Proc. Camb. Phil. Soc. (1970), **68**, 87
PCPS 68–11
Printed in Great Britain

The principal bundle action in the Eilenberg–Moore spectral sequence

By W. D. BARCUS

(*Received* 16 *August* 1969)

In (2) the author determined a formula for computing, by means of the Eilenberg–Moore spectral sequence, the action of the group in a principal G-bundle, under the hypothesis that $H^*(G)$ be universally transgressively generated. This rather severe restriction was due to the method of proof, which used the theory of Guy Hirsch. Since Hirsch's work can be derived as a special case of Brown's twisted tensor products (3), one might hope to obtain the formula for the group action in the general case from the latter theory, and this we shall do (Theorem 1·2 below).

1. *Statement of the theorem.* For simplicity, coefficients of all chain and homology groups will lie in a fixed field k, which will not be indicated; alternatively, one could make appropriate flatness assumptions. Tensor products will be taken over k. We shall work in the category of pointed topological spaces, with basepoint denoted by $*$. The identity map in any context will be written as 1.

Our general reference for algebraic notions will be (4). Let Λ be a connected, associative, differential graded (DG) co-algebra, and let M and N be right and left DG Λ-comodules, respectively. Λ_0 is a co-ideal of Λ, and we set $\overline{\Lambda} = \Lambda/\overline{\Lambda}_0$. The *co-bar complex* $\mathscr{F}(M, \Lambda, N)$ is defined by

$$\mathscr{F}_{-r}(M, \Lambda, N) = M \otimes \overline{\Lambda} \otimes \ldots \otimes \overline{\Lambda} \otimes N \quad (r \text{ factors } \overline{\Lambda};\ r \geqslant 0).$$

As usual, we write generators as $\alpha = m[\lambda_1, \ldots, \lambda_r]n$. Gradings are defined by

$$\deg_I \alpha = \deg m + \deg n + \Sigma \deg \lambda_i, \quad \deg_E \alpha = -r, \quad \deg \alpha = \deg_I \alpha + \deg_E \alpha,$$

and differentials as follows, where $\bar{x} = -(-1)^{\deg x} x$:

$$d_I \alpha = dm[\lambda_1, \ldots, \lambda_r]n - \Sigma_i \overline{m}[\overline{\lambda}_1, \ldots, \overline{\lambda}_{i-1}, d\lambda_i, \lambda_{i+1}, \ldots, \lambda_r]n - \overline{m}[\overline{\lambda}_1, \ldots, \overline{\lambda}_r]dn.$$

Write the co-module and co-algebra maps as $\nabla x = \Sigma x' \otimes x''$, $x = m, n$, or λ_i; then

$$d_E \alpha = \Sigma m'[m'', \lambda_1, \ldots, \lambda_r]n - \Sigma_i \Sigma \overline{m}[\overline{\lambda}_1, \ldots, \overline{\lambda}_{i-1}, \overline{\lambda}'_i, \overline{\lambda}''_i, \ldots, \lambda_r]n$$
$$- \Sigma \overline{m}[\overline{\lambda}_1, \ldots, \overline{\lambda}_r, \overline{n}']n''.$$

Finally, $d = d_I + d_E$. The total degree deg and the total differential d will be those under consideration.

$\mathscr{F}(k, \Lambda, k)$ is the co-bar complex defined by Adams (1), but with the differential replaced by its negative. It is a DGk algebra, with multiplication

$$\mu'([\lambda_1, \ldots, \lambda_r] \otimes [\lambda_{r+1}, \ldots, \lambda_s]) = [\lambda_1, \ldots, \lambda_s],$$

and $\mathscr{F}(M, \Lambda, k)$ is a right $\mathscr{F}(k, \Lambda, k)$-module, with action

$$\mu(m[\lambda_1, \ldots, \lambda_r] \otimes [\lambda_{r+1}, \ldots, \lambda_s]) = m[\lambda_1, \ldots, \lambda_s]$$

(as usual, in the notation we suppress the factors k).

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Consider an induced fibration

$$(1\cdot1) \qquad \begin{array}{ccc} X' = B' \prod_B X & \xrightarrow{\tilde{h}} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{h} & B \end{array}$$

with fibre $F = p^{-1}(*) = * \prod_B X$. We assume that B' and B are connected, and denote by CB', CB, CX', CX, and CF the normalized singular simplicial chain groups generated by simplexes with all vertices at the basepoint in the case of B' and B, all vertices in F in the case of X' and X, and unrestricted in the case of F. The standard Alexander–Whitney formula ((4), p. 218) turns CB, CB' and CX into DGk co-algebras, and the maps $h_\#$ and $p_\#$ further turn CB' and CX into right and left DG CB-comodules, respectively. A DGk map $\theta\colon CX' \to \mathscr{F}(CB', CB, CX)$ is defined by the composition

$$CX' \xrightarrow{\nabla} CX' \otimes CX' \xrightarrow{p'_\# \otimes h_\#} CB' \otimes CX = \mathscr{F}_0,$$

and the Eilenberg–Moore theorem ((4), 12·1) states that if B is simply connected, then θ is a chain equivalence.

In the case which concerns us, X is contractible. Then the map

$$\epsilon\colon \mathscr{F}(CB', CB, k) \to \mathscr{F}(CB', CB, CX),$$

induced by the inclusion $* \to X$, is also a chain equivalence ((4), 7.1).

We are now able to state the theorem. Let $p\colon X \to B$ be a Steenrod principal G-bundle with X contractible and B a simply connected paracompact space, and let $h\colon B' \to B$ be a map, where B' is assumed connected. Let $\mathscr{F}$ denote the co-bar complex, and let θ and ϵ be as above. Note that in the case $B' = *$, $B' \prod_B X = F = G$, while $CB' = k$.

(1·2) THEOREM. *The following diagram is commutative, where μ_* and m_* denote the module actions*:

$$(1\cdot3) \qquad \begin{array}{ccc} H(\mathscr{F}(CB', CB, k)) \otimes H(\mathscr{F}(k, CB, k)) & \xleftarrow{\epsilon_*^{-1}\theta_* \otimes \epsilon_*^{-1}\theta_*} & H(B' \prod_B X) \otimes H(G) \\ {\scriptstyle \mu_*}\downarrow & & \downarrow{\scriptstyle m_*} \\ H(\mathscr{F}(CB', CB, k)) & \xleftarrow{\epsilon_*^{-1}\theta_*} & H(B' \prod_B X) \end{array}$$

If M and N are left and right DG Λ-modules respectively, as above, then

$$\mathscr{F}(M, \Lambda, N) = M \square_\Lambda \mathscr{F}(\Lambda, \Lambda, N),$$

where $\square$ denotes the cotensor product (4). The natural map $N \to \mathscr{F}(\Lambda, \Lambda, N)$ (i.e. the composition of the comodule map $N \to \Lambda \otimes N$ with the inclusion) defines an injective resolution of N in the sense of (4). Hence, from the definition of cotor, we have a natural isomorphism $\mathrm{cotor}^\Lambda(M, N) \approx H(\mathscr{F}(M, \Lambda, N))$. The maps μ' and μ thus induce the structure of an algebra in $\mathrm{cotor}^\Lambda(k, k)$ and the structure of a right $\mathrm{cotor}^\Lambda(k, k)$-module in $\mathrm{cotor}^\Lambda(M, k)$. Then (1·2) can be rephrased as follows:

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Let $\epsilon: \mathrm{cotor}^{CB}(CB', k) \to \mathrm{cotor}^{CB}(CB', CX)$ be induced by the map $* \to X$, and let $\tau: H(B' \prod_B X) \to \mathrm{cotor}^{CB}(CB', CX)$ be defined as in ((4), p. 220). Then

(1·4) COROLLARY. *Under the hypotheses for* (1·2), *the following diagram is commutative*:

$$\begin{array}{ccc} \mathrm{cotor}^{CB}(CB', k) \otimes \mathrm{cotor}^{CB}(k, k) & \xleftarrow{\epsilon^{-1}\tau \otimes \epsilon^{-1}\tau} & H(B' \prod_B X) \otimes H(G) \\ \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle m_*} \\ \mathrm{cotor}^{CB}(CB', k) & \xleftarrow{\epsilon^{-1}\tau} & H(B' \prod_B X) \end{array}$$

In the corollary, as a result of ((4), 7·1), C may be taken to denote the singular simplicial chain group with no restriction on the vertices.

The Eilenberg–Moore spectral sequence $E_*(M, \Lambda, N)$ for $\mathrm{cotor}^{\Lambda}(M, N)$ arises from filtering $\mathscr{F}(M, \Lambda, N)$ by the external degree $\deg_E$, and $E_2(M, \Lambda, N) \approx \mathrm{cotor}^{H(\Lambda)}(H(M), H(N))$. Under this filtration, $\mathscr{F}(k, \Lambda, k)$ becomes a filtered DGk algebra, and $\mathscr{F}(M, \Lambda, k)$ becomes a filtered right $\mathscr{F}(k, \Lambda, k)$-module. Hence we have

(1·5) COROLLARY. *The maps* μ' and μ *defined above turn the Eilenberg–Moore spectral sequence* $E_*(k, CB, k)$ *into a spectral sequence of algebras converging to the algebra* $H(G)$, *and* $E_r(CB' CB, k)$ *into a right* $E_r(k, CB, k)$*-module, for* $r \geqslant 0$, *converging to the right* $H(G)$*-module* $H(B' \prod_B X)$.

Note that the product in $E_2(k, CB, k) = \mathrm{cotor}^{H(B)}(k, k)$ and the right $E_2(k, CB, k)$-module action in

$$E_2(CB', CB, k) = \mathrm{cotor}^{H(B)}(H(B'), k)$$

are also induced by the maps μ' and μ, taking $M = H(B')$ and $\Lambda = H(B)$.

2. *Proof of the theorem.* We shall use the method of acyclic models in the category $\mathscr{C}$ which has as objects maps $h: B' \to B$ of connected pointed spaces, and as maps, commutative squares.

We shall need the notion of a (weakly transitive) *lifting function* for a fibre map $p: X \to B$. Recall from (3) that this is a map $\lambda: \{(\alpha, x) \in PB \times X \mid \epsilon\alpha = px\} \to X$, where PB is the space of Moore paths on B and $\epsilon\alpha$ is the endpoint of the path α, satisfying the following conditions:

(2·1) (1) $p\lambda(\alpha, x) = \alpha(0)$.

(2) $\lambda(e_{px}, x) = x$, where e_y is the path of length zero at y.

(3) $\lambda(\alpha, \lambda(\beta, x)) = \lambda(\alpha\beta, x)$ whenever $\epsilon\alpha = * = \beta(0)$.

Given a lifting function λ for $p: X \to B$, if Y is a space and $p'': X \times Y \to B$ is defined by $p''(x, y) = p(x)$, then there is an obvious choice of lifting function λ'' for p: namely

$$\lambda''(\alpha, (x, y)) = (\lambda(\alpha, x), y). \tag{2·2}$$

The proof of the theorem will be based on the following diagram of chain maps, which is naturally defined on the object $h: B' \to B$ of $\mathscr{C}$. Here $p: EB \to B$ denotes the fibre space of Moore paths on B which end at $*$, with fibre ΩB and lifting function

$$\lambda(\alpha, \beta) = \alpha\beta.$$

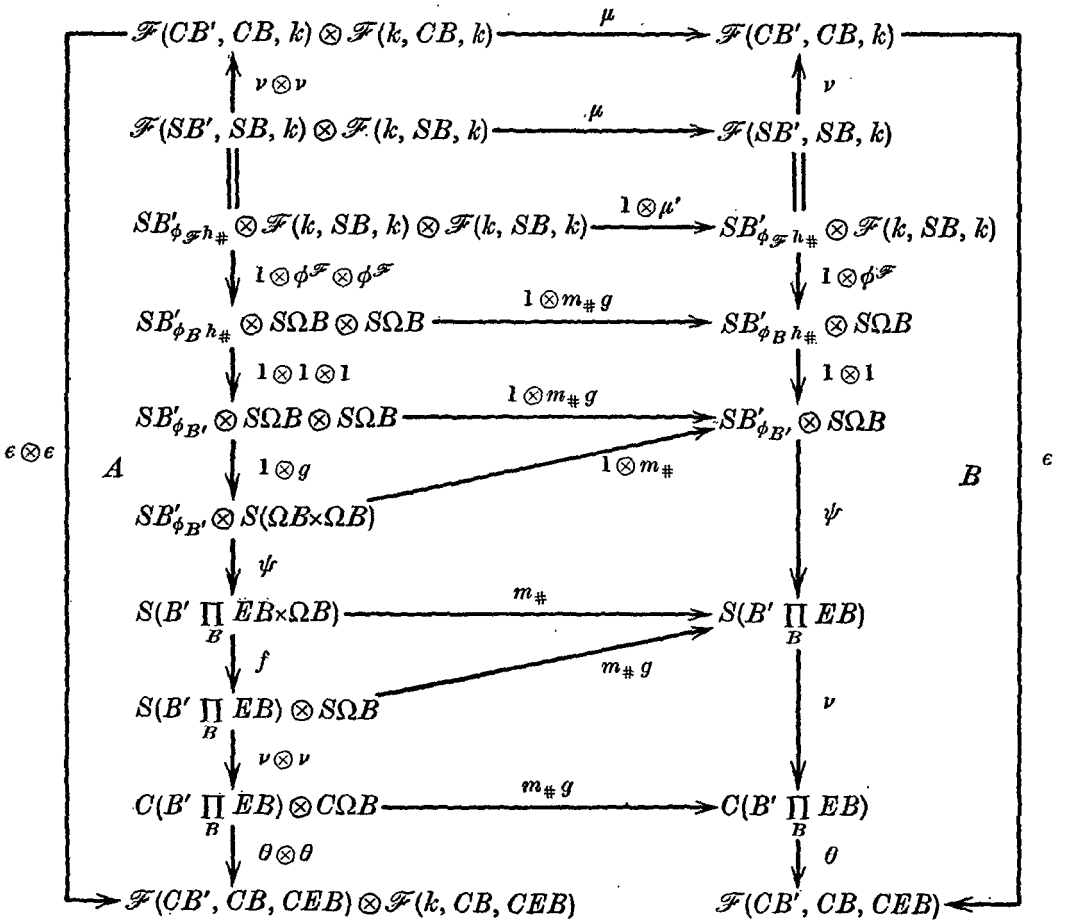
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The lifting function for the induced bundle $B' \prod_B EB$ is defined by

$$\lambda'(\alpha', (b', \beta)) = (\alpha'(0), (h\alpha')\beta), \quad \alpha' \in PB', b' \in B', \beta \in EB, \quad \epsilon\alpha' = b', \quad h(b') = \beta(0),$$

while that for $B' \prod_B EB \times \Omega B$ is obtained from λ' by (2·2). S denotes the un-normalized chain complex, with the same conventions concerning vertices as for C. ν is the natural projection of S onto C. ψ is the natural chain equivalence of ((3), 4·2), which we recall depends on the choice of lifting function, while ϕ_B and $\phi_{B'}$ are the twisting functions of ((3), 4·1) for the spaces B and B'. The twisting function $\phi_{\mathscr{F}}: SB \to \mathscr{F}(k, SB, k)$ is given by $\phi_{\mathscr{F}}(\sigma) = [\sigma]$, while the DGk-algebra map $\phi^{\mathscr{F}}: \mathscr{F}(k, SB, k) \to S\Omega B$ is determined by $\phi^{\mathscr{F}}[\beta_1, \ldots, \beta_r] = \phi_B\beta_1, \ldots, \phi_B\beta_r$ (multiplication of Moore paths). Our sign conventions here differ from those of (3). Finally, f and g are the canonical Eilenberg–Zilber maps ((5), pp. 241, 243), while μ and m are the appropriate products.

(2·3)



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(2·4) LEMMA. *The diagram* (2·3) *is homotopy-commutative.*

The proof will be given below.

If B is simply connected, then all of the vertical maps in (2·3) induce homology isomorphisms: for ψ, see ((3), 4·2); for $1\otimes\phi^{\mathscr{F}}$, ((3), section 10); for $1\otimes 1$, ((3), 3·7); for θ, ((4), 12·1); for ϵ, ((4), 7·1); while the remainder are well-known. The commutativity of diagram (1·3) with X replaced by EB, and G by ΩB, now follows from the lemma.

Suppose next that $p: X \to B$ is a Steenrod principal G-bundle with B paracompact. Then according to ((3), 1·6), p admits a lifting function λ such that for $\alpha \in PB$ and $x \in X$, with $\epsilon\alpha = px$,

$$\lambda(\alpha, xg) = \lambda(\alpha, x)\, g \quad \text{for all} \quad g \in G. \tag{2·5}$$

Define a map $q: EB \to X$ by $q(\beta) = \lambda(\beta, *)$. q is clearly fibre-preserving. Its restriction to the fibre, $q': \Omega B \to G$, is multiplicative, for if $\beta, \gamma \in \Omega B$, then

$$\begin{aligned} q'(\beta\gamma) &= \lambda(\beta\gamma, *) = \lambda(\beta, \lambda(\gamma, *)) && \text{by (2·1) (3)}, \\ &= \lambda(\beta, *)\,\lambda(\gamma, *) && \text{by (2·5)}, \\ &= q'(\beta)\, q'(\gamma). \end{aligned}$$

Similar equations show that q is right q'-semilinear. By naturality, the induced map $\tilde{q}: B' \prod_B EB \to B' \prod_B X$ is also q'-semilinear.

Suppose finally that X is contractible. Then $\tilde{q}$ and q' are weak homotopy equivalences, and the commutativity of (1·3) for the bundle $B' \prod_B X$ follows from that for $B' \prod_B EB$ by naturality. Thus Theorem (1·2) is a consequence of Lemma (2·4).

Proof of Lemma (2·4). The unlabelled squares and triangles in the diagram are all commutative with the exception that $m_{\#} gf \simeq m_{\#}$, which follows from the standard relation $gf \simeq 1$. Only the square containing ψ is not obvious: but we have defined our lifting functions so that they are preserved by the map of fibre spaces

$$m: B' \prod_B EB \times \Omega B \to B' \prod_B EB,$$

and hence commutativity of this square follows from the naturality of ψ ((3), 4·2, ii).

It remains to show that squares (A) and (B) are homotopy-commutative. Since the proofs are closely similar, we give only that for (B).

As indicated, we use the method of acyclic models in the category $\mathscr{C}$. The models $\mathscr{M}$ are indexed by ordered $(r+1)$-tuples of non-negative integers, $r \geqslant 0$, and

$$\mathscr{M}(m, q_1, \ldots, q_r)$$

is the inclusion map $\quad B_0' = \bar{\Delta}_m \to \bar{\Delta}_m \vee \bar{\Delta}_{q_1} \vee \ldots \vee \bar{\Delta}_{q_r} = B_0.$

Here $\bar{\Delta}$ is the space obtained by identifying all the vertices $v_0, \ldots, v_n$ of the standard n-simplex Δ_n to a point.

We shall prove:

(2·6) The functor $\mathscr{F}(SB', SB, k)$ is representable on models.

(2·7) The functor $\mathscr{F}(CB', CB, CEB)$ is acyclic on models in dimensions > 0.

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(2·8) Diagram (B) is homotopy-commutative in dimension O.

That (B) is homotopy-commutative in all dimensions then follows from the theory of acyclic models.

Proof of (2·6). The collapsing map $\Delta_n \to \bar{\Delta}_n$ followed by the inclusion defines simplexes $\iota_m, \iota_{q_1}, \ldots, \iota_{q_r}$ in SB_0, and ι'_m in SB'_0. Given an object $h: B' \to B$ in $\mathscr{C}$ and a generator $\sigma_m[\tau_{q_1}, \ldots, \tau_{q_r}] \in \mathscr{F}(SB', SB, k)$, define a map $\rho = (\sigma_m, \rho'): (B'_0, B_0) \to (B', B)$ by setting ρ' equal to $h\sigma_m$ on $\bar{\Delta}_m$, and equal to τ_{q_j} on $\bar{\Delta}_{q_j}, j = 1, \ldots, r$. Then

$$\rho_\#(\iota'_m[\iota_{q_1}, \ldots, \iota_{q_r}]) = \sigma_m[\tau_{q_1}, \ldots, \tau_{q_r}].$$

Thus $\mathscr{F}(SB', SB, k)$ is representable.

Proof of (2·7). Note that $\bar{\Delta}_n$ has the homotopy type of a wedge of circles. Since

$$H(\mathscr{F}(CB'_0, CB_0, C{\cdot}EB_0)) = \operatorname{cotor}^{CB_0}(CB'_0, CEB_0),$$

by ((4), 7·1) we may replace EB_0 by $*$, and B'_0 and B_0 by wedges of circles, without changing the homology and may further replace the resulting spaces by minimal subcomplexes of the total singular complexes. This reduces the problem to the computation of $H(\mathscr{F}(Q, R, k))$, where Q has a single 0-simplex $*$ and 1-simplexes $\sigma_1, \ldots, \sigma_s$ say, R has a single 0-simplex $*$ and 1-simplexes $\sigma_1, \ldots, \sigma_s, \sigma_{s+1}, \ldots, \sigma_t$, and the map $Q \to R$ is the inclusion. $\mathscr{F}(Q, R, k)$ is 0 in dimensions > 1, and has a k-basis consisting of the elements $\sigma_{i_0}[\sigma_{i_1}, \ldots, \sigma_{i_r}]$ in dimension 1, and the elements $*[\sigma_{i_1}, \ldots, \sigma_{i_r}]$ in dimension 0, with $1 \leqslant i_0 \leqslant s$, $1 \leqslant i_j \leqslant t$ for $j > 0$, and $r \geqslant 0$. The boundary is given by

$$d(\sigma_{i_0}[\sigma_{i_1}, \ldots, \sigma_{i_r}]) = [\sigma_{i_0}, \sigma_{i_1}, \ldots, \sigma_{i_r}].$$

As this carries basis elements to distinct basis elements, $\mathscr{F}(Q, R, k)$ is acyclic in dimensions > 0.

Proof of (2·8). The multiplication of Moore paths $\alpha, \beta, \in \Omega B$ will be written $\alpha\beta$, while for simplexes $\sigma_1, \ldots, \sigma_r \in S_1 B$, we define $\sigma_1 \circ \ldots \circ \sigma_r \in S_1 B$ by

$$(\sigma_1 \circ \ldots \circ \sigma_r)(t) = \sigma_1 \ldots \sigma_r(rt) \quad (0 \leqslant t \leqslant 1).$$

For a simplex $\sigma \in S_1 B$, we denote by $\bar{\sigma}$ the corresponding point in ΩB. $\iota \in S_1 B$ will denote the degenerate 1-simplex. Finally, we abbreviate $\zeta = \nu\psi(1 \otimes 1)(1 \otimes \phi^{\mathscr{F}})$ in diagram (2·3).

In dimension 0, $\mathscr{F}(SB', SB, k)$ has a k-basis consisting of the elements

$$*[\tau_1, \ldots, \tau_q], \quad q \geqslant 0,$$

with each τ_i a simplex in $S_1 B$, and we must show that $\theta\zeta \simeq \epsilon\nu$ when applied to each such element. For $q = 0$,

$$\theta\zeta(*[\,]) = *[\,]* = \epsilon\nu(*[\,]);$$

while for $q > 0$, it follows from the definitions of ϕ_B and ψ in (3) that

$$\theta\zeta(*[\tau_1, \ldots, \tau_q]) = *[\,](\bar{\tau}_1 - \bar{\iota}) \ldots (\bar{\tau}_q - \bar{\iota}),$$

and

$$\epsilon\nu(*[\tau_1, \ldots, \tau_q]) = *[\tau_1, \ldots, \tau_q]*.$$

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In order to show that these are chain-homotopic, we define two constructions.

(*a*) For simplexes $\sigma_1, \ldots, \sigma_r \in S_1 B$, $r \geqslant 2$, define $\chi(\sigma_1; \sigma_2, \ldots, \sigma_r) \in S_2 B$ as follows. Let $v = (r-1)/r\, v_0 + (1/r)\, v_2 \in \Delta_2$. Project Δ_2 linearly onto its first face so that v_1 maps to v, and then map this face to B by $\sigma_1 \circ \ldots \circ \sigma_r$. Then

$$d\chi = \sigma_2 \circ \ldots \circ \sigma_r - \sigma_1 \circ \ldots \circ \sigma_r + \sigma_1,$$

and

$$\nabla\chi = \chi \otimes 1 + 1 \otimes \chi + \sigma_1 \otimes \sigma_2 \circ \ldots \circ \sigma_r.$$

(*b*) For simplexes $\sigma_1, \ldots, \sigma_r \in S_1 B$, define $\gamma(\sigma_1, \ldots, \sigma_r) \in S_1 EB$ by

$$\left.\begin{aligned} \gamma(t)(s) &= \sigma_1 \ldots \sigma_r(rt), \quad s \leqslant rt \\ &= \sigma_1 \ldots \sigma_r(s), \quad s \geqslant rt \end{aligned}\right\} \quad (0 \leqslant t \leqslant 1, 0 \leqslant s \leqslant r).$$

Then

$$d\gamma = \gamma(1) - \gamma(0) = \bar{\iota} \ldots \bar{\iota} - \bar{\sigma}_1 \ldots \bar{\sigma}_r,$$

while

$$p\gamma(t) = \gamma(t)(0) = \sigma_1 \ldots \sigma_r(rt) = \sigma_1 \circ \ldots \circ \sigma_r(t),$$

so that $p\gamma = \sigma_1 \circ \ldots \circ \sigma_r$.

Now set

$$\alpha = \sum_{1 \leqslant i \leqslant q-1} \Sigma' * [\sigma_1, \ldots, \sigma_{i-1}, \chi(\sigma_i; \sigma_{i+1}, \ldots, \sigma_q)] * \in \mathscr{F}(CB', CB, k),$$

where Σ' means the sum over all monomials obtained by setting each $\sigma_j = \tau_j$ or ι, with the sign $(-1)^t$, where t is the number of times that ι occurs in the monomial. The boundary of the ith term is then

$$-\Sigma' * [\sigma_1, \ldots, \sigma_{i-1}, \sigma_i \circ \sigma_{i+1} \circ \ldots \circ \sigma_q] * + \Sigma' * [\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1} \circ \ldots \circ \sigma_q] * + \Sigma' * [\sigma_1, \ldots, \sigma_{i-1}, \sigma_i] * + \Sigma' * [\sigma_1, \ldots, \sigma_{i-1}, \sigma_i, \sigma_{i+1} \circ \ldots \circ \sigma_q] *.$$

In this expression the second and third sums are each 0. If $i > 1$, then the first sum in the ith term cancels with the last sum in the $(i-1)$st term. Thus we are left with two terms, and

$$d\alpha = -\Sigma' * [\sigma_1 \circ \ldots \circ \sigma_q] * + \Sigma' * [\sigma_1, \ldots, \sigma_q] *.$$

Since $*[\sigma_1, \ldots, \sigma_q] * = 0$ whenever some $\sigma_i = \iota$, the second sum in this expression is just $*[\tau_1, \ldots, \tau_q] *$.

If we set $\beta = \Sigma' * [\,] \gamma(\sigma_1, \ldots, \sigma_q)$, then

$$d\beta = \Sigma' * [\sigma_1 \circ \ldots \circ \sigma_q] * - \Sigma' * [\,] \bar{\sigma}_1 \ldots \bar{\sigma}_q.$$

Thus

$$d(\alpha + \beta) = *[\tau_1, \ldots, \tau_q] * - \Sigma' * [\,] \bar{\sigma}_1 \ldots \bar{\sigma}_q = (\epsilon\nu - \theta\zeta)(* [\tau_1, \ldots, \tau_q]).$$

This completes the proof of (2·8), and hence of Theorem (1·2).

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