Launching a projectile to cover maximal area

ROBERT KANTROWITZ and MICHAEL M. NEUMANN

1. *Introduction and motivation*

The launch and subsequent motion of a projectile provide a context for several quantities that yearn to be optimised. Most notable is the horizontal range of the projectile, a problem dating back to Galileo and still studied in modern times; see, for example [1], [2], [3], [4]. In a different direction, the articles [5] and [6] provide a solution to the problem of finding the angle of launch that results in the trajectory of longest arc length.

Here our goal is to launch a projectile through a non-resistive medium from the top of a tower with an angle of elevation that will yield the greatest area under its flight path. Problems concerning maximization of area date back to antiquity. In the 9th century B.C. Queen Dido needed to find, among all curves of given length, the one that enclosed the maximal area (section 1.3 of [7]). The modern interpretation of this seminal problem in the calculus of variations is the optimisation of a certain integral function subject to the canonical isoperimetric constraint. Troutman (section 1.3 of [7]) mentions the more current problem, due to Chaplygin, in which a pilot on reconnaissance seeks a flight path that will enclose maximal area. While the problem that we undertake may be similarly construed, our work entails nothing but elementary calculus.

We offer several natural approaches and compare the critical point equations that arise for each. It turns out that the tractability of the problem depends heavily on the choice of the method. Moreover, access to computer algebra systems could even influence one's perspective on the problem, since, as we already encountered in [2], Mathematica and Maple differ in their capabilities to solve the equations that arise. When the dust settles, we find that, among the competing techniques for handling the problem, a formula of Leibniz for *differentiation under the integral sign* leads quickly to a surprisingly simple expression for the optimal angle of launch. This was also the key for dealing with maximal arc length in [5] and [6]. A slight recasting allows for an application of the *Lagrange multiplier theorem*, to which the problem also responds.

2. *Framework for projectile motion*

We suppose throughout that the projectile is launched from a tower of height $h > 0$ at time $t = 0$ with muzzle speed $s > 0$. The initial position of the projectile is thus $(0, h)$, and we denote by $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ its angle of launch with respect to the horizontal line $y = h$ in the direction of the positive x-axis. We work under the assumption that the motion of the projectile is governed only by constant acceleration of magnitude $g > 0$ due to gravity in the direction of the negative y-axis. The position $\big(x(t, \theta), y(t, \theta)\big)$ of the projectile at time $t \geqslant 0$ is then given by the familiar

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parametric equations

$$x(t, \theta) = s \cos\theta\, t \qquad \text{and} \qquad y(t, \theta) = h + s \sin\theta\, t - \tfrac{1}{2}gt^2 \quad (1)$$

valid for all t for which the projectile remains airborne; specifically, the projectile stays in the air until it hits ground level $y = 0$. Elimination of the parameter t in (1) shows that the trajectory $\big(x(t, \theta), y(t, \theta)\big)$ traced by the projectile is a segment of the parabola

$$y = h + \tan\theta\, x - \frac{g}{2s^2} \sec^2\theta\, x^2.$$

3. *Fermat*

It is convenient at this early stage to sidestep explicit use of the trigonometric functions by invoking the substitution $u = \tan\theta$. Coupled with the assignment $c = \dfrac{g}{2s^2}$, the identity $\sec^2\theta = 1 + \tan^2\theta$ allows us to express a typical flight parabola as

$$p(x, u) = h + ux - c\big(1 + u^2\big)x^2, \quad (2)$$

for any real value of u and any $x \geqslant 0$. The solution of the quadratic equation $p(x, u) = 0$ for positive x is given by

$$d(u) = \frac{u + \sqrt{u^2 + 4ch\big(1 + u^2\big)}}{2c\big(1 + u^2\big)}, \quad (3)$$

marking the projectile's landing point $\big(d(u), 0\big)$ on the positive part of the x-axis.

The problem that we undertake is to maximise the integral function

$$A(u) = \int_0^{d(u)} p(x, u)\, dx = \int_0^{d(u)} \big(h + ux - c\big(1 + u^2\big)x^2\big)\, dx$$

for $-\infty < u < \infty$. Evaluating the integral is elementary because, after all, the integrand is nothing more than a quadratic polynomial in the variable x. Proceeding in this way gives rise to the formula

$$A(u) = h\, d(u) + \frac{u\, d(u)^2}{2} - \frac{c\big(1 + u^2\big)\, d(u)^3}{3}. \quad (4)$$

Our intention is to apply *Fermat's theorem* – a standard approach from calculus ensuring that a local extremum for A can only occur at a critical point. Using the definition of $d(u)$, we obtain the explicit representation

$$A(u) = \frac{u^2\big(u + \sqrt{u^2 + 4ch(1 + u^2)}\big) + 2ch\big(1 + u^2\big)\big(3u + 2\sqrt{u^2 + 4ch(1 + u^2)}\big)}{12c^2(1 + u^2)^2}.$$

Inasmuch as we are not enthusiastic about computing $A'(u)$ and solving the critical point equation $A'(u) = 0$ by hand, we turn to computer algebra systems. Both Mathematica and Maple provide expressions for $A'(u)$ that are long and complicated. After 75 minutes of running time, version 9 of Mathematica did not return a solution to the critical point equation, so the

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process was aborted. On the other hand, Maple 16 immediately returns the two solutions

$$u = \frac{3}{\sqrt{16hc + 3}} \quad \text{and} \quad u = \frac{-3}{\sqrt{16hc + 3}}, \tag{5}$$

of which only one turns out to be correct, as we shall see. Version 11 of Mathematica performs more capably by offering the two real solutions that Maple does, along with two imaginary solutions. So we need to explore different approaches.

4. *Archimedes*

In a remarkable discovery that predates Leibniz and Newton by some 2000 years, Archimedes applied Eudoxus' *method of exhaustion* to relate the area of a parabolic segment to the area of a certain inscribed triangle. The approach, described in [8, section 1.3] and [9, Chapter 1], entails summing a geometric series, anticipating yet another aspect of modern calculus. Inspired by Archimedes, rather than representing the area under the path of the projectile by an integral, we endeavour to express it in terms of the areas of certain triangles, hoping to arrive at a more manageable expression.

The classical result due to Archimedes may be stated as follows. Given a parabola ρ and real numbers α and β with $\alpha < \beta$, the area of the region R between the graph of ρ and the line segment from $(\alpha, \rho(\alpha))$ to $(\beta, \rho(\beta))$ is four thirds times the area of the triangle Δ with vertices $(\alpha, \rho(\alpha))$, $(\beta, \rho(\beta))$ and $\left(\frac{1}{2}(\alpha + \beta), \rho\left(\frac{1}{2}(\alpha + \beta)\right)\right)$.

Now it is well-known from vector calculus that the area of Δ may be found by computing half the length of the cross product of the vectors from $(\alpha, \rho(\alpha), 0)$ to $(\beta, \rho(\beta), 0)$ and from $(\alpha, \rho(\alpha), 0)$ to $\left(\frac{1}{2}(\alpha + \beta), \rho\left(\frac{1}{2}(\alpha + \beta)\right), 0\right)$, which amounts to half of the absolute value of the determinant

$$\begin{vmatrix} \beta - \alpha & \rho(\beta) - \rho(\alpha) \\ \frac{\alpha+\beta}{2} - \alpha & \rho\left(\frac{\alpha+\beta}{2}\right) - \rho(\alpha) \end{vmatrix}.$$

Thus, the area of R is two thirds of the absolute value of the preceding determinant.

We apply the result of Archimedes to $\alpha = 0$, $\beta = d(u)$, and $\rho(x) = p(x, u)$ for fixed real u.

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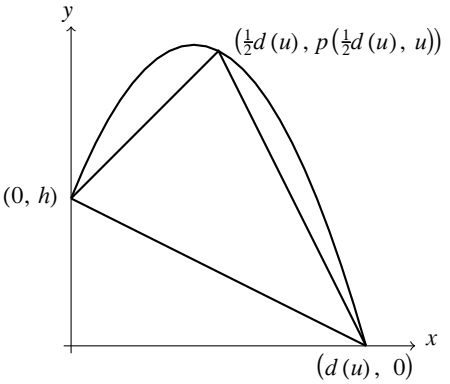


FIGURE 1

In this case, the above determinant becomes

$$\begin{vmatrix} d(u) & p(d(u),\, u) - p(0,\, u) \\ \frac{1}{2}d(u) & p\left(\frac{1}{2}d(u),\, u\right) - p(0,\, u) \end{vmatrix}. \tag{6}$$

To simplify the entries in the second column, we note that, by definition, $p\left(d(u),\, u\right) = 0$ for all u, which can be expressed as the operative identity

$$h + ud(u) = c\left(1 + u^2\right)d(u)^2. \tag{7}$$

From this it follows easily that

$$p\left(\frac{d(u)}{2},\, u\right) - p(0,\, u) = \frac{u\,d(u)}{2} - \frac{c\left(1 + u^2\right)d(u)^2}{4} = \frac{u\,d(u) - h}{4}$$

for all real u. The absolute value of the determinant in (6) is thus

$$\left| d(u)\frac{u\,d(u) - h}{4} + h\frac{d(u)}{2} \right| = \frac{h\,d(u) + u\,d(u)^2}{4}.$$

Finally, taking also the area of the triangle with vertices (0, 0), (0, h) and $\left(d(u),\, 0\right)$ into account, we arrive at the following counterpart of formula (4):

$$A(u) = \frac{h\,d(u)}{2} + \frac{4}{3}\cdot\frac{1}{2}\cdot\frac{h\,d(u) + u\,d(u)^2}{4} = \frac{2h\,d(u)}{3} + \frac{u\,d(u)^2}{6}. \tag{8}$$

Before proceeding, we incorporate the operative identity (7) to reconcile the apparent difference between formulas (4) and (8). Substitution into (4) leads to

$$h\,d(u) + \frac{u\,d(u)^2}{2} - \frac{(h + u\,d(u))\,d(u)}{3}$$

which evidently simplifies to (8). The computation of $A'(u)$ is less intimidating when the area function appears in the form (8), but we are still unenthusiastic about the prospect of solving the resulting critical point

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equation for u:

$$A'(u) = \frac{2h\,d'(u)}{3} + \frac{u\,d(u)\,d'(u)}{3} + \frac{d(u)^2}{6} = 0$$

Whereas neither version 9 nor version 11 of Mathematica is able to solve this equation for u, Maple 16 returns the same results (5) that it does for the Fermat approach in the preceding section. We switch to an important theorem of Leibniz.

5. *Leibniz*

Since the functions p and d are continuously differentiable, a classical theorem due to Leibniz about *differentiation under the integral sign* [7, Theorem A.14] allows us to obtain the derivative of A in the form

$$A'(u) = \int_0^{d(u)} p_u(x, u)\,dx + d'(u)\,p\big(d(u), u\big),$$

where p_u denotes the partial derivative of p with respect to u.

Because $p\big(d(u), u\big) = 0$ for all real numbers u, this immediately simplifies to

$$A'(u) = \int_0^{d(u)} p_u(x, u)\,dx$$

which turns out to be a gateway to a solution to our area problem. Continuing,

$$A'(u) = \int_0^{d(u)} \big(x - 2cux^2\big)dx = \frac{d(u)^2}{2} - \frac{2cu\,d(u)^3}{3} = \frac{d(u)^2}{6}\big(3 - 4cud(u)\big).$$

The fact that $d(u) > 0$ for all numbers u implies that $A'(u) = 0$ precisely when u satisfies $3 - 4cu\,d(u) = 0$. This simple equation reveals that any value of u for which $A'(u) = 0$ must satisfy $u > 0$, drawing attention to the fact that the second solution in (5) that was provided by Maple is not possible. Moreover, it lends the insight that solutions to the critical point equation are points of intersection of the distance function d and the branch of the hyperbola given by the formula $\frac{3}{4cu}$ for $u > 0$.

Beyond this information, the algebra involved to proceed is straightforward. Indeed,

$$\begin{aligned} & 3 = 4cu\,d(u) \\ \Leftrightarrow \quad & \frac{3}{4cu} = \frac{u + \sqrt{u^2 + 4ch(1 + u^2)}}{2c(1 + u^2)} \\ \Leftrightarrow \quad & 3 + u^2 = 2u\sqrt{u^2 + 4ch(1 + u^2)}. \end{aligned}$$

Squaring both sides and collecting terms we obtain the equation

$$(3 + 16ch)u^4 + (-6 + 16ch)u^2 - 9 = 0.$$

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With the assignments $v = u^2$ and $k = 3 + 16ch$, we solve the quadratic equation

$$kv^2 + (k - 9)v - 9 = 0$$

to arrive at

$$v = \frac{(9 - k) \pm \sqrt{(k - 9)^2 + 36k}}{2k} = \frac{(9 - k) \pm \sqrt{(k + 9)^2}}{2k},$$

which reduces to

$$v = \frac{9}{k} \qquad \text{or} \qquad v = -1.$$

Returning to u, c and h confirms that u must satisfy $u^2 = \dfrac{9}{3 + 16ch}$ and, because $u > 0$, we find that

$$u = \frac{3}{\sqrt{3 + 16ch}}$$

is the only solution to the critical point equation $A'(u) = 0$.

It is apparent from (3) that the positive function d tends to 0 as u approaches ∞ or $-\infty$, so that a glance at either of the expressions (4) or (8) reveals that the positive function A also tends to 0 as u approaches ∞ or $-\infty$. The sole critical point for A must therefore provide the absolute maximum for A on $(-\infty, \infty)$. Moreover, A is strictly increasing on $\left(-\infty, \dfrac{3}{\sqrt{3 + 16ch}}\right)$ and strictly decreasing on $\left(\dfrac{3}{\sqrt{3 + 16ch}}, \infty\right)$.

Finally, returning to the original variable θ, we find that the unique optimal angle of launch for maximising the area under the projectile is the positive angle

$$\theta_{\text{opt}} = \arctan\left(\frac{3}{\sqrt{3 + 16ch}}\right) = \arctan\left(\frac{3s}{\sqrt{3s^2 + 8gh}}\right).$$

Note that the optimal angle depends only on the dimensionless constant $ch = \dfrac{gh}{2s^2}$ and is, in fact, a decreasing function of ch. In particular, it follows that θ_{opt} is an increasing function of the initial speed s provided that g and h are kept fixed. Considering for a moment the case of shooting from ground level to ground level, allowing $h \to 0^+$ in this formula suggests that $\theta_{\text{opt}} = \arctan\left(\sqrt{3}\right) = \frac{\pi}{3}$. This may be verified directly by recasting the original problem with the simplifications $h = 0$ and $u = \tan\theta$ under the natural restriction $0 < \theta < \frac{\pi}{2}$.

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6. *Lagrange*

Another formulation of the problem at hand entails maximising the function $A(d,u) = \int_0^d p(x,u)dx$ on the open half plane $(0, \infty) \times (-\infty, \infty)$ subject to the constraint $p(d, u) = 0$. Posing the problem this way summons the *Lagrange multiplier theorem* which leads to the system of equations

$$\begin{aligned} A_d(d, u) &= \lambda p_d(d, u) \\ A_u(d, u) &= \lambda p_u(d, u) \\ p(d, u) &= 0, \end{aligned}$$

for positive d and real numbers u and λ. By the fundamental theorem of calculus and another application of differentiation under the integral sign, the preceding equations may be rewritten in the form

$$p(d, u) = \lambda\left(u - 2cd\left(1 + u^2\right)\right) \tag{9}$$

$$\int_0^d p_u(x, u)\, dx = \lambda\left(d - 2cud^2\right) \tag{10}$$

$$p(d, u) = 0. \tag{11}$$

Now, if $\lambda \neq 0$, then equations (9) and (11) lead immediately to the conclusion that $u - 2cd\left(1 + u^2\right) = 0$. In particular, this implies that $u > 0$. Writing equation (11) as $h + ud - c\left(1 + u^2\right)d^2 = 0$, we find that

$$\frac{h + ud}{cd^2} = 1 + u^2.$$

Substitution yields

$$u - 2cd\left(\frac{h + ud}{cd^2}\right) = 0,$$

whereupon one is led to the equation

$$2h + ud = 0,$$

a contradiction since h, u and d are all greater than 0. We conclude that $\lambda = 0$.

In this case, we begin by interpreting equation (11) as providing the information that $d = d(u)$. Equation (10) then informs us that

$$\int_0^{d(u)} p_u(x, u)\, dx = 0,$$

which is exactly the critical point equation that was obtained by applying the Leibniz formula in the preceding section. The subsequent steps to the solution are thus the same.

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ROBERT KANTROWITZ

Department of Mathematics, Hamilton College, 198 College Hill Road, Clinton, NY 13323, USA

e-mail: *rkantrow@hamilton.edu*

MICHAEL M. NEUMANN

Department of Mathematics and Statistics, Mississippi State University, Mississippi State, MS 39762, USA

e-mail: *neumann@math.msstate.edu*

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