Visualising alternating geometric series

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When first learning about geometric series, students often wonder why these series are termed ‘geometric’. The geometry of some geometric series is readily apparent for some common series such as

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots .$$

This sum can readily be visualised as 1 by presenting students with a square of side length 1 and considering the sum as the sum of the areas of the rectangular regions that make up the square as given in Figure 1.

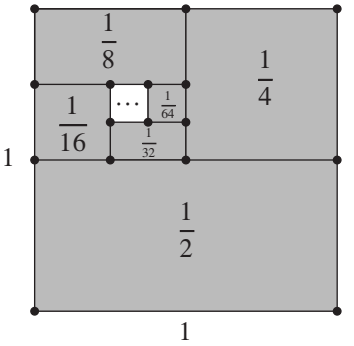


FIGURE 1: Visualising the sum $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$

But what of other geometric sums? The reader might pause for a moment and consider what other infinite sums may be visualised geometrically. General methods are known for visualising any alternating geometric series of the form $\sum_{n=0}^{\infty}(-r)^n$ for $r \in (0, 1)$, using self-similarity using triangles (see [1]) and or integrals that approximate improper integrals (see [2]). In this Article, we offer several alternating geometric sum visualisations using self-similarity with different classes of geometric figures and indicate a method for visualising the sum of others by generalisation.

We consider alternating geometric series of the form

$$\sum_{n=0}^{\infty}\left(-\frac{a}{b}\right)^n$$

where a and b are positive integers and $a < b$. Here, we are saying that the common ratio is $-\frac{a}{b}$ and that the series is convergent since $\left|-\frac{a}{b}\right| < 1$.

Numerically, these series can be shown to be convergent by the usual formula for a convergent geometric series:

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$$\sum_{n=0}^{\infty}\left(-\frac{a}{b}\right)^n = \frac{1}{1-\left(-\frac{a}{b}\right)} = \frac{1}{1+\frac{a}{b}} = \frac{1}{\frac{b+a}{b}} = \frac{b}{a+b}.$$

The sums of many of these series may be visualised using self-similar geometric figures.

Sums with common ratio $-\frac{a}{4}$

We claim that all sums with common ratio $-\frac{a}{4}$ with $a = 1, 2, 3$ may be visualised geometrically using self-similar subsets of triangles. As an example, we offer $a = 1$ and leave $a = 2, 3$ for the reader.

Consider the sum $\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n$, presented by Sinha [3]. To represent one unit of area, we begin with a triangle together with a similar triangle having area a quarter of the area of the first, as illustrated in Figure 2. Note that the larger triangle has area four-fifths.

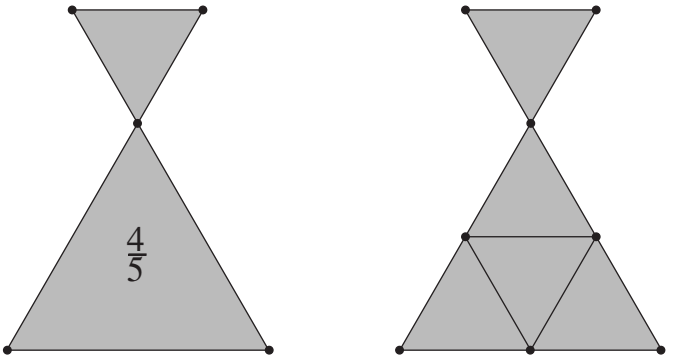


FIGURE 2: The unit whole with the larger triangle $\frac{4}{5}$ the whole.

Now we remove the midpoint triangles from the larger and the smaller triangles, thereby removing a quater the area of each triangle and hence, the unit whole. Then, we add the midpoint triangles of those removed in the last iteration, increasing the area of each (and hence the area of the whole) by one-sixteenth. This process continues ad infinitum and is illustrated in Figure 4 with the sum equalling the area of the final shaded region.

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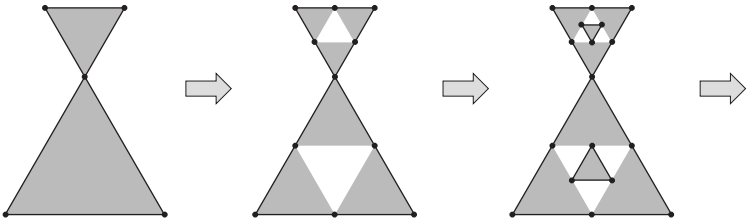


FIGURE 3: Visualising the sum $\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n$.

We notice that the areas removed from the larger triangle are equal to the areas added to the smaller. Now, to visualise the sum, we consider the figure after many iterations. Since the area removed from the larger triangle equals the area added to the smaller, we may visualise the smaller triangle being translated to coincide with the midpoint triangle of the larger triangle, for a final sum (the shaded area) equal to the area of the larger triangle, or four-fifths, illustrated in Figure 4. So,

$$1 - \frac{1}{4} + \frac{1}{16} - \ \ldots = \sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n = \frac{4}{5}.$$

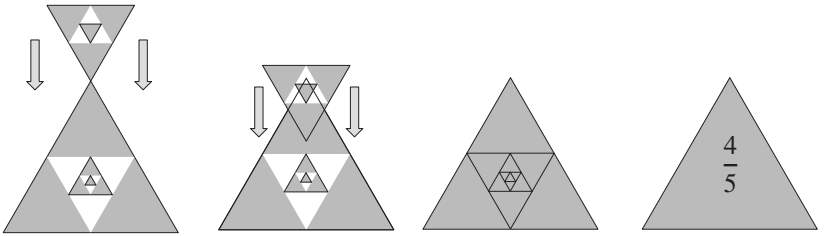


FIGURE 4: Visualising the final sum

Sums with common ratio $-\dfrac{a}{9}$

We claim that all sums with common ratio $-\frac{a}{9}$ with $a = 1, 2, 3, \ldots, 8$ may be visualised geometrically using self-similar subsets of squares. As examples, we offer $a = 1$ and $a = 4$, and leave other examples for the reader.

Consider the sum $\sum_{n=0}^{\infty}\left(-\frac{1}{9}\right)^n$ illustrated in Figure 5.

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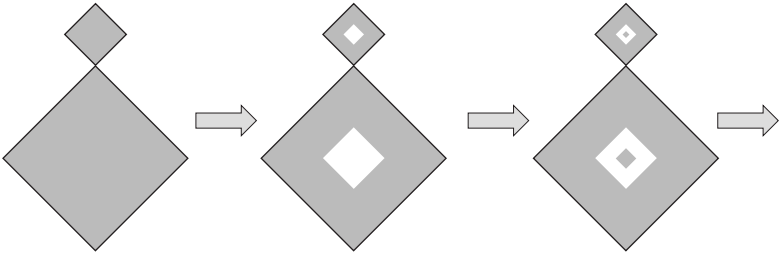


FIGURE 5: Visualising the sum $\sum_{n=0}^{\infty}\left(-\frac{1}{9}\right)^n$

The continuation of this figure is a visualisation of the sum

$$1 - \frac{1}{9} + \frac{1}{81} - \ldots = \sum_{n=0}^{\infty}\left(-\frac{1}{9}\right)^n = \frac{9}{10}.$$

We now consider the sum $\sum_{n=0}^{\infty}\left(-\frac{4}{9}\right)^n$ illustrated in Figure 6.

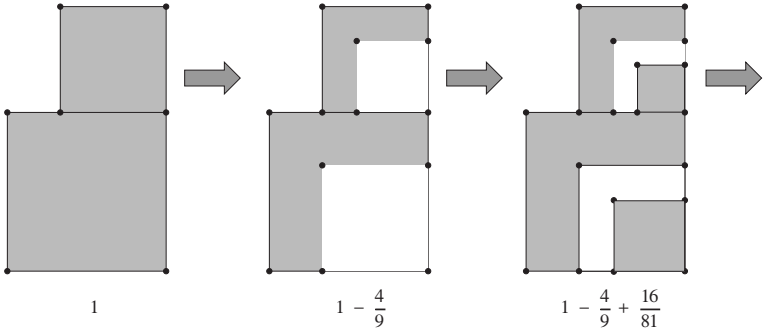


FIGURE 6: Visualising the sum $\sum_{n=0}^{\infty}\left(-\frac{4}{9}\right)^n$

The continuation of this figure is a visualisation of the sum

$$1 - \frac{4}{9} + \frac{16}{81} - \ldots = \sum_{n=0}^{\infty}\left(-\frac{4}{9}\right)^n = \frac{9}{13}.$$

Sums with common ratio $-\dfrac{a}{27}$

We claim that all sums with common ratio $-\frac{a}{27}$, where a is a positive integer with $a < 27$ may be visualised geometrically using self-similar subsets of cubes. As an example, we offer $a = 2$ and leave other examples for the reader.

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Consider the sum $\sum_{n=0}^{\infty}\left(-\frac{2}{27}\right)$ illustrated in Figure 7.

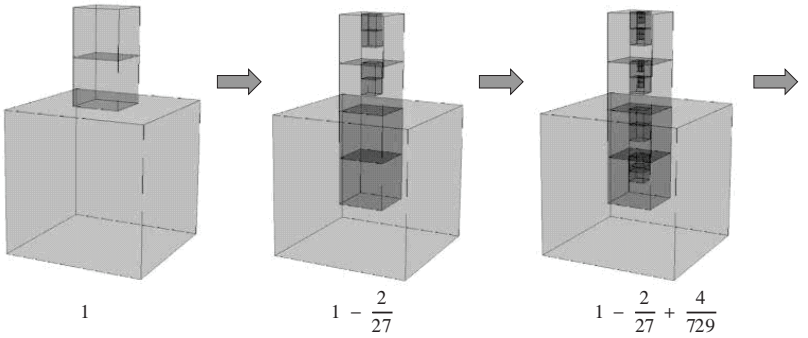


FIGURE 7: Visualising the sum $\sum_{n=0}^{\infty}\left(-\frac{2}{27}\right)^n$

The continuation of this figure is a visualisation of the sum

$$1 - \frac{2}{27} + \frac{4}{729} - \ldots = \sum_{n=0}^{\infty}\left(-\frac{2}{27}\right)^n = \frac{27}{29}.$$

Rep-tiles: Self-replicating tiles

A *rep-tile* (or *reptile*) is a shape that can be dissected into smaller copies of the same shape and is labelled rep-n if the dissection uses n copies [4]. We label the rep-tile regular if the n copies are congruent. Note that rep-tiles are self-similar, allowing for this visualisation.

Consider the regular rep-4 rep-tile illustrated in Figure 8.

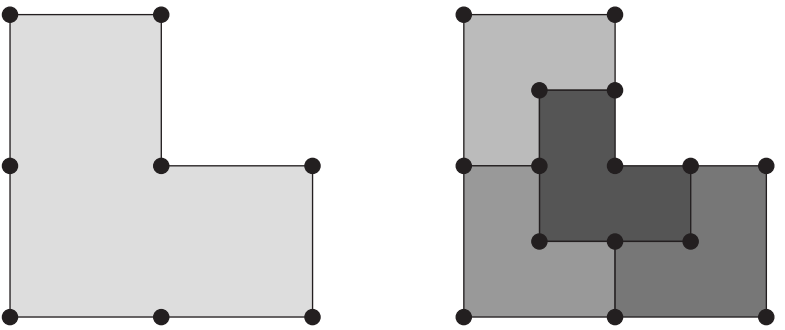


FIGURE 8: Regular rep-4 rep-tile and its decomposition

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If we combine this rep-tile with 1 copy of the smaller similar rep-tile, we can visualise a geometric sum of the form $\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)$ illustrated in Figure 9.

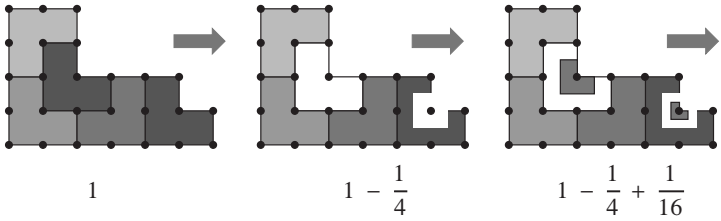


FIGURE 9: Visualising the sum $\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n$

The continuation of this figure is a visualisation of the sum

$$1 - \frac{1}{4} + \frac{1}{16} - \ \ldots = \sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n = \frac{4}{5}.$$

Consider the regular rep-5 rep-tile (right triangle) illustrated in Figure 10 with sides in ratio $1 : 2 : \sqrt{5}$.

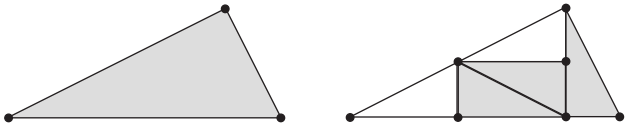


FIGURE 10: Regular rep-5 rep-tile and its decomposition

If we combine this rep-tile with one copy of the smaller similar rep-tile, we can visualise a geometric sum of the form $\sum_{n=0}^{\infty}\left(-\frac{1}{5}\right)^n$ illustrated in Figure 11.

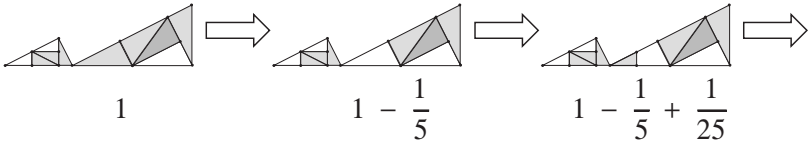


FIGURE 11: Visualising the sum $\sum_{n=1}^{\infty}\left(-\frac{1}{5}\right)^n$

The continuation of this figure is a visualisation of the sum

$$1 - \frac{1}{5} + \frac{1}{25} - \ \ldots = \sum_{n=0}^{\infty}\left(-\frac{1}{5}\right)^n = \frac{5}{6}.$$

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General case

This can be generalised (without proof) as follows:

Consider any rep-b rep-tile and choose any positive integer a with $a < b$. The rep-b rep-tile together with a copies of the smaller rep-tile allows one to visualise a series having form

$$\sum_{n=0}^{\infty} \left(-\frac{a}{b}\right)^n.$$

So any series of this form may be visualised given the existence of a rep-b rep-tile. All previous sums thus far have been visualised used rep-tiles.

Here is a place for rich exploration. As an example, how might the reader represent the following sum visually using rep-tiles?

$$1 - \frac{2}{5} + \frac{4}{25} - \frac{8}{125} \pm \ldots .$$

As another example, what sum does the sphinx rep-tile with two additional smaller copies in Figure 12 allow one to visualise?

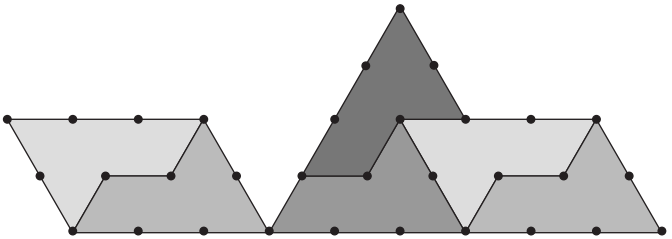


FIGURE 12: Sphinx rep-tile suggesting the sum of an alternating series

Further explorations

While using rep-tiles to visualise alternating geometric series seems to be sufficient, it is not necessary. For example, consider a circle of radius r. Then, it is easily seen that a circle of radius $\frac{1}{2}r$ has a quarter the area of the circle of radius r. So sums with common ratio $-\frac{1}{4}a$ may be visualised using circles.

Consider the sum $\sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n$ illustrated in Figure 13.



1 $\qquad$ $1 - \frac{1}{4}$ $\qquad$ $1 - \frac{1}{4} + \frac{1}{16}$

FIGURE 13: Visualising the sum $\sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n$

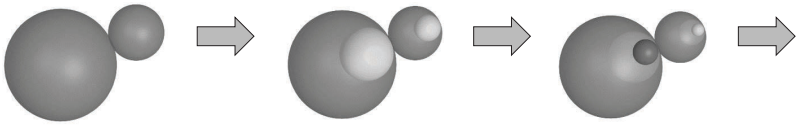
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The continuation of this figure is a visualisation of the sum

$$1 - \frac{1}{4} + \frac{1}{16} - \ldots = \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n = \frac{4}{5}.$$

Now consider a sphere of radius r. Then it is easily seen that a sphere of radius $\frac{1}{2}r$ has one-eighth the volume of the sphere of radius r. So sums with common ratio $-\frac{1}{8}a$ may be visualised using spheres.

Consider the sum $\sum_{n=0}^{\infty}\left(-\frac{1}{8}\right)^n$ illustrated in Figure 14.



1 $\qquad$ $1 - \frac{1}{8}$ $\qquad$ $1 - \frac{1}{8} + \frac{1}{64}$

FIGURE 14: Visualising the sum $\sum_{n=0}^{\infty}\left(-\frac{1}{8}\right)^n$

The continuation of this figure is a visualisation of the sum

$$1 - \frac{1}{8} + \frac{1}{64} - \ldots = \sum_{n=0}^{\infty} \left(-\frac{1}{8}\right)^n = \frac{8}{9}.$$

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