ON SELF-RECIPROCAL FUNCTIONS FOR FOURIER–BESSEL INTEGRAL TRANSFORMS

By AFZAL AHMAD and V. LAKSHMIKANTH

Communicated by L. J. Slater

Received 20 September 1960

Following Hardy and Titchmarsh (1) a function $f(x)$ is said to be self-reciprocal if it satisfies the Fourier–Bessel integral transform

$$f(x) = \int_0^\infty J_p(xt)\,(xt)^{\frac{1}{2}} f(t)\,dt, \tag{1}$$

where $J_p(x)$ is a Bessel function of order $p \geqslant -\frac{1}{2}$. This integral is denoted by R_p. The special cases $p = \frac{1}{2}$ and $-\frac{1}{2}$, we denote by R_s and R_c, respectively.

The aim of this note is to obtain certain self-reciprocal functions. We shall use the following result, proved elsewhere (2).

Theorem. *If $h(x)$ is a kernel function which transforms R_p into R_{p+1}, then*

$$f(x) = x^{\frac{1}{2}} h(\tfrac{1}{2}x^2)$$

is R_{2p+1}, and conversely.

It is well known that the function $x^{2p+\frac{3}{2}} e^{-\frac{1}{2}x^2}$ is R_{2p+1}. Hence the converse of the above result shows that

$$h(x) = x^{p+\frac{1}{2}} e^{-x} \tag{2}$$

transforms R_p into R_{p+1} and vice versa. Putting $p = 0$, we have the kernel $x^{\frac{1}{2}} e^{-x}$ which transforms R_0 (R_1) into R_1 (R_0). Consider the well-known integral

$$\int_0^\infty e^{-xt} J_0(bt)\,dt = (x^2 + b^2)^{-\frac{1}{2}}, \tag{3}$$

which, after rearrangement, reduces to

$$\int_0^\infty (xt)^{\frac{1}{2}} e^{-xt} J_0(bt)\, t^{-\frac{1}{2}}\,dt = x^{\frac{1}{2}}(x^2 + b^2)^{-\frac{1}{2}}. \tag{4}$$

We know (3) that

$$\prod_{i=1}^{n} J_{p_i}(a_i t)\, t^{-(\Sigma p_i + \frac{1}{2})} \tag{5}$$

is R_1. Hence $J_0(bt)\, t^{-\frac{1}{2}}$ is also R_1. We obtain from (2), (4) and (5) that

$$x^{\frac{1}{2}}(x^2 + b^2)^{-\frac{1}{2}} \tag{6}$$

is R_0. Putting $b = 0$, we find that $x^{-\frac{1}{2}}$ is R_0.

Consider the following integral (5)

$$\int_0^\infty e^{-xt} f(t)\,dt = x^{-1}[f(0) + f'(0)\,x^{-1} + f''(0)\,x^{-2} + \ldots] \tag{7}$$

https://doi.org/10.1017/S0305004100035921 Published online by Cambridge University Press

which can be rewritten as

$$\int_0^\infty (xt)^{p+\frac{1}{2}} e^{-xt} f(t)\, t^{-(p+\frac{1}{2})}\, dt = x^{p-\frac{1}{2}}[f(0)+f'(0)\,x^{-1}+f''(0)\,x^{-2}+\ldots]. \tag{8}$$

It is clear from (2) and (8) that, if $f(t)\,t^{-(p+\frac{1}{2})}$ is R_p, then

$$x^{p-\frac{1}{2}}[f(0)+f'(0)\,x^{-1}+f''(0)\,x^{-2}+\ldots] \tag{9}$$

is R_{p+1}. The integral (5)

$$\int_0^\infty e^{-x^2t^2} f(t)\, dt = \tfrac{1}{2}\pi^{\frac{1}{2}}x^{-1}\left[f(0)+\frac{1}{2!}\tfrac{1}{2}f''(0)\,x^{-2}+\ldots\right] + 2x^{-1}\left[f'(0)\,x^{-1}+\frac{1}{3!}f^{(3)}(0)\,x^{-3}+\frac{2}{5!}f^{(5)}(0)\,x^{-5}+\ldots\right] \tag{10}$$

which reduces, if we put $x^2 = x$ and $t^2 = t$, to

$$\int_0^\infty (xt)^{p+\frac{1}{2}} e^{-xt} f(t^{\frac{1}{2}})\, t^{-(p+\frac{3}{2})}\, dt = \pi^{\frac{1}{2}}x^{p-\frac{1}{2}}\left[f(0)+\frac{1}{2!}\tfrac{1}{2}f''(0)\,x^{-2}+\frac{1}{4!}\frac{1.3}{2^2}f^{(4)}(0)\,x^{-4}+\ldots\right] + 4x^{p-\frac{1}{2}}\left[f'(0)\,x^{-1}+\frac{1}{3!}f^{(3)}(0)\,x^{-3}+\frac{2!}{5!}f^{(5)}(0)\,x^{-5}+\ldots\right]. \tag{11}$$

If $f(t^{\frac{1}{2}})\,t^{-(p+\frac{3}{2})}$ is R_p, it follows from (2) and (11) that the infinite series on the right-hand side of (11) is R_{p-1}. The following integral (5)

$$\int_0^\infty J_1(xt) f(t)\, dt = f(0)\,x^{-1}+x^{-1}\left[f'(0)\,x^{-1}+\tfrac{1}{2}f^{(3)}(0)\,x^{-3} + \frac{1.3}{2^2.2!}f^{(5)}(0)\,x^{-5}-\frac{1.3.5}{2^3\,3!}f^{(7)}(0)\,x^{-7}+\ldots\right] \tag{12}$$

can be written as

$$\int_0^\infty J_1(xt)\,(xt)^{\frac{1}{2}} f(t)\, t^{-\frac{1}{2}}\, dt = x^{-\frac{1}{2}}\left[f(0)+f'(0)\,x^{-1}+\tfrac{1}{2}f^{(3)}(0)\,x^{-3}+\frac{1.3}{2^2\,2!}f^{(5)}(0)\,x^{-5}-\frac{1.3.5}{2^3\,3!}f^{(7)}(0)\,x^{-7}+\ldots\right]. \tag{13}$$

It is clear from (13) that if $f(t)\,t^{-\frac{1}{2}}$ is R_1, the infinite series

$$x^{-\frac{1}{2}}\left[f(0)+f'(0)\,x^{-1}+\tfrac{1}{2}f^{(3)}(0)\,x^{-3}+\frac{1.3}{2^2.2!}f^{(5)}(0)\,x^{-5}-\frac{1.3.5}{2^3.3!}f^{(7)}(0)\,x^{-7}+\ldots\right] \tag{14}$$

is also R_1. The integral (5)

$$\int_0^\infty J_0(xt) f(t)\, dt = x^{-1}\left[f(0)-\tfrac{1}{2}f''(0)\,x^{-2}+\frac{1.3}{2^2.2!}f^{(4)}(0)\,x^{-4}-\frac{1.3.5}{2^3.3!}f^{(6)}(0)\,x^{-6}+\ldots\right] \tag{15}$$

can be written as

$$\int_0^\infty (xt)^{\frac{1}{2}} J_0(xt) f(t)\, t^{-\frac{1}{2}}\, dt = x^{-\frac{1}{2}}\left[f(0)-\tfrac{1}{2}f''(0)\,x^{-2}+\frac{1.3}{2^2.2!}f^{(4)}(0)\,x^{-4}-\frac{1.3.5}{2^3.3!}f^{(6)}(0)\,x^{-6}+\ldots\right]. \tag{16}$$

https://doi.org/10.1017/S0305004100035921 Published online by Cambridge University Press

It follows from (16) that if $f(t)\,t^{-\frac{1}{2}}$ is R_0, the infinite series of the right-hand side of (16) is also R_0.

Let the integral (5)

$$\int_0^\infty \sin xt f(t)\,dt = x^{-1}[f(0)+f'(0)x^{-2}+f''(0)\,x^{-4}+\ldots] \tag{17}$$

be thrown into the form

$$\int_0^\infty \sin xt(xt)^{-\frac{1}{2}} f(t)\,t^{\frac{1}{2}}\,dt = x^{-\frac{3}{2}}[f(0)+f'(0)\,x^{-2}+f''(0)\,x^{-4}+\ldots]. \tag{18}$$

Also the integral (5)

$$\int_0^\infty \sin xt f(t)\,t^{-1}\,dt = \tfrac{1}{2}\pi f(0)+f'(0)\,x^{-1}-\tfrac{1}{3}f^{(3)}(0)\,x^{-3}+\tfrac{1}{5}f^{(5)}(0)\,x^{-5}-\ldots] \tag{19}$$

can be written in the form

$$\int_0^\infty (xt)^{-\frac{1}{2}} \sin xt f(t)\,t^{-\frac{1}{2}}\,dt = x^{-\frac{1}{2}}[\tfrac{1}{2}\pi f(0)+f'(0)\,x^{-1}-\tfrac{1}{3}f^{(3)}(0)\,x^{-3}+\tfrac{1}{5}f^{(5)}(0)\,x^{-5}-\ldots]. \tag{20}$$

It is known that the kernel $x^{-\frac{1}{2}}\sin x$ transforms R_0 into R_1. Hence if follows from (18) that if $f(t)\,t^{\frac{1}{2}}$ is R_0, the infinite series on the right-hand side of (18) is R_1. Similarly, the infinite series on the right-hand side of (20) is R_1, if $f(t)\,t^{-\frac{1}{2}}$ is R_0. Also if $f(t)\,t^{-1}$ is R_s, then the right-hand side of (19) is also R_s.

We know that the kernel $x^{\frac{1}{2}(p-q+1)}J_{\frac{1}{2}(p+q)}(x)$ transforms R_p into R_q. Putting $q=1$, we find that

$$x^{\frac{1}{2}p}J_{\frac{1}{2}(p+1)}(x) \tag{21}$$

transforms R_p into R_1. Consider the integral (6)

$$\int_0^\infty J_p(xt)\,J_p(bt)\,J_p(ct)\,t^{-p+1}\,dt = \frac{2^{p-1}\Delta^{2p-1}}{\Gamma(p+\frac{1}{2})\,\Gamma(\frac{1}{2})\,(xbc)^p}, \tag{22}$$

where x, b, c represent the sides of the triangle of area Δ. Putting $p=\frac{1}{2}(p+1)$, the integral (22) reduces, after rearrangement, to the form

$$\int_0^\infty (xt)^{\frac{1}{2}p}\,J_{\frac{1}{2}(p+1)}(xt)\,J_{\frac{1}{2}(p+1)}(bt)\,J_{\frac{1}{2}(p+1)}(ct)\,t^{-p+\frac{1}{2}}\,dt = kx^{-\frac{1}{2}}, \tag{23}$$

where

$$k = \frac{2^{\frac{1}{2}(p-1)}\Delta^{\frac{1}{2}p}}{\Gamma_{\frac{1}{2}(p+2)}\,\Gamma(\frac{1}{2})\,(bc)^{\frac{1}{2}(p+1)}}.$$

Since $x^{-\frac{1}{2}}$ is R_1 in particular, it follows from (21) and (23) that

$$J_{\frac{1}{2}(p+1)}(bt)\,J_{\frac{1}{2}(p+1)}(ct)\,t^{-p+\frac{1}{2}}$$

is R_p. If $p=1$, the R_p function (23) reduces to

$$J_1(bt)\,J_1(ct)\,t^{-\frac{1}{2}} \tag{24}$$

which is known (4) to be R_1.

https://doi.org/10.1017/S0305004100035921 Published online by Cambridge University Press

REFERENCES

(1) HARDY, G. H. and TITCHMARSH, E. C. Self reciprocal functions. *Quart. J. Math.* (2), 1 (1930), 196–231.
(2) LAKSHMIKANTH, V. On the relation between the self-reciprocal functions and kernels (submitted for publication).
(3) LAKSHMIKANTH, V. Some self-reciprocal functions. *Proc. Nat. Acad. Sci.* (*India*), Part A, 28 (1959), 246–8.
(4) LAKSHMIKANTH, V. Some self-reciprocal functions and kernels. *Proc. Camb. Phil. Soc.* 57 (1961), 690–2.
(5) WILLS, H. F. A formula for expanding an integral as series. *Phil. Mag.* (7), 39 (1948), 455–9.
(6) WATSON, G. N. *Theory of Bessel Functions* (Cambridge, 1922).

OSMANIA UNIVERSITY
HYDERABAD (A.P.)
INDIA

https://doi.org/10.1017/S0305004100035921 Published online by Cambridge University Press