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LARGE DEVIATION PRINCIPLE FOR EPIDEMIC MODELS

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Abstract

We consider a general class of epidemic models obtained by applying the random time changes of Ethier and Kurtz (2005) to a collection of Poisson processes and we show the large deviation principle for such models. We generalise the approach followed by Dolgoarshinnykh (2009) in the case of the SIR epidemic model. Thanks to an additional assumption which is satisfied in many examples, we simplify the recent work of Kratz and Pardoux (2017).

Keywords: Poisson process; large deviation principle; law of large numbers

2010 Mathematics Subject Classification: Primary 60F10; 60H10; 60J75; 92C60

1. Introduction

In this paper we are interested in a class of Poisson-driven stochastic differential equations (SDEs) which arise in many fields such as chemical kinetics and ecological and epidemic models. We consider a d-dimensional processes of the type

$$Z^N(t) := Z^{N,z}(t) := \frac{[Nz]}{N} + \frac{1}{N}\sum_{j=1}^{k} h_j P_j\left(\int_0^t N\beta_j(Z^N(s))\,\mathrm{d}s\right), \tag{1}$$

where $(P_j)_{1\le j\le k}$ are independent and identically distributed standard Poisson processes and the $h_j \in \{-1, 0, 1\}^d$ denote the k distinct jump directions with jump rates $\beta_j(z)$ and $z \in A$, where A is a compact subset of $\mathbb{R}^d$, which will be assumed to satisfy Assumption 1 below.

In the main application which we have in mind (see our examples at the end of the paper), the components of the vector $Z^N(t)$ are the proportions of the population in the various compartments corresponding to the various disease status of the individuals (susceptible, infectious, and so on). In most of those models, the set A is given as

$$A = \left\{z \in \mathbb{R}^d_+ :\ \sum_{i=1}^{d} z_i \le 1\right\}. \tag{2}$$

We refer the reader to [2] for a presentation of many such epidemics models.

As we shall recall below, it is plain that under mild assumptions, as $N \to \infty$, $Z^N(t) \to Y^z(t)$ almost surely (a.s.), locally uniformly for $t > 0$, where $Y^z(t)$ solves the ordinary differential

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equation (ODE)

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = b(Y(t)), \qquad Y(0) = z, \tag{3}$$

with $b(y) = \sum_{j=1}^{k} \beta_j(y) h_j$. In this paper we want to investigate the large deviations from this law of large numbers.

Let us now be more precise about the initial condition $Z^N(0) = [Nz]/N$. In the models we have in mind, since each component of $Z^N(t)$ is a proportion in a population of total population size equal to N, we want $Z^N(t)$ to take its values in the set $A^{(N)} = \{z \in A,\ Nz \in \mathbb{Z}_+^d\}$. In particular, we want the initial condition $Z^N(0)$ to belong to this set $A^{(N)}$. If that is not the case, some of the components of the vector $Z^N(t)$ may become negative, while jumping from a/N to $(a-1)/N$, $0 < a < 1$, which is not very natural. For that reason, we will use the following convention concerning the initial condition. For some fixed $z \in A$, for $1 \le i \le d$, $N \ge 1$, $Z_i^N(0) = [Nz_i]/N$. Consequently, the initial condition of the Z^N equation depends upon N.

In all what follows, $D_{T,A}$ denotes the set of functions from $[0, T]$ into A which are right-continuous and have left limits at any $t \in [0, T]$ and let $\mathcal{AC}_{T,A}$ be the subspace of absolutely continuous functions. We equip $D_{T,A}$ with Skorohod's topology; see [1, p. 124].

We denote by $\mathcal{B}$ the Borel σ-field on $D_{T,A}$ and $\mathbb{P}_z^N$ the probability measure on paths with the initial condition $Z^N(0) = [Nz]/N$, defined by

$$\mathbb{P}_z^N(B) = \mathbb{P}(Z^N \in B) \quad \text{for all } B \in \mathcal{B}.$$

Our goal is to show that the probability measures $\mathbb{P}_z^N$, $N > 1$, satisfy a large deviation principle with a good rate function I_T that we will define below in Subsection 2.2. In other words, for any G open subset of $D_{T,A}$ and F closed subset of $D_{T,A}$ with $G \subset F$, we want to show the following inequalities:

$$-\inf_{\phi \in G} I_T(\phi) \le \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}_z^N(G) \le \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_z^N(F) \le -\inf_{\phi \in F} I_T(\phi). \tag{4}$$

The topic of large deviation principles is the subject of many treatises; see, in particular, [3], [5], [8], [10], and [17]. In some of these references the authors studied large deviations for Poisson processes, e.g. [17]. However, in this paper it is assumed that the rates of the Poisson processes are bounded away from 0, and, hence, their logarithms are bounded. The case of Poisson processes with vanishing rates was studied in [18]. However, their assumptions are not satisfied in our situation, as is explained in [12]. Our results have already been established in [12]. However, our argument here is simpler, and the proofs are shorter. It is based upon an idea from [4] and our assumptions are slightly different from those in [12]. We also refer the reader to [9] and [15] for large deviation results concerning specific epidemic models (the latter one for a population of two types whose individuals move in space by jumps).

Our approach forces us to make the following assumption.

Assumption 1. *We suppose that there exists $z_0 \in \mathbb{R}^d$ such that the collection of mappings $\Phi_a\colon A \mapsto \mathbb{R}^d$ defined by $\Phi_a(z) = z + a(z_0 - z)$, defined for each $0 < a < 1$, is such that $z^a = \Phi_a(z) \in A$ for all $z \in A$, and, moreover, for some $0 < c_2 < c_1$ and all $z \in A$,*

$$|z - z^a| \le c_1 a, \qquad \operatorname{dist}(z^a, \partial A) \ge c_2 a.$$

We define, for all $a > 0$,

$$B^a = \{z \in A\colon \operatorname{dist}(z, \partial A) \ge c_2 a\} \quad \text{and} \quad R^a = \{\phi \in \mathcal{AC}_{T,A}\colon \phi_t \in B^a \text{ for all } t \in [0, T]\}.$$

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Remark 1. For any convex set A, Assumption 1 is satisfied with $z_0 \in \mathring{A}$, the interior of A. The same construction is possible for many not necessarily convex sets, provided A is compact, and there is a point z_0 in its interior which is such that, for each $z \in \partial A$, the segment joining z_0 and z does not touch any other point of the boundary ∂A. We also note that for A given by (2) and $z_0 \in A$, the constants c_1 and c_2 can be defined by

$$c_1 = \sup_{z\in A} |z - z_0|, \qquad c_2 = \sin(\theta_0) \inf_{z\in\partial A} |z - z_0| \le \inf_{z\in\partial A} |z - z_0| \times \sin(\theta(z)),$$

where $\theta(z)$ is the most acute angle between the tangent to the boundary ∂A at z and the vector $z_0 - z$ and θ_0 is an angle such that, for all $z \in \partial A$, $\theta_0 \le \theta(z) \le \pi/2$.

For all $a > 0$, we let $C_a = \inf_j \inf_{z\in B^a} \beta_j(z)$ and we formulate our assumptions on the β_j.

Assumption 2. (i) *The rate functions β_j are Lipschitz continuous with the Lipschitz constant equal to C.*

(ii) *For any $1 \le j \le k$, $\beta_j(z) > 0$ if $z \in \mathring{A}$, and β_j is bounded by a positive constant σ.*

(iii) *There exist two constants λ_1 and λ_2 such that whenever $z \in A$ is such that $\beta_j(z) < \lambda_1$, $\beta_j(z^a) > \beta_j(z)$ for all $a \in (0, \lambda_2)$.*

(iv) *There exists a constant $\nu \in (0, \frac{1}{2})$ such that $\lim_{a\to 0} a^\nu \log C_a = 0$.*

Note that our assumptions are easily verified in all the classical epidemic models; see below Section 5.

Remark 2. We have not made any restriction concerning the set of vectors $\{h_1, \dots, h_k\}$. In all examples which we have in mind, $\{\sum_{j=1}^k \alpha_j h_j,\ \alpha \in \mathbb{R}_+^k\} = \mathbb{R}^d$, which ensures that the process Z_t^N can move in all directions in A. Note that at any rate, there is no restriction as to which β_j vanish at some given point $z \in \partial A$. This is a major difference with the assumptions of [18]. Note also that in some sense our assumptions are weaker than those of [12], except for our Assumption 1.

For all $\phi, \psi \in D_{T,A}$, we will define the distance between ϕ and ψ by

$$\|\phi - \psi\|_T = \sup_{t\le T} |\phi_t - \psi_t|,$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$.

The paper is structured as follows. In Section 2 we formulate the law of large numbers and the Girsanov theorem, we define a good rate function for our large deviation principle, and we establish some properties that it satisfies. The proof of the lower bound (first inequality in (4)) is detailed in Section 3. In Section 4 we state the upper bound (third inequality in (4)), which follows from known results. In Section 5 we apply the large deviations principle to the study of the exit time from a domain, and we give four applications to the time of extinction of an endemic situation in four distinct epidemic models.

2. Some important results

We start with (1), whose existence and uniqueness is essentially obvious, the solution being constant between its jumps, and Z_t^N has a jump of size h_j/N when $\int_0^t N\beta_j(Z^N(s))\,\mathrm{d}s$ hits a jump time of the Poisson process P_j, $1 \le j \le k$. The fact that the process $Z^N(t)$ lives in the compact set A is due to the fact that β_j vanishes on any piece of the boundary ∂A where h_j points towards the exterior of A; see the examples in Section 5.

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2.1. Law of large numbers and change of measure

We first recall the law of large numbers; see [14].

Theorem 1. *Let $Z^{N,z}(t)$ be the solution of the Poissonian SDE (1) with the initial condition $[Nz]/N$. Assume that Assumption 2(i) holds. Then*

$$\lim_{N\to\infty} \|Z^{N,z} - Y^z\|_T = 0 \quad a.s.,$$

where $Y^z(\cdot)$ is the unique solution of ODE (3).

We shall need the following Girsanov theorem. Let Q denote the random number of jumps of Z^N in the interval $[0, T]$, τ_p be the time of the pth jump for $p = 1, \dots, Q$, and define

$$\delta_p(j) = \begin{cases} 1 & \text{if the } p\text{th jump is in the direction } h_j, \\ 0 & \text{otherwise.} \end{cases}$$

We shall denote $\mathcal{F}_t^N = \sigma\{Z^N(s),\ 0 \le s \le t\}$. Consider another set of rates $\tilde\beta_j(z)$, $1 \le j \le k$. The rates $\beta_j(t, z)$ above depend only upon the second variable z. In the next statement, we introduce new rates $\tilde\beta_j(t, z)$, which in the next section will depend upon the two variables t and z, and are supposed to be a.s. continuous at the jump times of $Z^N(t)$. Combining Theorem III.5.19 of [11] and Theorem 2.4 of [19], we have the following theorem ($\tilde{\mathbb{P}} \ll \mathbb{P}$ means that $\tilde{\mathbb{P}}$ is absolutely continuous with respect to $\mathbb{P}$ and $Z^N(\tau_p^-) = \lim_{t\to\tau_p,\, t<\tau_p} Z^N(t)$).

Theorem 2. *Let $\widetilde{\mathbb{P}}^N$ denote the law of Z^N when the rates are rates $\tilde\beta_j(\cdot)$. Then provided that $\sup_{0\le t\le T,\, z\in A} \tilde\beta_j(t, z)/\beta_j(z) < \infty$, which implies, in particular, that $\{z\colon \beta_j(z) = 0\} \subset \{z\colon \tilde\beta_j(t, z) = 0\}$ for all $0 \le t \le T$, $\widetilde{\mathbb{P}}^N|_{\mathcal{F}_T^N} \ll \mathbb{P}^N|_{\mathcal{F}_T^N}$, and with the convention that $\frac{0}{0} = 1$,*

$$\begin{aligned} \xi_T &= \frac{\mathrm{d}\widetilde{\mathbb{P}}^N|_{\mathcal{F}_T^N}}{\mathrm{d}\mathbb{P}^N|_{\mathcal{F}_T^N}} \\ &= \left(\prod_{p=1}^{Q}\prod_{j=1}^{k}\left[\frac{\tilde\beta_j(\tau_p, Z^N(\tau_p^-))}{\beta_j(Z^N(\tau_p^-))}\right]^{\delta_p(j)}\right) \exp\left\{N\sum_{j=1}^{k}\int_0^T (\beta_j(Z^N(t)) - \tilde\beta_j(t, Z^N(t)))\,\mathrm{d}t\right\}. \end{aligned} \tag{5}$$

Corollary 1. *For any nonnegative random variable X, $\mathbb{E}(X) \ge \tilde{\mathbb{E}}(\xi_T^{-1} X)$.*

Proof. As $X \ge 0$, we write

$$\mathbb{E}(X) \ge \mathbb{E}(X\mathbf{1}_{\{\xi_T\neq 0\}}) = \tilde{\mathbb{E}}(\xi_T^{-1} X\mathbf{1}_{\{\xi_T\neq 0\}}) = \tilde{\mathbb{E}}(\xi_T^{-1} X).$$

Note that ξ_T^{-1} is well defined $\widetilde{\mathbb{P}}$-a.s., since $\widetilde{\mathbb{P}}(\xi_T = 0) = 0$. □

It is not difficult to see that under $\tilde{\mathbb{P}}$, there exist again mutually independent standard Poisson processes $\tilde P_j$, $1 \le j \le k$, such that

$$Z^N(t) := Z^{N,z}(t) := \frac{[Nz]}{N} + \frac{1}{N}\sum_{j=1}^{k} h_j \tilde P_j\left(\int_0^t N\tilde\beta_j(Z^N(s))\,\mathrm{d}s\right).$$

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2.2. The rate function

For all $\phi \in \mathcal{AC}_{T,A}$, let $\mathcal{A}_d(\phi)$ be the set of $\mathbb{R}_+^k$-valued Borel measurable functions μ which are such that

$$\phi_t = \phi_0 + \sum_{j=1}^{k} h_j \int_0^t \mu_s^j \, \mathrm{d}s \quad \text{for all } t > 0.$$

We define the rate function

$$I_T(\phi) := \begin{cases} \inf_{\mu \in \mathcal{A}_d(\phi)} I_T(\phi \mid \mu) & \text{if } \phi \in \mathcal{AC}_{T,A}, \\ \infty & \text{otherwise,} \end{cases}$$

where

$$I_T(\phi \mid \mu) = \int_0^T \sum_{j=1}^{k} f(\mu_t^j, \beta_j(\phi_t)) \, \mathrm{d}t,$$

with $f(\nu, \omega) = \nu \log(\nu/\omega) - \nu + \omega$ and the conventions $\log(\nu/0) = \infty$ for all $\nu > 0$ and $0 \log(0/0) = 0 \log(0) = 0$.

Note that under our standing assumptions, set $\mathcal{A}_d(\phi)$ can be empty for many $\phi \in \mathcal{AC}_{T,A}$. Recall the usual convention that the infimum over an empty set is $+\infty$.

It was shown in [12] that $I_T(\phi) = \int_0^T L(\phi_t, \phi_t') \, \mathrm{d}t$, where, for all $z \in A$, $y \in \mathbb{R}^d$, $L(z, y) = \sup_{\theta \in \mathbb{R}^d} \langle \theta, y \rangle - \sum_{j=1}^{k} \beta_j(z)(\mathrm{e}^{\langle \theta, h_j \rangle} - 1)$.

The following Theorem is Proposition 4.23 of [12].

Theorem 3. *It holds that I_T is a good rate function.*

Lemma 1. *Let $s > 0$, $\phi \in D_{T,A}$, and $\mu \in \mathcal{A}_d(\phi)$ such that $I_T(\phi \mid \mu) \le s$. Then, for all $0 \le t_1, \ t_2 \le T$ such that $t_2 - t_1 \le 1/\sigma$,*

$$\int_{t_1}^{t_2} \mu_t^j \, \mathrm{d}t \le \frac{s+1}{-\log(\sigma(t_2 - t_1))}, \qquad j = 1, \ldots, k.$$

Proof. For all $0 \le t_1, \ t_2 \le T$,

$$\int_{t_1}^{t2} f(\mu_t^j, \beta_j(\phi_t)) \, \mathrm{d}t \le I_T(\phi \mid \mu) \le s,$$

and, since the function $h(x) = x \log(x/\sigma) - x$ is convex in x,

$$\begin{aligned} h\left(\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \mu_t^j \, \mathrm{d}t \right) &\le \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} h(\mu_t^j) \, \mathrm{d}t \\ &\le \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left(\mu_t^j \log \frac{\mu_t^j}{\beta_j(\phi_t)} - \mu_t^j + \beta_j(\phi_t) \right) \mathrm{d}t \\ &\le \frac{s}{t_2 - t_1}. \end{aligned}$$

It is easy to show that, for all $\alpha > 0$, $h(x) \ge \alpha x - \sigma \exp\{\alpha\}$ and then, for all $\alpha > 0$,

$$\int_{t_1}^{t_2} \mu_t^j \, \mathrm{d}t \le \frac{1}{\alpha}(s + (t_2 - t_1)\sigma \exp\{\alpha\}).$$

Let $t_2 - t_1 < 1/\sigma$. The result follows by choosing $\alpha = -\log(\sigma(t_2 - t_1))$. □

For $\phi \in D_{T,A}$, let ϕ^a be defined by $\phi_t^a := \Phi_a(\phi_t)$. We note that $\phi^a \in R^a$.

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Lemma 2. *For all $\phi \in D_{T,A}$, we have* $\limsup_{a\to 0} I_T(\phi^a) \le I_T(\phi)$.

Proof. It clearly suffices to treat the case where $I_T(\phi) < \infty$. For any $\eta > 0$, there exists $\mu \in \mathcal{A}_d(\phi)$ such that $I_T(\phi \mid \mu) \le I_T(\phi) + \eta$. Let $\mu^a = (1-a)\mu$. Then $\mu^a \in \mathcal{A}_d(\phi^a)$. We will now show that

$$\limsup_{a\to 0} I_T(\phi^a \mid \mu^a) \le I_T(\phi \mid \mu),$$

which clearly implies the result since

$$\limsup_{a\to 0} I_T(\phi^a) \le \limsup_{a\to 0} I_T(\phi^a \mid \mu^a) \le I_T(\phi \mid \mu) \le I_T(\phi) + \eta.$$

We note that

$$f(\mu_t^{j,a}, \beta_j(\phi_t^a)) = \mu_t^{j,a} \log\left(\frac{\mu_t^{j,a}}{\beta_j(\phi_t^a)}\right) - \mu_t^{j,a} + \beta_j(\phi_t^a) = f(\mu_t^j, \beta(\phi_t)) + H(a,t,j),$$

where

$$H(a,t,j) = (1-a)\mu_t^j \log\left(\frac{\beta_j(\phi_t)}{\beta_j(\phi_t^a)}\right) - a\mu_t^j \log\left(\frac{\mu_t^j}{\beta_j(\phi_t)}\right) + (1-2a)\mu_t^j \log(1-a) + a\mu_t^j.$$

It remains to show that $\limsup_{a\to 0} \int_0^T H(a,t,j)\,\mathrm{d}t \le 0$. We first consider the first term in right-hand side of the equation above. From Assumption 2(iii), it follows that

$$\int_0^T \mu_t^j \log\left(\frac{\beta_j(\phi_t)}{\beta_j(\phi_t^a)}\right)\mathrm{d}t \le \int_0^T \mathbf{1}_{\{\beta_j(\phi_t)\ge\lambda_1\}} \log\left(\frac{\beta_j(\phi_t)}{\beta_j(\phi_t^a)}\right)\mathrm{d}t,$$

and the right-hand side tends to 0 as $a \to 0$, since μ_t^j is integrable from our assumption and Lemma 1. The other terms tend to 0 thanks to $I_T(\phi) < \infty$. □

Lemma 3. *Let $a > 0$ and $\phi \in R^a$ be such that $I_T(\phi) < \infty$. For all $\eta > 0$, there exists $L > 0$ and $\phi^L \in R^{a/2}$ such that $\|\phi - \phi^L\|_T < c_1 a/2$, and $\mu^L \in \mathcal{A}_d(\phi^L)$ such that $I_T(\phi^L \mid \mu^L) \le I_T(\phi) + \eta$, with $\mu_t^{L,j} < L$, $j = 1, \dots, k$.*

Proof. Let $\eta > 0$ and $\mu \in \mathcal{A}_d(\phi)$ be such that $I_T(\phi \mid \mu) < I_T(\phi) + \eta/2$. For $L > 0$, let $\mu_t^{L,j} = \mu_t^j \wedge L$ and let ϕ^L be a solution of the ODE

$$\frac{\mathrm{d}\phi_t^L}{\mathrm{d}t} = \sum_{j=1}^k \mu_t^{L,j} h_j.$$

It follows from the monotone convergence theorem that $\|\phi - \phi^L\|_T \to 0$ as $L \to \infty$. Since $\phi \in R^a$, for large enough L, $\phi^a \in R^{a/2}$. It is easy to show that $I_T(\phi^L \mid \mu^L) \to I_T(\phi \mid \mu)$ as $L \to \infty$. Hence the result. □

Let $\varepsilon > 0$ be such that $T/\varepsilon \in \mathbb{N}$ and let ϕ^ε be the polygonal approximation of ϕ defined for $t \in [\ell\varepsilon, (\ell+1)\varepsilon)$ by

$$\phi_t^\varepsilon = \phi_{\ell\varepsilon}\frac{(\ell+1)\varepsilon - t}{\varepsilon} + \phi_{(\ell+1)\varepsilon}\frac{t - \ell\varepsilon}{\varepsilon}. \tag{6}$$

Lemma 4. *Let $\eta > 0$ be arbitrary. Let $0 < a < 1$, $\phi \in R^a$, and $\mu \in \mathcal{A}_d(\phi)$ be such that $\mu_t^j < L$, $j = 1, \dots, k$, for some $L > 0$ and $I_T(\phi \mid \mu) < \infty$. Then there exists a_η such that, for all $0 < a \le a_\eta$, there exists a $\varepsilon_a > 0$ and, for all $\varepsilon < \varepsilon_a$, the polygonal approximation ϕ^ε*

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belongs to $R^{a/2}$ *and* $\|\phi - \phi^\varepsilon\|_T < c_2 a/2 < c_1 a/2$. *Moreover, there exists* $\mu^\varepsilon \in \mathcal{A}_d(\phi^\varepsilon)$ *such that* $\mu_t^{\varepsilon,j} < L$, $j = 1, \dots, k$, *and* $I_T(\phi^\varepsilon \mid \mu^\varepsilon) \le I_T(\phi \mid \mu) + \eta$.

Proof. Since ϕ is uniformly continuous on $[0, T]$, there exists $\varepsilon_a > 0$ such that, for all $\varepsilon < \varepsilon_a$,

$$\sup_{|t-t'|<2\varepsilon} |\phi_t - \phi_{t'}| < c_2 \frac{a e^{-a^{-\nu}}}{4}.$$

Consequently, $\|\phi - \phi^\varepsilon\|_T < c_2 a/2$ and $\phi^\varepsilon \in R^{a/2}$.

For $t \in [\ell\varepsilon, (\ell+1)\varepsilon)$,

$$\frac{\mathrm{d}\phi_t^\varepsilon}{\mathrm{d}t} = \frac{\phi_{(\ell+1)\varepsilon} - \phi_{\ell\varepsilon}}{\varepsilon} = \frac{1}{\varepsilon} \sum_{j=1}^{k} h_j \int_{\ell\varepsilon}^{(\ell+1)\varepsilon} \mu_t^j \, \mathrm{d}t.$$

Therefore, μ_t^ε defined, for $t \in [\ell\varepsilon, (\ell+1)\varepsilon)$, by

$$\mu_t^{\varepsilon,j} = \frac{1}{\varepsilon} \int_{\ell\varepsilon}^{(\ell+1)\varepsilon} \mu_r^j \, \mathrm{d}r, \qquad j = 1, \dots, k,$$

is constant on $[\ell\varepsilon, (\ell+1)\varepsilon)$ and belongs to $\mathcal{A}_d(\phi^\varepsilon)$. We also note that $\mu_t^{\varepsilon,j} \le L$ for all $j = 1, \dots, k$. Moreover, if $0 < \nu \le L$ and $\omega \ge C_a$ then

$$\left| \frac{\partial f(\nu, \omega)}{\partial \omega} \right| = \left| -\frac{\nu}{\omega} + 1 \right| \le \frac{L}{C_a} + 1.$$

By Assumption 2(iv), there exists $\tilde{a}_\eta > 0$ such that, for all $a < \tilde{a}_\eta$,

$$\frac{L}{C_a} + 1 \le L e^{a^{-\nu}} + 1.$$

Then, for $t \in [\ell\varepsilon, (\ell+1)\varepsilon)$ and $a < a_0$,

$$|f(\mu_t^{\varepsilon,j}, \beta_j(\phi_t^\varepsilon)) - f(\mu_t^{\varepsilon,j}, \beta_j(\phi_{\ell\varepsilon}))| \le \frac{c_2}{4} C(L+1)a = \frac{Va}{2},$$
$$|f(\mu_t^{j}, \beta_j(\phi_t)) - f(\mu_t^{j}, \beta_j(\phi_{\ell\varepsilon}))| \le \frac{c_2}{4} C(L+1)a = \frac{Va}{2}.$$

The above implies that

$$\begin{aligned}
\int_{\ell\varepsilon}^{(\ell+1)\varepsilon} f(\mu_t^{\varepsilon,j}, \beta_j(\phi_t^\varepsilon)) \, \mathrm{d}t &\le \int_{\ell\varepsilon}^{(\ell+1)\varepsilon} f(\mu_t^{\varepsilon,j}, \beta_j(\phi_{\ell\varepsilon})) \, \mathrm{d}t + \frac{\varepsilon V a}{2} \\
&= \varepsilon f(\mu_{\ell\varepsilon}^{\varepsilon,j}, \beta_j(\phi_{\ell\varepsilon})) + \frac{\varepsilon V a}{2} \\
&\le \int_{\ell\varepsilon}^{(\ell+1)\varepsilon} f(\mu_t^{j}, \beta_j(\phi_{\ell\varepsilon})) \, \mathrm{d}t + \frac{\varepsilon V a}{2} \\
&\le \int_{\ell\varepsilon}^{(\ell+1)\varepsilon} f(\mu_t^{j}, \beta_j(\phi_t)) \, \mathrm{d}t + \varepsilon V a,
\end{aligned}$$

where the second inequality follows from Jensen's inequality. Therefore,

$$I_T(\phi^\varepsilon \mid \mu^\varepsilon) \le I_T(\phi \mid \mu) + VTa.$$

The result follows by choosing $a < \min\{a_0, \eta / VT\}$. □

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3. The lower bound

Our proof of the lower bound relies essentially upon the next lemma.

Lemma 5. *For $z \in A$, $\phi \in \mathcal{AC}_{T,A}$, $\phi_0 = z$, for any $0 < a < 1$ and $\varepsilon > 0$ small enough such that the polygonal approximation $\phi^{a,\varepsilon}$ of ϕ^a defined as in (6) satisfies $\mathrm{d}(\phi^{a,\varepsilon}, \partial A) \le c_2 a/2$, and $\mu^{a,\varepsilon} \in \mathcal{A}_d(\phi^{a,\varepsilon})$ being chosen as in Lemma 4, the following holds. For any $\eta > 0$ and suitably small $0 < \delta \le c_2 a/4$, there exist $N_{\eta,\delta} \in \mathbb{N}$ such that, for all y, $|y - z| < \delta/2$, and any $N > N_{\eta,\delta}$,*

$$\mathbb{P}_y(\|Z^N - \phi^{a,\varepsilon}\|_T < \delta) \ge \exp\{-N(I_T(\phi^{a,\varepsilon} \mid \mu^{a,\varepsilon}) + \eta)\}.$$

Proof. We first define new rates $\tilde{\beta}_j(t, z) = \mu_t^{a,\varepsilon,j} \varkappa(z)$, where $\varkappa \in C(A; [0, 1])$ is such that

$$\varkappa(z) = \begin{cases} 1 & \text{if } d(z, \partial A) \ge c_2 a/4, \\ 0 & \text{if } d(z, \partial A) \le c_2 a/8, \end{cases}$$

so that on the event $\{\|Z^N - \phi^{a,\varepsilon}\|_T < \delta\}$, $\tilde{\beta}_j(t, Z_t^N) = \mu_t^{a,\varepsilon,j}$ for $0 \le t \le T$, while the assumption of Theorem 2 is satisfied. It is clear that the new probability measure $\tilde{\mathbb{P}}_y$ depends upon both a and ε, and also that dependence will not appear explicitly for the sake of notational simplicity.

With $(8kTLC/C_a)\delta$, we define the following events B_j^δ, $j = 1, \dots, k$, for controlling the likelihood ratio. For $\gamma > 0$, let

$$B_j^\delta = \left\{ \left| \sum_{p=1}^{Q} \delta_p(j) \log\left(\frac{\beta_j(Z^N(\tau_p^-))}{\mu^{a,\varepsilon,j}_{\lceil \tau_p/\varepsilon \rceil \varepsilon}} \right) - N \sum_{\ell=1}^{T/\varepsilon} \mu_{\ell\varepsilon}^{a,\varepsilon,j} \log\left(\frac{\beta_j(\phi^{a,\varepsilon}_{\ell\varepsilon})}{\mu^{a,\varepsilon,j}_{\ell\varepsilon}} \right) \varepsilon \right| \le N\gamma \right\},$$

where Q was first introduced just before Theorem 2.

On the event $\{\|Z^N - \phi^{a,\varepsilon}\|_T < \delta\} \cap (\bigcap_{j=1}^k B_j^\delta) = \{\|Z^N - \phi^{a,\varepsilon}\|_T < \delta\} \cap B^\delta$, with ξ_T defined by (5),

$$\begin{aligned}
\xi_T^{-1} &= \exp\left\{ \sum_{p=1}^{Q} \sum_{j=1}^{k} \delta_p(j) \log\left(\frac{\beta_j(Z^N(\tau_p^-))}{\mu^{a,\varepsilon,j}_{\tau_p^-}} \right) + N \int_0^T \sum_{j=1}^{k} (\mu_t^{a,\varepsilon,j} - \beta_j(Z^N(t)))\, \mathrm{d}t \right\} \\
&\ge \exp\left\{ -N \sum_{\ell=1}^{T/\varepsilon} \sum_{j=1}^{k} \mu_{\ell\varepsilon}^{a,\varepsilon,j} \log\left(\frac{\mu_{\ell\varepsilon}^{a,\varepsilon,j}}{\beta_j(\phi^{a,\varepsilon}_{\ell\varepsilon})} \right) \varepsilon \right. \\
&\qquad \left. + N \int_0^T \sum_{j=1}^{k} (\mu_t^{a,\varepsilon,j} - \beta_j(Z^N(t)))\, \mathrm{d}t - kN\gamma \right\} \\
&\ge \exp\left\{ -N \sum_{\ell=1}^{T/\varepsilon} \sum_{j=1}^{k} \mu_{\ell\varepsilon}^{a,\varepsilon,j} \log\left(\frac{\mu_{\ell\varepsilon}^{a,\varepsilon,j}}{\beta_j(\phi^{a,\varepsilon}_{\ell\varepsilon})} \right) \varepsilon \right. \\
&\qquad \left. + N \int_0^T \sum_{j=1}^{k} (\mu_t^{a,\varepsilon,j} - \beta_j(\phi_t^{a,\varepsilon}))\, \mathrm{d}t - N(kTC\delta + k\gamma) \right\}.
\end{aligned}$$

The first inequality holds since the $\mu_t^{a,\varepsilon,j}$ are piecewise constant, and the second one follows from the Lipschitz continuity of the β_j. Since the derivative of $\phi^{a,\varepsilon}$ is bounded, we can compare

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the sum in the above exponential and an integral, thus,

$$\begin{aligned}\xi_T^{-1} &\geq \exp\Bigg\{-N\int_0^T \sum_{j=1}^k \Bigg[\mu_t^{a,\varepsilon,j}\log\Bigg(\frac{\mu_t^{a,\varepsilon,j}}{\beta_j(\phi_t^{a,\varepsilon})}\Bigg) - \mu_t^{a,\varepsilon,j} + \beta_j(\phi_t^{a,\varepsilon})\Bigg]\mathrm{d}t \\ &\qquad - N[kTC(\delta+\varepsilon)+k\gamma]\Bigg\} \\ &= \exp\{-N[I_T(\phi^{a,\varepsilon} \mid \mu^{a,\varepsilon}) + kTC(\delta+\varepsilon)+k\gamma]\}\end{aligned}$$

on the event $\{\|Z^N - \phi^{a,\varepsilon}\|_T < \delta\} \cap B^\delta$. Then, for any $\eta > 0$, there exists $\delta > 0$, $\varepsilon > 0$ small enough, and we have

$$\xi_T^{-1} \geq \exp\big\{-N\big(I_T(\phi^{a,\varepsilon} \mid \mu^{a,\varepsilon}) + \tfrac{1}{2}\eta\big)\big\}.$$

Moreover, from Corollary 1,

$$\begin{aligned}\mathbb{P}_y(\|Z^N - \phi^{a,\varepsilon}\|_T < \delta) &\geq \widetilde{\mathbb{E}}_y(\xi_T^{-1}\mathbf{1}_{\{\{\|Z^N-\phi^{a,\varepsilon}\|_T<\delta\}\cap B^\delta\}}) \\ &\geq \exp\big\{-N\big(I_T(\phi^{a,\varepsilon} \mid \mu^{a,\varepsilon}) + \tfrac{1}{2}\eta\big)\big\}\widetilde{\mathbb{P}}_y(\{\|Z^N - \phi^{a,\varepsilon}\|_T < \delta\}\cap B^\delta).\end{aligned}$$

The result is now a consequence of the next lemma. □

Lemma 6. *For $z \in A$, $\phi \in \mathcal{AC}_{T,A}$, $\phi_0 = z$, for any $0 < a < 1$, small enough $\varepsilon > 0$, and $0 < a \leq c_2 a/4$, the polygonal approximation $\phi^{a,\varepsilon}$ of ϕ^a has the property that, for all y, $|y - z| < \delta/2$,*

$$\lim_{N\to\infty} \widetilde{\mathbb{P}}_y(\{\|Z^N - \phi^{a,\varepsilon}\|_T < \delta\}\cap B^\delta) = 1.$$

Proof. It is enough to prove both that $\lim_{N\to\infty}\widetilde{\mathbb{P}}_y(\|Z^N - \phi^{a,\varepsilon}\|_T < \delta) = 1$ and that, for all $1 \leq j \leq k$, $\lim_{N\to\infty}\widetilde{\mathbb{P}}_y(\{\|Z^N - \phi^{a,\varepsilon}\|_T < \delta\}\cap (B_j^\delta)^c) = 0$. We first sketch the proof of the first limit. With the notation $\tilde{M}_j(t) = \tilde{P}_j(t) - t$, we have

$$\begin{aligned}Z^N(t) &= \frac{[Ny]}{N} + \sum_{j=1}^k h_j\int_0^t \mu_s^{a,\varepsilon,j}\,\mathrm{d}s + \frac{1}{N}\sum_{j=1}^k \tilde{M}_j\Bigg(\int_0^t N\mu_s^{a,\varepsilon,j}\,\mathrm{d}s\Bigg) \\ &= \phi_t^{a,\varepsilon} + \frac{[Ny]}{N} - z + \frac{1}{N}\sum_{j=1}^k \tilde{M}_j\Bigg(\int_0^t N\mu_s^{a,\varepsilon,j}\,\mathrm{d}s\Bigg),\end{aligned}$$

and it follows from the arguments, e.g. of [14], that under $\widetilde{\mathbb{P}}_y$, the last term on the right-hand side tends to 0 a.s. as $N \to \infty$.

We now establish that $\widetilde{\mathbb{P}}_y(\|Z^N - \phi^{a,\varepsilon}\|_T < \delta \cap (B_j^\delta)^c) \to 0$ as $N \to \infty$. We have $\sup_p |Z^N(\tau_p) - \phi_{\tau_p}^{a,\varepsilon}| < \delta$ on $D_\delta = \{\|Z^N - \phi^{a,\varepsilon}\|_T < \delta\}$ and we can choose ε such that $\sup_p |\phi_{\tau_p}^{a,\varepsilon} - \phi_{\lceil\tau_p/\varepsilon\rceil\varepsilon}^{a,\varepsilon}| < \delta$ and, thus, $\sup_p |Z^N(\tau_p) - \phi_{\lceil\tau_p/\varepsilon\rceil\varepsilon}^{a,\varepsilon}| < 2\delta$. Hence, on D_δ,

$$\begin{aligned}&\Bigg|\sum_{p=1}^Q \delta_p(j)\log\Bigg(\frac{\beta_j(Z^N(\tau_p^-))}{\mu_{\lceil\tau_p/\varepsilon\rceil\varepsilon}^{a,\varepsilon,j}}\Bigg) - \sum_{p=1}^Q \delta_p(j)\log\Bigg(\frac{\beta_j(\phi_{\lceil\tau_p/\varepsilon\rceil\varepsilon}^{a,\varepsilon})}{\mu_{\lceil\tau_p/\varepsilon\rceil\varepsilon}^{a,\varepsilon,j}}\Bigg)\Bigg| \\ &\quad\leq \Bigg|\sum_{p=1}^Q \delta_p(j)\log\Bigg(\frac{\beta_j(Z^N(\tau_p^-))}{\beta_j(\phi_{\lceil\tau_p/\varepsilon\rceil\varepsilon}^{a,\varepsilon})}\Bigg)\Bigg| \\ &\quad\leq \frac{2C\delta}{C_a}Q,\end{aligned}$$

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since $|\beta_j(Z^N(\tau_p^-)) - \beta_j(\phi^{a,\varepsilon}_{\lceil \tau_p/\varepsilon\rceil\varepsilon})| < 2C\delta$. With Q_ℓ denoting the number of jumps of Z^N in the interval $[(\ell-1)\varepsilon, \ell\varepsilon)$, we have (note that on the event $\tau_p \in ((\ell-1)\varepsilon, \ell\varepsilon)$, $\lceil \tau_p/\varepsilon\rceil = \ell$)

$$\begin{aligned}
&\left|\sum_{p=1}^{Q} \delta_p(j) \log\left(\frac{\beta_j(Z^N(\tau_p^-))}{\mu^{a,\varepsilon,j}_{\lceil \tau_p/\varepsilon\rceil\varepsilon}}\right) - N\sum_{\ell=1}^{T/\varepsilon} \mu^{a,\varepsilon,j}_{\ell\varepsilon} \log\left(\frac{\beta_j(\phi^{a,\varepsilon}_{\ell\varepsilon})}{\mu^{a,\varepsilon,j}_{\ell\varepsilon}}\right)\varepsilon\right| \\
&\quad\le \left|\sum_{p=1}^{Q} \delta_p(j) \log\left(\frac{\beta_j(\phi^{a,\varepsilon}_{\lceil \tau_p/\varepsilon\rceil\varepsilon})}{\mu^{a,\varepsilon,j}_{\lceil \tau_p/\varepsilon\rceil\varepsilon}}\right) - N\sum_{\ell=1}^{T/\varepsilon} \mu^{a,\varepsilon,j}_{\ell\varepsilon} \log\left(\frac{\beta_j(\phi^{a,\varepsilon}_{\ell\varepsilon})}{\mu^{a,\varepsilon,j}_{\ell\varepsilon}}\right)\varepsilon\right| \\
&\qquad + \left|\sum_{p=1}^{Q} \delta_p(j) \log\left(\frac{\beta_j(Z^N(\tau_p^-))}{\mu^{a,\varepsilon,j}_{\lceil \tau_p/\varepsilon\rceil\varepsilon}}\right) - \sum_{p=1}^{Q} \delta_p(j) \log\left(\frac{\beta_j(\phi^{a,\varepsilon}_{\lceil \tau_p/\varepsilon\rceil\varepsilon})}{\mu^{a,\varepsilon,j}_{\lceil \tau_p/\varepsilon\rceil\varepsilon}}\right)\right| \\
&\quad\le \left|\sum_{\ell=1}^{T/\varepsilon} \log\left(\frac{\beta_j(\phi^{a,\varepsilon}_{\ell\varepsilon})}{\mu^{a,\varepsilon,j}_{\ell\varepsilon}}\right)\left(\sum_{p=1}^{Q_\ell} \delta_{\ell,p}(j) - N\mu^{a,\varepsilon,j}_{\ell\varepsilon}\varepsilon\right)\right| + \frac{2C\delta}{C_a} Q,
\end{aligned}$$

where

$$\delta_{\ell,p}(j) = \begin{cases} 1 & \text{if the } p\text{th jump in the time interval } [(\ell-1)\varepsilon, \ell\varepsilon) \text{ is in the direction } h_j, \\ 0 & \text{otherwise.} \end{cases}$$

As the jump rates are constant on the interval $[(\ell-1)\varepsilon, \ell\varepsilon)$ under $\widetilde{\mathbb{P}}^N$, $\sum_{p=1}^{Q_\ell} \delta_{\ell,p}(j)$ is the number of jumps of a Poisson process P_j on this interval. So it is a Poisson random variable with mean $N\mu^{a,\varepsilon,j}_{\ell\varepsilon}\varepsilon$. From Chebyshev's inequality, we deduce that

$$\begin{aligned}
&\widetilde{\mathbb{P}}_y\left(\left|\log\left(\frac{\beta_j(\phi^{a,\varepsilon}_{\ell\varepsilon})}{\mu^{a,\varepsilon,j}_{\ell\varepsilon}}\right)\left(\sum_{p=1}^{Q_\ell} \delta_{\ell,p}(j) - N\mu^{a,\varepsilon,j}_{\ell\varepsilon}\varepsilon\right)\right| > \frac{N\gamma\varepsilon}{2T}\right) \\
&\qquad\le \frac{4T^2 \sup_{\ell\le T/\varepsilon}(\log^2(\beta_j(\phi^{a,\varepsilon}_{\ell\varepsilon})/\mu^{a,\varepsilon,j}_{\ell\varepsilon}) N\mu^{a,\varepsilon,j}_{\ell\varepsilon}\varepsilon)}{N^2\gamma^2\varepsilon^2}.
\end{aligned}$$

As $C_a \le \beta_j(\phi^{a,\varepsilon}_t) \le \sigma$, $\mu^{a,\varepsilon,j}_t \le L$, we have

$$\sup_{\ell\le T/\varepsilon}\left(\log^2\left(\frac{\beta_j(\phi^{a,\varepsilon}_{\ell\varepsilon})}{\mu^{a,\varepsilon,j}_{\ell\varepsilon}}\right)\mu^{a,\varepsilon,j}_{\ell\varepsilon}\right) \le C(L,a).$$

Thus,

$$\begin{aligned}
&\widetilde{\mathbb{P}}_y(\{\|Z^N - \phi^{a,\varepsilon}\|_T < \delta\} \cap (B_j^\delta)^c) \\
&\quad\le \widetilde{\mathbb{P}}_y\left(\left|\sum_{\ell=1}^{T/\varepsilon} \log\left(\frac{\beta_j(\phi^{a,\varepsilon}_{\ell\varepsilon})}{\mu^{a,\varepsilon,j}_{\ell\varepsilon}}\right)\left(\sum_{p=1}^{Q_\ell} \delta_{\ell,p}(j) - N\mu^{a,\varepsilon,j}_{\ell\varepsilon}\varepsilon\right)\right| + \frac{2C\delta}{C_a} Q > N\gamma\right) \\
&\quad\le \frac{4T^2 C(L,a)}{N\gamma^2\varepsilon} + \widetilde{\mathbb{P}}_y\left(\frac{2C\delta}{C_a} Q \ge \frac{N\gamma}{2}\right).
\end{aligned}$$

Under the probability $\widetilde{\mathbb{P}}_y$, Q, the number of jumps during the time interval $[0, T]$, is the sum of T/ε Poisson random variables, the ℓth one having mean $N\sum_{j=1}^{k} \mu^{a,\varepsilon,j}_{\ell\varepsilon}\varepsilon$. Consequently,

$$\widetilde{\mathbb{E}}(Q) = \widetilde{\mathrm{var}}(Q) = N\sum_{j=1}^{T/\varepsilon}\sum_{j=1}^{k} \mu^{a,\varepsilon,j}_{\ell\varepsilon}\varepsilon \le NkTL.$$

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Finally, from $\gamma = (8kTLC/C_a)\delta$, we deduce that

$$\widetilde{\mathbb{P}}_y\left(\frac{2C\delta}{C_a}Q \geq \frac{N\gamma}{2}\right) \leq \widetilde{\mathbb{P}}_y(Q - \widetilde{\mathbb{E}}(Q) > NkTL) \leq \frac{1}{NkTL}.$$

The result follows. □

From Lemma 5, we deduce the next result, whose proof follows the argument of Lemma 3 of [4]. Note, however, the local uniform continuity in the initial condition.

Proposition 1. *For $z \in A$, $\phi \in \mathcal{AC}_{T,A}$ with $\phi_0 = z$, and any $\eta > 0$, $\delta > 0$, there exists $N_{\eta,\delta}$ such that, for all $N > N_{\eta,\delta}$,*

$$\inf_{y:\,|y-z|<\delta/2} \mathbb{P}_y(\|Z^N - \phi\|_T < \delta) \geq \exp\{-N(I_T(\phi) + \eta)\}.$$

Proof. For $\delta, \eta > 0$, let $\phi \in \mathcal{AC}_{T,A}$ with $\phi_0 = z$ be such that $I_T(\phi) < \infty$. Then, from Lemma 2, if $a_\eta > 0$ is small enough, for all $a < a_\eta$ there exists $\phi^a \in R^a$ with $\|\phi - \phi^a\|_T < c_1 a$ and $I_T(\phi^a) \leq I_T(\phi) + \eta/4$. As $I_T(\phi^a) < \infty$, we deduce from Lemma 3 that there exists $L > 0$ such that $\phi^{a,L} \in R^{a/2}$ satisfies $\|\phi^a - \phi^{a,L}\|_T < c_1 a/2$ and $I_T(\phi^{a,L} \mid \mu^{a,L}) \leq I_T(\phi^a) + \eta/4$, where $\mu^{a,L} \in \mathcal{A}_d(\phi^{a,L})$ is such that $\mu_t^{a,L,j} < L$, $j = 1, \ldots, k$. Now we can deduce from Lemma 4 that, for all $\varepsilon > 0$, the polygonal approximation $\phi^{a,L,\varepsilon}$ of $\phi^{a,L}$ satisfies $\|\phi^{a,L} - \phi^{a,L,\varepsilon}\|_T < c_1 a/2$ and $I_T(\phi^{a,L,\varepsilon} \mid \mu^{a,L,\varepsilon}) \leq I_T(\phi^{a,L} \mid \mu^{a,L}) + \eta/4$, where $\mu^{a,L,\varepsilon} \in \mathcal{A}_d(\phi^{a,L,\varepsilon})$ is such that $\mu_t^{a,L,\varepsilon,j} < L$, $j = 1, \ldots, k$. Choosing $a < \delta/(4c_1)$, we have

$$\begin{aligned}
\inf_{y:\,|y-z|<\delta/2} \mathbb{P}_y(\|Z^N - \phi\|_T < \delta) &\geq \inf_{y:\,|y-z|<\delta/2} \mathbb{P}_y\big(\|Z^N - \phi^{a,L,\varepsilon}\|_T < \tfrac{1}{2}\delta\big) \\
&\geq \exp\big\{-N\big(I_T(\phi^{a,L,\varepsilon} \mid \mu^{a,L,\varepsilon}) + \tfrac{1}{4}\eta\big)\big\} \\
&\geq \exp\big\{-N\big(I_T(\phi^{a,L} \mid \mu^{a,L}) + \tfrac{1}{2}\eta\big)\big\} \\
&\geq \exp\{-N(I_T(\phi) + \eta)\},
\end{aligned}$$

where we have used Lemma 5 in the third inequality. □

The following theorem follows rather easily from the previous proposition.

Theorem 4. *For any open subset G of $D_{T,A}$ and $z \in A$,*

$$\liminf_{N\to\infty,\, y\to z} \frac{1}{N} \log \mathbb{P}_y^N(G) \geq - \inf_{\phi\in G,\, \phi_0 = z} I_T(\phi).$$

The next corollary follows as in [3, Corollary 5.6.15].

Corollary 2. *For any open subset G of $D_{T,A}$ and any compact subset K of A,*

$$\liminf_{N\to\infty} \frac{1}{N} \log \inf_{z\in K} \mathbb{P}_z(Z^N \in G) \geq - \sup_{z\in K} \inf_{\phi\in G,\, \phi_0 = z} I_T(\phi).$$

4. The upper bound

If we define $b(x) = \sum_{j=1}^k \beta_j(x) h_j$, $a(x) = 0$, $\mu_x(dv) = \sum_{j=1}^k \beta_j(x) \delta_{h_j}(dv)$, where δ_{h_j} denotes the Dirac measure on $\mathbb{R}^d$ at h_j, we see that we are in the framework of [6], and their assumptions are satisfied. Consequently, the following upper bound is a consequence of their Theorem 1.1.

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Theorem 5. *For any open subset F of $D_{T,A}$ and any compact subset K of A,*

$$\limsup_{N\to\infty} \frac{1}{N} \log \sup_{z\in K} \mathbb{P}_z(Z^N \in F) \le - \inf_{z\in K} \inf_{\phi\in F,\, \phi_0=z} I_T(\phi).$$

5. Time of exit from a domain

Let O be the domain of attraction of a stable point z^* of the dynamical system (3) and $\widetilde{\partial O}$ be the part of boundary of O that the stochastic system (1) can cross. We need to formulate a theorem which gives us an approximate value for the exit time τ_O^N from O for large N as well as the exponential asymptotic of its mean $\mathbb{E}_z(\tau_O^N)$. For the models of infectious disease, τ_O^N is the time to extinction of the disease. This is the most important application of our large deviations result. To this end, for $z, y \in A$, where A is defined by (2), we define the following functionals:

$$V(z, y, T) := \inf_{\substack{\phi\in D_{T,A}\\ \phi_0=z,\, \phi_T=y}} I_T(\phi), \qquad V(z, y) := \inf_{T>0} V(z, y, T), \qquad \bar{V} := \inf_{y\in\widetilde{\partial O}} V(z^*, y).$$

The following theorem is a consequence of the large deviation principle established above, the law of large numbers, and some technical arguments. The proof, which can be found in Section 7 of [12], requires the following technical assumptions.

Assumption 3. (i) *For all $z \in O$, the solution $Y^z(\cdot)$ of ODE (3) satisfies*

$$Y^z(t) \in O \quad \text{for all } t > 0 \quad \text{and} \quad \lim_{t\to\infty} Y^z(t) = z^*.$$

(ii) *We have $\bar{V} < \infty$.*

(iii) *For all $\rho > 0$, there exist constants $T(\rho)$ and $\varepsilon(\rho) > 0$ with $T(\rho), \varepsilon(\rho) \downarrow 0$ as $\rho \downarrow 0$ such that, for all $x \in \widetilde{\partial O} \cup \{z^*\}$ and all $z, y \in B_\rho(x) \cap A$, there exists a $\phi = \phi(\rho, z, y)\colon [0, T(\rho)] \to A$ with $\phi_0 = z$, $\phi_{T(\rho)} = y$, and $I_{T(\rho)}(\phi) < \varepsilon(\rho)$.*

(iv) *For all $z \in \widetilde{\partial O}$, there exists a $\delta_0 > 0$ such that, for all $\delta < \delta_0$, there exists $z^\delta \in A \setminus \bar{O}$ with $|z - z^\delta| > \delta$.*

(v) *There exists a collection $\{O_\rho,\ \rho > 0\}$ such that*

- *$\bar{O}_\rho \subset O$ for all $\rho > 0$;*
- *$\mathrm{d}(O_\rho, \widetilde{\partial O}) \to 0$ as $\rho \to 0$;*
- *for all $\rho > 0$, O_ρ satisfies the four above assumptions and, for all $z \in \partial O_\rho$, the solution $Y^z(\cdot)$ of ODE (3) is such that $\lim_{t\to\infty} Y^z(t) = z^*$.*

Note that all these assumptions are satisfied in the infectious disease models which we will present below. Assumption 3(iii) is not difficult to verify and Assumption 3(iv) allows us to consider a trajectory which crosses the characteristic boundary $\widetilde{\partial O}$, in such a way that all paths in a sufficiently small tube around that trajectory do exit O. Assumption 3(iv) is not satisfied in the first two examples below. However, the result applies, thanks to an argument which is detailed in Section 7 of [12] for the SIRS model.

Theorem 6. *Under Assumptions 1–3, given $\eta > 0$, for all $z \in O$,*

$$\lim_{N\to\infty} \mathbb{P}_z(\exp\{N(\bar{V} - \eta)\} < \tau_O^N < \exp\{N(\bar{V} + \eta)\}) = 1.$$

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Moreover, for all $\eta > 0$, $z \in O$, and large enough N,

$$\exp\{N(\bar{V} - \eta)\} \le \mathbb{E}_z(\tau_O^N) \le \exp\{N(\bar{V} + \eta)\}.$$

5.1. The SIS model

We consider a population of fixed size N, which is composed of susceptible and infected individuals. The proportion of infected individuals obeys the SDE

$$I^N(t) = I^N(0) + \frac{1}{N} P_1\left(N\beta \int_0^t I^N(s)(1 - I^N(s))\,\mathrm{d}s\right) - \frac{1}{N} P_2\left(N\alpha \int_0^t I^N(s)\,\mathrm{d}s\right),$$

where β is the rate at which infected individuals infect susceptibles, and α the rate at which an infected individual recovers. In this case $O = \mathring{A} = (0, 1)$. We assume that $\beta > \alpha$, in which case the law of large number ODE limit

$$\frac{\mathrm{d}i}{\mathrm{d}t}(t) = \beta i(t)(1 - i(t)) - \alpha i(t)$$

has the unique stable equilibrium $i^* = 1 - \alpha/\beta$, which is the endemic equilibrium. Our results show that the time taken by the random perturbation to estinguish the disease is of the order of $\exp(N\bar{V})$, where $\bar{V} = \inf_{T>0} \inf_{\phi_0 = i^*,\, \phi_T = 0} I_T(\phi)$ (Assumption 3 is not satisfied, but the needed extension is easy to justify). This is the value function of an optimal control problem, which in this case can be computed explicitly, and we obtain $\bar{V} = \log(\beta/\alpha) - 1 + \alpha/\beta$. Note that the optimal T is infinite, and $\bar{V}$ depends only upon the ratio $R_0 = \beta/\alpha$, which is called the 'basic reproduction number,' and it is an increasing function of that ratio. We refer the reader to [2] for more details.

5.2. The SIRS model

In this model the individuals who recover are 'retired': they cannot be infected, until they lose their immunity. The model becomes

$$I^N(t) = I^N(0) + \frac{1}{N} P_1\left(N\beta \int_0^t I^N(s)(1 - I^N(s) - R^N(s))\,\mathrm{d}s\right) - \frac{1}{N} P_2\left(N\alpha \int_0^t I^N(s)\,\mathrm{d}s\right),$$

and

$$R^N(t) = R^N(0) + \frac{1}{N} P_1\left(N\alpha \int_0^t I^N(s)\,\mathrm{d}s\right) - \frac{1}{N} P_3\left(N\gamma \int_0^t R^N(s)\,\mathrm{d}s\right).$$

Again, if the basic reproduction number $R_0 = \beta/\alpha > 1$, the law of large numbers ODE

$$\frac{\mathrm{d}i}{\mathrm{d}t}(t) = \beta i(t)(1 - i(t) - r(t)) - \alpha i(t), \qquad \frac{\mathrm{d}r}{\mathrm{d}t}(t) = \alpha i(t) - \gamma r(t)$$

has a unique stable endemic equilibrium $(i^*, r^*) = ((\beta - \alpha)/\beta(\alpha + \gamma))(\gamma, \alpha)$. Here $A = \{i \ge 0, r \ge 0 \colon i + r \le 1\}$ and $O = \mathring{A}$. As shown in [12], although Assumption 3 is again not quite satisfied, Theorem 6 applies. Here, unfortunately, it does not seem possible to compute explicitly the quantity $\bar{V}$.

5.3. The SIV model

Consider a model with vaccination for a population of fixed size N, composed of susceptible, infected, and vaccinated individuals. We assume that the vaccinated individuals are not

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fully protected, and can be infected, but this happens at a smaller rate than the infection of susceptibles. They can also loose their partial immunity and become susceptible again. In each class of individuals, there is the same death rate. The proportions of infected and vaccinated individuals follow the following SDE:

$$\begin{aligned} I^N(t) &= I^N(0) + \frac{1}{N}P_1\left(N\beta\int_0^t I^N(s)(1-I^N(s)-V^N(s))\,\mathrm{d}s\right) \\ &\quad + \frac{1}{N}P_2\left(N\chi\beta\int_0^t I^N(s)V^N(s)\,\mathrm{d}s\right) - \frac{1}{N}P_3\left(N\gamma\int_0^t I^N(s)\,\mathrm{d}s\right) \\ &\quad - \frac{1}{N}P_6\left(N\mu\int_0^t I^N(s)\,\mathrm{d}s\right), \\ V^N(t) &= V^N(0) - \frac{1}{N}P_2\left(N\chi\beta\int_0^t I^N(s)V^N(s)\,\mathrm{d}s\right) - \frac{1}{N}P_4\left(N\theta\int_0^t I^N(s)V^N(s)\,\mathrm{d}s\right) \\ &\quad + \frac{1}{N}P_5\left(N\eta\int_0^t (1-I^N(s)-V^N(s))\,\mathrm{d}s\right) - \frac{1}{N}P_7\left(N\mu\int_0^t V^N(s)\,\mathrm{d}s\right). \end{aligned}$$

Under appropriate assumptions on the parameters of the model (see [13]), its law of large numbers ODE limit

$$\begin{aligned} \frac{\mathrm{d}i}{\mathrm{d}t}(t) &= (\beta-\mu-\gamma)i(t) - \beta(1-\chi)i(t)v(t) - \beta i^2(t), \\ \frac{\mathrm{d}v}{\mathrm{d}t}(t) &= \eta - \eta i(t) - (\eta+\mu+\theta)v(t) - \chi\beta i(t)v(t) \end{aligned}$$

has two endemic equilibria $z^* = (z_1^*, z_2^*)$ and $\tilde{z} = (\tilde{z}_1, \tilde{z}_2)$. Also z^* is locally stable while $\tilde{z}$ is unstable. These two equilibria are completed with the disease free equilibrium $\bar{z}$ ($\bar{z}_1 = 0$, $\bar{z}_2 = \eta/\mu + \theta + \eta$) which is locally stable. In Figure 1 we show the basin of attraction O of the equilibrium z^* delimited by the characteristic boundary and containing the point z^*. Here $A = \{i \geq 0,\ v \geq 0 \colon i + v \leq 1\}$. Assumptions 1–3 are satisfied and Theorem 6 applies. Again it is not possible to compute explicitly the quantity $\bar{V}$ but its numerical computation will be the object of a forthcoming publication.

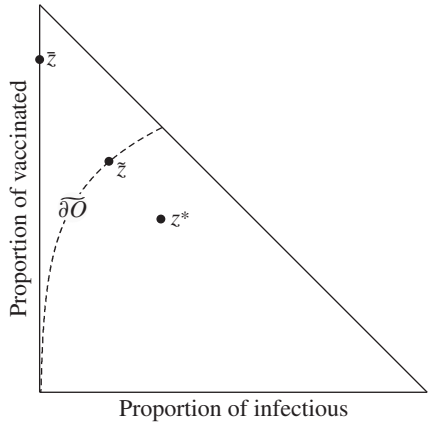


FIGURE 1: Attraction region O of the equilibrium z^* in the SIV model.

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5.4. The S_0IS_1 model

We consider here a model with two levels of susceptibility in which the population has a fixed size N composed of susceptibles S_0 who are individuals who have never been infected and may contract the infection, the infected I, and the susceptibles S_1 with at least one past infection. We assume that the infected individuals can recover and become again susceptible (of type S_1), and be infected with a rate which is different from that of the type S_0 individuals. The death rate is the same in all classes. The proportions of infected and vaccinated individuals follow the following SDE:

$$\begin{aligned} I^N(t) &= I^N(0) + \frac{1}{N}P_1\left(N\beta\int_0^t I^N(s)(1 - I^N(s) - S_1^N(s))\,\mathrm{d}s\right) \\ &\quad - \frac{1}{N}P_2\left(N\alpha\int_0^t I^N(s)\,\mathrm{d}s\right) \\ &\quad - \frac{1}{N}P_3\left(N\mu\int_0^t I^N(s)\,\mathrm{d}s\right) + \frac{1}{N}P_4\left(Nr\beta\mu\int_0^t I^N(s)S_1^N(s)\,\mathrm{d}s\right), \\ S_1^N(t) &= S_1^N(0) + \frac{1}{N}P_2\left(N\alpha\int_0^t I^N(s)\,\mathrm{d}s\right) - \frac{1}{N}P_4\left(Nr\beta\mu\int_0^t I^N(s)S_1^N(s)\,\mathrm{d}s\right) \\ &\quad - \frac{1}{N}P_5\left(N\mu\int_0^t S_1^N(s)\,\mathrm{d}s\right). \end{aligned}$$

If the parameters of this model are chosen in an appropriate way (see [16]), its law of large number ODE limit

$$\begin{aligned} \frac{\mathrm{d}i}{\mathrm{d}t}(t) &= -(\alpha + \mu - \beta)i(t) + (r-1)\beta i(t)s_1(t) - \beta i^2(t), \\ \frac{\mathrm{d}s_1}{\mathrm{d}t}(t) &= \alpha i(t) - \mu s_1(t) - r\beta i(t)s_1(t), \end{aligned}$$

there exist two positive endemic equilibria; the first one z^* is locally asymptotically stable and the second one $\tilde{z}$ is unstable, in addition to the disease free equilibrium $\bar{z} = (0, 0)$ which is again locally asymptotically stable. In Figure 2 we show the basin of attraction O of the

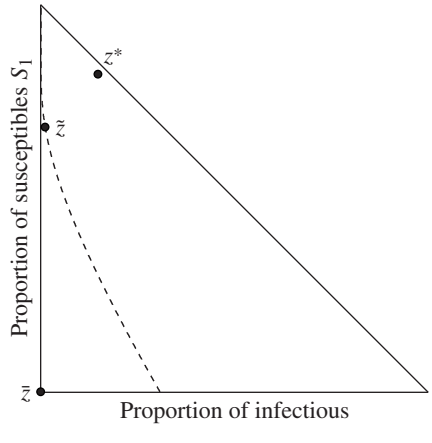


FIGURE 2: Attraction region O of the equilibrium in the S_0IS_1 model.

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equilibrium z^* delimited by the characteristic boundary and containing the point z^*. Here $A = \{i \geq 0,\ s_1 \geq 0 \colon i + s_1 \leq 1\}$. Assumptions 1–3 are satisfied and Theorem 6 applies.

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