and assumptions of Theorem 1.6. Let $p$ be the projection

$$p : X_{V,W} \longrightarrow \mathbb{P}(W'); \quad [v] \longmapsto [v + W]$$

and let $Y_{V,W}$ be the $G$-invariant subvariety of $X^2_{V,W}$,

$$Y_{V,W} := \{(x,\ x') \in X^2_{V,W} \mid p(x) = p(x'),\ x \neq x'\}.$$

LEMMA 5.1. *Same notation and assumptions as in Theorem 1.6.*
*There is no $\mu$-stationary probability measure $\widetilde{\nu}$ on $Y_{V,W}$.*

*Proof.* Suppose that such a measure $\widetilde{\nu}$ does exist. Consider again the natural projection $\rho : G \longrightarrow \mathrm{GL}(W) \times \mathrm{GL}(W')$ introduced in (4.6). Let $\overline{G} := \rho(G)$ be the image of $G$ by $\rho$, a reductive subgroup of $\mathrm{GL}(W) \times \mathrm{GL}(W')$, and let $\overline{\mu} := \rho_* \mu$ be the image of $\mu$ by $\rho$, a Zariski dense probability measure on $\overline{G}$. Now consider the map

$$f : Y_{V,W} \longrightarrow \overline{Y}; \quad ([w_1,\ w'], [w_2,\ w']) \longmapsto [w_1 - w_2,\ w'],$$

where $\overline{Y} := \mathbb{P}(W \oplus W') \smallsetminus (\mathbb{P}(W) \cup \mathbb{P}(W'))$. Let $\overline{\nu} = f_* \widetilde{\nu}$ be the probability measure on $\overline{Y}$ that is the image of $\widetilde{\nu}$ by $\rho$. Since the map $f$ is equivariant, the probability measure $\overline{\nu}$ is $\overline{\mu}$-stationary. According to Proposition A.6, such a measure $\overline{\nu}$ is supported by a compact $\overline{G}$-orbit in $\overline{Y}$. This contradicts the following Lemma 5.2. □

LEMMA 5.2. *There are no compact $\overline{G}$-orbits in $\overline{Y}$.*

*Proof.* Such a compact orbit would be of the form $\overline{G}/\overline{H}$, where $\overline{H}$ is an algebraic subgroup of $\overline{G}$ containing a conjugate of the group $\overline{A}\,\overline{N}$ with $\overline{A}$ a maximal split subtorus of $\overline{G}$ and $\overline{N}$ a maximal unipotent subgroup normalized by $\overline{A}$. Since $W$ and $W'$ are proximal irreducible representations of $\overline{G}$, there is only one $\overline{N}$-invariant line $\mathbb{R}v$ in $W$ and one $\mathbb{R}v'$ in $W'$. Hence, the $\overline{N}$-invariant lines in $W \oplus W'$ are included in the plane $\mathbb{R}v \oplus \mathbb{R}v'$. Since, by Lemma 4.8, the highest weights $\chi$ and $\chi'$ of $W$ and $W'$ are distinct, the lines $\mathbb{R}v$ and $\mathbb{R}v'$ are the only $\overline{A}$-invariant lines in $\mathbb{R}v \oplus \mathbb{R}v'$. Therefore, a compact $\overline{G}$-orbit in $\mathbb{P}(W \oplus W')$ is contained in $\mathbb{P}(W) \cup \mathbb{P}(W')$. □

5.2. *Proof of uniqueness.* We can now show the uniqueness of the $\mu$-stationary probability measure $\nu$ on $X_{V,W}$. The same proof will tell us that its limit probability measures $\nu_b$ are Dirac measures.

PROPOSITION 5.3. *Same notation and assumptions as in Theorem 1.6. If $\lambda_1 < \lambda'_1$, the $\mu$-stationary probability measure $\nu$ on $X_{V,W}$ is unique. Moreover, the limit measures $\nu_b$ are $\beta$-almost surely Dirac measures.*

*Proof of Proposition 5.3.* Let $\nu_1$ and $\nu_2$ be two $\mu$-stationary probability measures on $X_{V,W}$. By Corollary 4.2, the joining measure $\nu_1 \boxtimes \nu_2$ on $X^2_{V,W}$ is $\mu$-stationary.

Let us show that its support is contained in the subvariety

$$Z_{V,W} := \{(x,\ x') \in X^2_{V,W} \mid p(x) = p(x')\},$$

https://doi.org/10.1017/etds.2018.18 Published online by Cambridge University Press

where $p : X_{V,W} \to \mathbb{P}(W')$ is again the canonical projection. Since the action of $G$ on $W'$ is irreducible and proximal, there exists a unique $\mu$-stationary measure $\nu'_0$ on $\mathbb{P}(W')$ called the *Furstenberg measure*. Its limit probability measures $\nu'_{0,b}$ are $\beta$-almost surely Dirac measures $\delta_{\xi_b}$ for some $\xi_b \in \mathbb{P}(W')$. See [**4**, Proposition 3.7] for more detail on the Furstenberg measure. Since $\nu'_0$ is unique, we have the equalities

$$p_*\nu_1 = p_*\nu_2 = \nu'_0.$$

Therefore, for $\beta$-almost every $b \in B$, we have

$$p_*\nu_{1,b} = p_*\nu_{2,b} = \delta_{\xi_b}$$

and hence

$$\nu_{1,b} \otimes \nu_{2,b}(Z_{V,W}) = 1.$$

By the very definition (4.1) of the joining measure, integrating this equality gives

$$\nu_1 \boxtimes \nu_2(Z_{V,W}) = 1.$$

By definition, this set $Z_{V,W}$ is the union $Z_{V,W} = Y_{V,W} \cup \Delta_{X_{V,W}}$. By Lemma 5.1, the $G$-variety $Y_{V,W}$ does not support $\mu$-stationary measures. Therefore, the joining measure $\nu_1 \boxtimes \nu_2$ is supported on the diagonal $\Delta_{X_{V,W}}$. Hence, for $\beta$-almost every $b$ in $B$, the measure $\nu_{1,b} \otimes \nu_{2,b}$ is also supported on the diagonal: $\nu_{1,b} \otimes \nu_{2,b}(\Delta_{X_{V,W}}) = 1$. Therefore, by Lemma 4.3, the limit probability measures $\nu_{1,b}$ and $\nu_{2,b}$ are both equal to the same Dirac measures. Hence, by Lemma 4.1, one has $\nu_1 = \nu_2$. □

5.3. *Limit of means of transition probabilities.* In this section, we prove that the sequence of means of the transition probabilities $\mu^{*n} * \delta_x$ on $X_{V,W}$ always has a limit.

COROLLARY 5.4. *Same notation and assumptions as in Theorem 1.6. Let $x \in X_{V,W}$.*

(a) *When $\lambda_1 \geq \lambda'_1$, one has the weak convergence $(1/n)\sum_{j=1}^n \mu^{*j} * \delta_x \xrightarrow[n\to\infty]{} 0$.*

(b) *When $\lambda_1 < \lambda'_1$, one has the weak convergence $(1/n)\sum_{j=1}^n \mu^{*j} * \delta_x \xrightarrow[n\to\infty]{} \nu$.*

*Proof of Corollary 5.4.* Every accumulation point of the sequence of probability measures $(1/n)\sum_{j=1}^n \mu^{*j} * \delta_x$ is a $\mu$-stationary finite measure.

When $\lambda_1 \geq \lambda'_1$, by Proposition 4.4, such a measure is necessarily 0.

When $\lambda_1 < \lambda'_1$, by Corollary 3.6, the corresponding Markov chain is recurrent in law; hence, no mass is lost and the accumulation points are thus $\mu$-stationary probability measures. By Proposition 5.3, there is only one such measure. □

## A. *Appendix. Limit laws on reductive groups*

In this appendix, we recall some facts about random walks on reductive groups, which are mainly detailed in [**4**].

A.1. *Cartan decomposition.* Let $G$ be a Zariski connected real algebraic reductive group. Let $A$ be a maximal split subtorus of $G$, $\mathfrak{a}$ be the Lie algebra of $A$, $\mathfrak{a}_+ \subset \mathfrak{a}$ be a Weyl chamber and $A_+ = \exp \mathfrak{a}_+$. There exists a maximal compact subgroup $K$ of $G$

https://doi.org/10.1017/etds.2018.18 Published online by Cambridge University Press

such that $G$ has a *Cartan decomposition* $G = KA_+K$. The *Cartan projection* of $G$ is the unique map $\kappa : G \to \mathfrak{a}_+$ such that, for all $g \in G$,

$$g \in K \exp(\kappa(g)) K.$$

The Cartan projection is useful because of the following lemma.

LEMMA A.1. [**4**, Lemma 6.33] *Let $G$ be a Zariski connected real algebraic reductive group, $\rho$ be an irreducible algebraic representation of $G$ in a real vector space $W$ and $\chi \in \mathfrak{a}^*$ be the* highest weight of $W$. *There exists a norm on $W$ such that, for all $g \in G$,*

$$\chi(\kappa(g)) = \log(\|\rho(g)\|). \tag{A.1}$$

A.2. *Limit laws.* Let $\mu$ be a Zariski dense probability measure with compact support on $G$. Let $B = G^{\mathbb{N}^*}$ and $\beta = \mu^{\otimes \mathbb{N}^*}$. We now recall two limit laws for the Cartan projection.

Define the *Lyapunov vector* (see [**4**, Theorem 10.9]) $\sigma_\mu \in \mathfrak{a}_+$ as the limit

$$\sigma_\mu := \lim_{n\to\infty} \frac{1}{n} \int_G \kappa(g)\, d\mu^{*n}(g). \tag{A.2}$$

THEOREM A.2. (Law of large numbers [**4**, Theorem 10.9]) *Let $G$ be a Zariski connected real reductive group and $\mu$ a Zariski dense probability measure with compact support on $G$. For $\beta$-almost every $b \in B$, we have the convergence*

$$\frac{1}{n} \kappa(b_n \cdots b_1) \xrightarrow[n\to\infty]{} \sigma_\mu.$$

Define the *covariance 2-tensor* (see [**4**, Proposition 14.18]) $\Phi_\mu \in S^2\mathfrak{a}$ as the limit

$$\Phi_\mu := \lim_{n\to\infty} \frac{1}{n} \int_G (\kappa(g) - n\sigma_\mu)^{\otimes 2}\, d\mu^{*n}(g). \tag{A.3}$$

We have that $\Phi_\mu$ is a symmetric 2-tensor on $\mathfrak{a}$. We denote by $\mathfrak{a}_\mu \subset \mathfrak{a}$ the *linear span of* $\Phi_\mu$, which is the smallest subspace $\mathfrak{a}_\mu$ of $\mathfrak{a}$ such that we have $\Phi_\mu \in S^2\mathfrak{a}_\mu$. We can see $\Phi_\mu$ as an inner product over $\mathfrak{a}_\mu$. We then denote by $K_\mu$ the closed unit ball of $\mathfrak{a}_\mu$ for the metric corresponding to $\Phi_\mu$. When $G$ is semisimple, the covariance 2-tensor $\Phi_\mu$ is *non-degenerate*, i.e. one has $\mathfrak{a}_\mu = \mathfrak{a}$.

THEOREM A.3. (Law of the iterated logarithm [**4**, Theorem 13.17]) *Let $G$ be a Zariski connected real reductive group and $\mu$ a Zariski dense probability measure with compact support on $G$. Then, for $\beta$-almost every $b \in B$, the set of accumulation points of the sequence*

$$\left( \frac{\kappa(b_n \cdots b_1) - n\sigma_\mu}{\sqrt{2n \log\log n}} \right)_{n\geq 1} \quad \textit{is exactly } K_\mu.$$

A.3. *Opposition involution.* In §4, we need a variation of Theorems A.2 and A.3, where the order in the product of the $b_i$ is inverted.

COROLLARY A.4. *Let $G$ be a Zariski connected real reductive group and $\mu$ a Zariski dense probability measure with compact support on $G$.*

https://doi.org/10.1017/etds.2018.18 Published online by Cambridge University Press

(a) *For $\beta$-almost every $b \in B$, we have the convergence*

$$\frac{1}{n}\,\kappa(b_1 \cdots b_n) \xrightarrow[n\to\infty]{} \sigma_\mu.$$

(b) *For $\beta$-almost every $b \in B$, the set of accumulation points of the sequence*

$$\left(\frac{\kappa(b_1 \cdots b_n) - n\sigma_\mu}{\sqrt{2n \log\log n}}\right)_{n\geq 1} \quad \textit{is exactly } K_\mu.$$

The proof will use the probability measure $\check{\mu}$ on $G$, which is the image of $\mu$ by the map $g \mapsto g^{-1}$. Recall that there exists a linear map $\iota : \mathfrak{a} \to \mathfrak{a}$ called the *opposition involution* (see [**4**, §8.2]) such that, for all $g \in G$, we have

$$\kappa(g^{-1}) = \iota(\kappa(g)). \tag{A.4}$$

LEMMA A.5. *The Lyapunov vector $\sigma_{\check{\mu}}$ of $\check{\mu}$, its covariance 2-tensor $\Phi_{\check{\mu}}$ and the closed unit ball $K_{\check{\mu}}$ of the linear span $\mathfrak{a}_{\check{\mu}}$ of $\Phi_{\check{\mu}}$ are given by*

$$\sigma_{\check{\mu}} = \iota(\sigma_\mu), \quad \Phi_{\check{\mu}} = \iota(\Phi_\mu), \quad K_{\check{\mu}} = \iota(K_\mu).$$

*Proof.* These identities follow from (A.2), (A.3) and (A.4). □

*Proof of Corollary A.4.* We apply the law of large numbers Theorem A.2 to the probability measure $\check{\mu}$. For $\beta$-almost every $b \in B$, the sequence $(1/n)\kappa(b_n^{-1} \cdots b_1^{-1})$ converges to $\sigma_{\check{\mu}}$. Applying the opposition involution $\iota$, we find that, for $\beta$-almost every $b \in B$, the sequence $(1/n)\kappa(b_1 \cdots b_n)$ converges to $\iota(\sigma_{\check{\mu}})$, which is equal to $\sigma_\mu$ by Lemma A.5.

In the same manner, Theorem A.3 tells us that the set of accumulation points of the sequence $(\kappa(b_1 \cdots b_n) - n\sigma_\mu)/(\sqrt{2n \log\log n})$ is exactly $\iota(K_{\check{\mu}})$, which is equal to $K_\mu$ by Lemma A.5. □

A.4. *Stationary measures on projective spaces.* In §5, we use the classification of stationary measures in [**3**, Theorem 1.7], as follows.

PROPOSITION A.6. *Let $V$ be a real vector space, $G \subset \mathrm{GL}(V)$ be a reductive algebraic subgroup of $\mathrm{GL}(V)$ and $\mu$ be a Zariski dense probability measure on $G$.*

*Then every $\mu$-stationary probability measure $\nu$ on $\mathbb{P}(V)$ is supported by a compact $G$-orbit.*

*Conversely, every compact $G$-orbit in $\mathbb{P}(V)$ supports a unique $\mu$-stationary probability measure $\nu$.*

## REFERENCES


[**1**] Y. Benoist and J.-F. Quint. Mesures stationnaires et fermés invariants des espaces homogènes I. *Ann. Math.* **174** (2011), 1111–1162.

[**2**] Y. Benoist and J.-F. Quint. Random walks on finite volume homogeneous spaces. *Invent. Math.* **187** (2012), 37–59.

[**3**] Y. Benoist and J.-F. Quint. Random walks on projective spaces. *Compos. Math.* **150** (2014), 1579–1606.

[**4**] Y. Benoist and J.-F. Quint. *Random Walks on Reductive Groups*. Springer, Berlin, 2016.


https://doi.org/10.1017/etds.2018.18 Published online by Cambridge University Press

[**5**] P. Bougerol, M. Babillot and L. Élie. The random difference equation $x_n = a_n x_{n-1} + b_n$ in the critical case. *Ann. Probab.* **25** (1997), 478–493.
[**6**] P. Bougerol and N. Picard. Strict stationarity of generalized autoregressive processes. *Ann. Probab.* **20** (1992), 1714–1730.
[**7**] N. Bourbaki. *Groupes et Algèbres de Lie*. Springer, Berlin, 2006, Chs. 7 and 8.
[**8**] L. Breiman. *Probability (Classics in Applied Mathematics, 7)*. SIAM, Philadelphia, 1992.
[**9**] C. Bruère. Un critère de récurrence pour certains espaces homogènes. *Preprint*, 2016, arXiv:1607.05698, *Bulletin de la SMF*, to appear.
[**10**] A. Eskin and G. Margulis. Recurrence properties of random walks on finite volume homogeneous spaces. *Random Walks and Geometry*. de Gruyter, Berlin, 2004, pp. 431–444.
[**11**] H. Furstenberg. Noncommuting random products. *Trans. Amer. Math. Soc.* **108** (1963), 377–428.
[**12**] I. Goldsheid and G. Margulis. Lyapunov indices of a product of random matrices. *Russian Math. Surveys* **44** (1989), 11–81.
[**13**] Y. Guivarc'h and A. Raugi. Frontière de Furstenberg, propriété de contraction et théorèmes de convergence. *Z. Wahrscheinlichkeitsth. verw. Geb.* **69** (1985), 187–242.
[**14**] F. Ledrappier. *Quelques propriétés des exposants caractéristiques*. Springer, Berlin, 1984, pp. 305–396.
[**15**] S. Meyn and R. Tweedie. *Markov Chain and Stochastic Stability*. Springer, Berlin, 1993.
[**16**] V. Oseledets. A multiplicative ergodic theorem. *Tr. Mosk. Mat.* **19** (1968), 179–210.

https://doi.org/10.1017/etds.2018.18 Published online by Cambridge University Press