Math. Proc. Camb. Phil. Soc. (2001), **131**, 507

Printed in the United Kingdom © 2001 *Cambridge Philosophical Society*

The quantization dimension of distributions

By KLAUS PÖTZELBERGER
Institut für Statistik, WU Wien, Augasse 2-6, *A*-1090 *Wien, Austria.*

(*Received* 2 *November* 1998; *revised* 20 *August* 1999)

Abstract

We show that the asymptotic behaviour of the quantization error allows the definition of dimensions for probability distributions, the upper and the lower quantization dimension. These concepts fit into standard geometric measure theory, as the upper quantization dimension is always between the packing and the upper box-counting dimension, whereas the lower quantization dimension is between the Hausdorff and the lower box-counting dimension.

1. *Introduction*

The Theorem of Zador ([**9**]), which is of fundamental importance in the theory of optimal quantization, says that

$$\lim_{n\to\infty} n^{2/d}\Delta_n(\|.\|^2, P) = C_{\mathscr{L}}(d)\left(\int\left(\frac{dP}{d\mathscr{L}^d}\right)^{d/(2+d)} d\mathscr{L}^d\right)^{1+2/d}, \tag{1}$$

where the quantization error is defined by

$$\Delta_n(\|.\|^2, P) = \inf_{|O|\leqslant n}\int \min_{p\in O}\|x-p\|^2 dP(x). \tag{2}$$

$\mathscr{L}^d$ is the d-dimensional Lebesgue measure and $dP/d\mathscr{L}^d$ the Radon–Nikodym derivative of the absolutely continuous component of P with respect to $\mathscr{L}^d$. If P is singular with respect to $\mathscr{L}^d$ then $dP/d\mathscr{L}^d = 0$. The quantization constant is denoted by $C_{\mathscr{L}}(d)$, which depends on the dimension d, but not on the distribution P. (This constant is known for $d = 1$ and $d = 2$. For $d \geqslant 3$ only estimates exist.)

This theorem may be viewed as a first result on the relation between the asymptotic behaviour of the quantization error and dimensional properties of the distribution in case the absolutely continuous component of P (with respect to Lebesgue measure) does not vanish. Analogous results have been established for special distributions which are singular with respect to Lebesgue measure. We notice that in these examples a set E with $P(E) = 1$ exists for which Hausdorff dimension and upper box-counting dimension are identical.

Results have been obtained by Graf and Luschgy for instance for the uniform distribution on the Cantor set ([**4**, **5**]). Reference [**10**] discusses the idea of a quantization dimension for the Cantor set on an informal level.

We define the lower quantization dimension $\underline{\dim}_Q(P)$ and the upper quanti-

https://doi.org/10.1017/S0305004101005357 Published online by Cambridge University Press

zation dimension $\overline{\dim}_Q(P)$ as the points of discontinuity of the mappings $s \mapsto \liminf_{n\to\infty} n^{2/s}\Delta_n(\|.\|^2, P)$ and $s \mapsto \limsup_{n\to\infty} n^{2/s}\Delta_n(\|.\|^2, P)$. It is the aim of this paper to present results concerning the quantization dimensions, which allow a comparison of these concepts with popular notions of dimension of a distribution, such as Hausdorff dimension, packing dimension or box-counting dimension. It will be proved that for a suitable set E with $P(E) = 1$, $\underline{\dim}_Q(P) \in [\dim_{\mathscr{H}}(E), \underline{\dim}_B(E)]$ and $\dim_Q(P) \in [\dim_{\mathscr{P}}(E), \dim_B(E)]$.

Furthermore, if E, the support of P, is a bounded s-set (i.e. $0 < \mathscr{H}^s(E) < \infty$) which additionally is of exact upper box-counting dimension $\overline{\dim}_B(E) = s$ and if $P \ll \mathscr{H}^s$, then $\liminf_{n\to\infty} n^{2/s}\Delta_n(\|.\|^s, P) > 0$ and $\limsup_{n\to\infty} n^{2/s}\Delta_n(\|.\|^s, P) < \infty$, but in general $n^{2/s}\Delta_n(\|.\|^s, P)$ does not converge. Examples of such distributions are invariant distributions on attractors of similarity transformations satisfying a separation condition, such as the open set condition.

The Theorem of Zador as a result on the convergence of $n^{2/s}\Delta_n(\|.\|^s, P)$ may be generalized for distributions with a regular s-dimensional support s is then necessarily integral (see [**3**]). Equation (1) holds, if d is replaced by s, $\mathscr{L}^d$ by s-dimensional Hausdorff measure $\mathscr{H}^s$ and $C_{\mathscr{L}}(d)$ by a suitable constant. The proof of this result will be given in a forthcoming paper.

Reference [**7**] provides a frame for classification and quantization. It includes approaches such as vector quantization or unsupervised learning in the theory of artificial neural nets. For a given distribution P on $\mathbb{R}^d$, a partition $\mathscr{B} = (B_1, \ldots, B_n)$ containing maximal information is sought. The information of a partition $\mathscr{B}$ is measured by

$$I^f(\mathscr{B}, P) = \int f(\mathbb{E}(X \mid \mathscr{B}))dP, \tag{3}$$

where $f: \mathbb{R}^d \to \mathbb{R}$ is an integrable convex function and $\mathbb{E}(X \mid \mathscr{B})$ is the conditional expectation of the identity $X(x) = x$, given the σ-field generated by the partition. A partition $\mathscr{B}$ is called optimal if it maximizes the information over all partitions of size n. Optimal partitions $\mathscr{B}$ match optimal quantizers $O = \{p_1, \ldots, p_n\}$ and vice versa (see [**7**]). In the case of $f(x) = \|x\|^2$, $B_i = \{x \mid \|x - p_i\| \leqslant \|x - p_k\|, k = 1, \ldots, n$ and no smaller argument has this property$\}$ and $p_i = \int_{B_i} xdP/P(B_i)$. The means p_i are called prototypes in case $\mathscr{B}$ is optimal.

Notation. Let a finite Borel measure G on $\mathbb{R}^d$ and a set E be given. $G_{|E}$ and $G_{\uparrow E}$ denote the restriction of G to E and the conditional distribution of G given E, i.e. $G_{|E}(A) = G(A \cap E)$ and for $0 < G(E) < \infty$, $G_{\uparrow E} = G(A \cap E)/G(E)$. $\mathscr{L}^s$ and $\mathscr{H}^s$ denote s-dimensional Lebesgue measure and s-dimensional Hausdorff measure. $B(x, r)$ denotes the open ball with center x and radius r. Let $O = \{p_1, \ldots, p_n\}$ be a set of points, not necessarily prototypes. We define

$$\Delta_n(\|.\|^2, O, P) = \int \min_{p\in O} \|x - p\|^2 dP(x). \tag{4}$$

2. *The quantization dimension for distributions of bounded support*

In order to integrate the concept of the quantization dimension into geometric measure theory we restrict the results to probability measures with bounded support and to the squared norm as information function. The quantization dimension of

https://doi.org/10.1017/S0305004101005357 Published online by Cambridge University Press

a distribution is a dimension characteristic of the support, similar to Hausdorff, packing or box-counting dimensions. In the case when the support is unbounded, the asymptotic behaviour of the quantization error depends additionally on regularity conditions, such as the existence of moments. The choice of the classical information function $f = \|.\|^2$ is no restriction compared to the case $f \in C^{(2)}$ with f'' regular. If f'' is not of full rank, the asymptotic behaviour of $\Delta_n(f, P) := \int f dP - \max\{I^f(\mathscr{B}, P) \mid |\mathscr{B}| = n\}$ simply depends not only on the geometry of the support of E, but to a large extent on features of f.

We define the lower and the upper quantization dimension of P, $\underline{\dim}_Q(P)$ and $\overline{\dim}_Q(P)$ as the unique real numbers s_u and $s_o \in [0, \infty]$ resp., satisfying

$$\liminf_{n\to\infty} n^{2/s}\Delta_n(\|.\|^2, P) = \begin{cases} \infty & \text{for } s < s_u, \\ 0 & \text{for } s > s_u, \end{cases}$$
$$\limsup_{n\to\infty} n^{2/s}\Delta_n(\|.\|^2, P) = \begin{cases} \infty & \text{for } s < s_o, \\ 0 & \text{for } s > s_o. \end{cases}$$

If $\underline{\dim}_Q(P) = \overline{\dim}_Q(P)$ we denote the common value by $\dim_Q(P)$, the quantization dimension of P. Equation (1) bounds $\overline{\dim}_Q(P)$ by d.

Reference [**3**] discusses various concepts of dimension for distributions. The (upper) Hausdorff dimension $\dim^*_{\mathscr{H}}(P)$ and the packing dimension $\dim^*_{\mathscr{P}}(P)$ are defined by

$$\dim^*_{\mathscr{H}}(P) = \inf\{\dim_{\mathscr{H}}(E) \mid E \subseteq \mathbb{R}^d, \text{ Borel and } P(E) = 1\}, \tag{5}$$
$$\dim^*_{\mathscr{P}}(P) = \inf\{\dim_{\mathscr{P}}(E) \mid E \subseteq \mathbb{R}^d, \text{ Borel and } P(E) = 1\}. \tag{6}$$

Analogously we define the lower and the upper box-counting dimension by

$$\underline{\dim}^*_B(P) = \inf\{\underline{\dim}_B(E) \mid E \subseteq \mathbb{R}^d, \text{ Borel and } P(E) = 1\}, \tag{7}$$
$$\overline{\dim}^*_B(P) = \inf\{\overline{\dim}_B(E) \mid E \subseteq \mathbb{R}^d, \text{ Borel and } P(E) = 1\}. \tag{8}$$

Note that for $s = \dim^*_{\mathscr{H}}(P)$ (and analogously for $\dim^*_{\mathscr{P}}(P)$, $\overline{\dim}^*_B(P)$ and $\underline{\dim}^*_B(P)$) a Borel set E exists with $P(E) = 1$ and $\dim_{\mathscr{H}}(E) = s$. Upper and lower local dimensions provide an alternative definition of $\dim^*_{\mathscr{H}}(P)$ and $\dim^*_{\mathscr{P}}(P)$ (see [**5**]),

$$\dim^*_{\mathscr{H}}(P) = \inf\{s \mid \underline{\dim}_{\mathrm{loc}}(P, x) \leqslant s \quad P - \text{a.e.}\}, \tag{9}$$
$$\dim^*_{\mathscr{P}}(P) = \inf\{s \mid \overline{\dim}_{\mathrm{loc}}(P, x) \leqslant s \quad P - \text{a.e.}\}. \tag{10}$$

We recall the decomposition of measures into exact and diffuse components (see [**3**, proposition 10·10]). Only for distributions P with vanishing diffuse component exact results as Theorem 1(1)–(5) are possible.

THEOREM 1. *Let P be a probability distribution on $[0, 1]^d$. Then*

(1) $\dim^*_{\mathscr{H}}(P) \leqslant \underline{\dim}_Q(P) \leqslant \underline{\dim}^*_B(P)$.

(2) $\dim^*_{\mathscr{P}}(P) \leqslant \overline{\dim}_Q(P) \leqslant \overline{\dim}^*_B(P)$.

(3) *If $\dim^*_{\mathscr{H}}(P) = \overline{\dim}^*_B(P)$, then $\underline{\dim}_Q(P) = \overline{\dim}_Q(P) = \dim^*_{\mathscr{H}}(P)$.*

(4) *Let $E \subseteq [0, 1]^d$ be a Borel s-set, i.e. $0 < \mathscr{H}^s(E) < \infty$, and $P \ll \mathscr{H}^s_{|E}$. Then*

$$\liminf_{n\to\infty} n^{2/s}\Delta_n(\|.\|^2, P) > 0. \tag{11}$$

(5) *Let $E \subseteq [0, 1]^d$ be a Borel set of exact upper box-counting dimension s with $P(E) = 1$. Then*

$$\limsup_{n\to\infty} n^{2/s}\Delta_n(\|.\|^2, P) < \infty. \tag{12}$$

https://doi.org/10.1017/S0305004101005357 Published online by Cambridge University Press

Proof. First, we prove $\overline{\dim}_Q(P) \leqslant \overline{\dim}^*_B(P)$. It is sufficient to show $\Delta_n(\|.\|^2, P) = O(1/n^{2/\alpha})$ for all $\alpha > \overline{\dim}^*_B(P)$, which implies $\limsup_{n\to\infty} n^{2/\alpha}\Delta_n(\|.\|^2, P) = 0$ for $\alpha > \overline{\dim}^*_B(P)$. Let $s = \overline{\dim}^*_B(P)$ and $\alpha > s$. A Borel set E exists with $P(E) = 1$ and $\overline{\dim}_B(E) < (s + \alpha)/2$. Let $r_0 > 0$ such that $N_K(E, r) \leqslant r^{-\alpha}$ for all $r \in]0, r_0]$. $N_K(E, r)$ denotes the covering number of E by r-meshed cubes, i.e. the minimal number of r-meshed cubes required to cover E (see Appendix). Let $n \geqslant r_0^{-\alpha}$ and $r = n^{-1/\alpha}$. Note that $\tilde{n} := N_K(E, r) \leqslant n$. Let $\{S_i | i \leqslant \tilde{n}\}$ be a cover of E with r-meshed cubes. $O_{\tilde{n}} = \{p_1, \ldots, p_{\tilde{n}}\}$ denotes the set of centers of the cubes. $x \in S_i$ implies $\|x - p_i\|^2 \leqslant r^2 d/4$ and thus

$$\Delta_n(\|.\|^2, P) \leqslant \Delta_{\tilde{n}}(\|.\|^2, O_{\tilde{n}}, P) \leqslant \sum_{i=1}^{\tilde{n}} P(S_i) r^2 d/4 = \frac{d}{4}\left(\frac{1}{n}\right)^{2/\alpha}.$$

$\underline{\dim}_Q(P) \leqslant \underline{\dim}^*_B(P)$ can be proved the same way.

If E has exact upper box-counting dimension s, then $\overline{\dim}_B(E) = s$ and a constant C exists such that $N_K(E, r) \leqslant Cr^{-s}$. In that case we choose $r = (C/n)^{1/s}$, which leads to

$$\Delta_n(\|.\|^2, P) \leqslant \sum_{i=1}^{\tilde{n}} P(S_i) r^2 d/4 = \frac{d}{4}\left(\frac{C}{n}\right)^{2/s}.$$

This finishes the proof of Theorem 1(5).

Before providing the proof of the remaining inequality in Theorem 1(1) (i.e. $\underline{\dim}^*_{\mathscr{H}}(P) \leqslant \dim_Q(P)$) and the exact result Theorem 1(4), we comment on the idea of the proof. Let s be the Hausdorff dimension of P, $\alpha < s$ and $(B_1, \ldots, B_n)$ an optimal partition with corresponding prototypes $(p_1, \ldots, p_n)$. Let $B(p_i, r_i)$ be a ball containing p_i, such that $P(B_i \backslash B(p_i, r_i))$ is large compared to $P(B_i)$, for instance equal to $P(B_i)/2$, and such that r_i is of order $\operatorname{diam}(B_i)$ in magnitude. $\|x - p_i\|$ is then of order r_i for $x \in B_i \backslash B(p_i, r_i)$ and $\Delta_n(\|.\|^2, P)$ of order $\sum_{i=1}^n P(B_i) r_i^2$. Equation (9) implies that r_i^α is at least $P(B_i)$ and thus $\Delta_n(\|.\|^2, P) \geqslant c \sum_{i=1}^n P(B_i)^{1+2/\alpha} \geqslant c/n^{2/\alpha}$ for a suitable positive constant c. However, (9) is a local result, therefore the arguments hold only on a subset $E_0 \subseteq E$ of positive probability.

Let $\alpha < s$, E a Borel set with $P(E) > 0$ and $\dim_{\mathscr{H}}(E) > \alpha$. Equation (9) implies that for P-a.e. $x \in E$ a $r_x > 0$ exists, such that for all convex sets U, $\operatorname{diam}(U) \leqslant r_x$ and $x \in U$ imply $P(U) \leqslant \operatorname{diam}(U)^\alpha$. But then a Borel set $E_0 \subseteq E$ with $P(E_0) > 0$ and a r_0 exist, such that for $x \in E_0$ and all convex sets U with $\operatorname{diam}(U) \leqslant r_0$ and $x \in U$, $P(U) \leqslant \operatorname{diam}(U)^\alpha$ holds. Let $(B_1, \ldots, B_n)$ be an optimal partition for $P_0 := P_{\restriction E_0}$ with conditional means $(p_1, \ldots, p_n)$. Let $2r_i = r_0 \wedge (P(B_i \cap E_0)/2)^{1/\alpha}$, $J_1 = \{i | r_i = (P(B_i \cap E_0)/2)^{1/\alpha}/2\}$ and $J_2 = \{i | r_i = r_0/2\}$. Then,

$$\begin{aligned} P_0(B_i \backslash B(p_i, r_i)) &\geqslant \frac{1}{P(E_0)}(P(B_i \cap E_0) - P(B(p_i, r_i) \cap E_0)) \\ &\geqslant \frac{1}{P(E_0)}(P(B_i \cap E_0) - (2r_i)^\alpha) \\ &\geqslant \frac{1}{P(E_0)}(P(B_i \cap E_0) - P(B_i \cap E_0)/2) = P_0(B_i)/2. \end{aligned}$$

https://doi.org/10.1017/S0305004101005357 Published online by Cambridge University Press

The second inequality is a consequence of the definition of r_i if $B(p_i, r_i) \cap E_0 \neq \varnothing$ and holds trivially otherwise. Thus

$$\begin{aligned} n^{2/\alpha}\Delta_n(\|.\|^2, P) &\geqslant P(E_0)n^{2/\alpha}\Delta_n(\|.\|^2, P_0) \\ &= P(E_0)n^{2/\alpha}\sum_{i=1}^{n}\int_{B_i}\|x-p_i\|^2 dP_0(x) \\ &\geqslant P(E_0)n^{2/\alpha}\sum_{i=1}^{n}\int_{B_i\setminus B(p_i,r_i)}\|x-p_i\|^2 dP_0(x) \\ &\geqslant P(E_0)n^{2/\alpha}\sum_{i=1}^{n} r_i^2 P_0(B_i\setminus B(p_i,r_i)) \\ &\geqslant \frac{P(E_0)n^{2/\alpha}r_0^2 P_0(\bigcup_{i\in J_2} B_i)}{8} + \frac{P(E_0)^{1+2/\alpha}n^{2/\alpha}}{2^{3+2/\alpha}}\sum_{i\in J_1} P_0(B_i)^{1+2/\alpha}. \end{aligned}$$

If $P_0(\bigcup_{i\in J_2} B_i) \geqslant \frac{1}{2}$, the first summand is at least $P(E_0)r_0^2/16$. Otherwise, the second summand satisfies

$$\begin{aligned} &\frac{P(E_0)^{1+2/\alpha}n^{2/\alpha}P_0(\bigcup_{j\in J_1} B_j)^{1+2/\alpha}}{2^{3+2/\alpha}}\sum_{i\in J_1}\left(\frac{P_0(B_i)}{P_0(\bigcup_{j\in J_1} B_j)}\right)^{1+2/\alpha} \\ &\qquad\geqslant \frac{P(E_0)^{1+2/\alpha}}{2^{4+4/\alpha}}\left(\frac{n}{|J_1|}\right)^{2/\alpha} \geqslant \frac{P(E_0)^{1+2/\alpha}}{2^{4+4/\alpha}}. \end{aligned}$$

This concludes the proof of $\dim^*_{\mathscr{H}}(P) \leqslant \underline{\dim}_Q(P)$.

Theorem 1(4) is the exact version of the result just proved. Note that $\overline{D}_c^s(E, x) = 1$ holds P-a.e. A Borel set $E_0 \subseteq E$ and positive constants c_1, c_2, r_0 with $r_0 < 1$ exist, such that $P(E_0) > 0$,

$$\frac{dP}{d\mathscr{H}^s_{|E}}(x) \geqslant c_1$$

on E_0 and

$$\mathscr{H}^s(U \cap E) \leqslant c_2|U|^s$$

for $x \in E_0$ and all convex sets U with $x \in U$ and $\operatorname{diam}(U) \leqslant r_0$. In particular, for $x \in E_0$ and convex sets U with $x \in U$ of diameter at most r_0,

$$\mathscr{H}^s(U \cap E_0) \leqslant c_2|U|^s.$$

We may choose $r_0 < (\mathscr{H}^s(E_0)/c_2)^{1/s}$. Again, let $(B_1, \ldots, B_n)$ be an optimal partition for $P_0 := P_{\restriction E_0}$ with prototypes $(p_1, \ldots, p_n)$. We define $c_3 = 1 - c_2 r_0^s/\mathscr{H}^s(E_0)$ and $r_i = (\mathscr{H}^s(B_i \cap E_0)(1-c_3)/c_2)^{1/s}/2$. Then $2r_i \leqslant r_0$, $c_3 > 0$ and

$$\begin{aligned} P(E_0)P_0(B_i\setminus B(p_i,r_i)) &\geqslant c_1\mathscr{H}^s((B_i\setminus B(p_i,r_i)) \cap E_0) \\ &\geqslant c_1(\mathscr{H}^s(B_i \cap E_0) - \mathscr{H}^s(B(p_i,r_i) \cap E_0)) \\ &\geqslant c_1(\mathscr{H}^s(B_i \cap E_0) - c_2(2r_i)^s) \\ &= c_1(\mathscr{H}^s(B_i \cap E_0) - (1-c_3)\mathscr{H}^s(B_i \cap E_0)) \\ &= c_1c_3\mathscr{H}^s(B_i \cap E_0). \end{aligned}$$

https://doi.org/10.1017/S0305004101005357 Published online by Cambridge University Press

Thus

$$\begin{aligned}
n^{2/s}\Delta_n(\|.\|^2, P) &\geqslant n^{2/s}P(E_0)\Delta_n(\|.\|^2, P_0)\\
&= n^{2/s}P(E_0)\sum_{i=1}^{n}\int_{B_i}\|x-p_i\|^2 dP_0(x)\\
&\geqslant n^{2/s}P(E_0)\sum_{i=1}^{n}\int_{B_i\setminus B(p_i,r_i)} r_i^2 dP_0(x)\\
&\geqslant n^{2/s}\sum_{i=1}^{n} r_i^2 c_1 c_3 \mathscr{H}^s(B_i\cap E_0)\\
&\geqslant n^{2/s}\frac{c_1c_3}{4}((1-c_3)/c_2)^{2/s}\sum_{i=1}^{n}\mathscr{H}^s(B_i\cap E_0)^{1+2/s}\\
&=: c_4 n^{2/s}\sum_{i=1}^{n}\mathscr{H}^s(B_i\cap E_0)^{1+2/s}.
\end{aligned}$$

Now, using Hölder's inequality, we have

$$c_4 n^{2/s}\sum_{i=1}^{n}\mathscr{H}^s(B_i\cap E_0)^{1+2/s} \geqslant c_4\mathscr{H}^s(E_0)^{1+2/s}.$$

With that we have finished the proof of part 4.

To prove $\dim^*_{\mathscr{P}}(P) \leqslant \overline{\dim}_Q(P)$, let $\alpha_0 = \overline{\dim}_Q(P)$ and $\alpha_1 = \dim^*_{\mathscr{P}}(P)$. We assume $\alpha_0 < \alpha_1$ and deduce a contradiction. The assumption implies that for all $\alpha \in]\alpha_0, \alpha_1[$ and every sequence (n_i) with $n_i \to \infty$,

$$\lim_{i\to\infty} n_i^{2/\alpha}\Delta_{n_i}(\|.\|^2, P) = 0.$$

Let $\beta > 0$ with $\alpha_0 < 2(\alpha_1-\beta)/(2+\beta)$. An $\alpha_2 > \alpha_1$ exists, such that

$$\alpha_0 < \frac{2(\alpha_1-\beta)}{2+\beta+\alpha_2-\alpha_1} =: \gamma. \tag{13}$$

We denote $E_0 = \{x|\overline{\dim}_{\mathrm{loc}}(P,x) > \alpha_1 - \beta/2\}$. The definition of $\dim^*_{\mathscr{P}}$ implies $P(E_0) > 0$ and according to Lemma A2(4), $\dim_{\mathscr{P}}(E_0) \geqslant \alpha_1 - \beta/2$ holds. For $x \in E_0$ a $\tilde{r}_x > 0$ exists, such that for all cubes S with $x \in S$ of side-length at most $|S| \leqslant \tilde{r}_x$, $P(S) \geqslant |S|^{\alpha_2}$ holds. Let (r_i) be a sequence of positive numbers with $r_i \downarrow 0$ and $E_i = \{x \in E_0 | \tilde{r}_x \geqslant r_i\}$. Then $E_0 = \bigcup E_i$. We will prove that $\overline{\dim}_B(E_i) \leqslant \alpha_1 - \beta$. Lemma A1(5) implies $\dim_{\mathscr{P}}(E_0) \leqslant \alpha_1 - \beta < \alpha_1 - \beta/2$, in contradiction to $\overline{\dim}_Q(P) < \dim^*_{\mathscr{P}}(P)$.

Fix i and choose an arbitrary positive and decreasing nullsequence (δ_j) with $\delta_1 < r_i$. Let

$$n_j = \left[\frac{1}{3^d 2} N_K(E_i, \delta_j)\right].$$

(n_j) is unbounded. Let $O = \{p_1, \ldots, p_{n_j}\}$ be a set of n_j prototypes in $[0,1]^d$. The $N_K(E_i, \delta_j)$ cubes, which have nonempty intersection with E_i consist of

(1) at most $n_j 3^d$ cubes that contain a point p_k, or which are neighbouring such a cube;

https://doi.org/10.1017/S0305004101005357 Published online by Cambridge University Press

(2) the remaining cubes. Their distance to the nearest prototype is at least δ_j. There are at least $N_K(E_i, \delta_j) - n_j 3^d$ such cubes. Let $\tilde{E}$ denote the union of these cubes. Then

$$P(\tilde{E}) \geqslant (N_K(E_i, \delta_j) - n_j 3^d)\delta_j^{\alpha_2} \geqslant N_K(E_i, \delta_j)\delta_j^{\alpha_2}/2.$$

Let $\alpha \in]\alpha_0, \gamma[$. Since $\alpha > \alpha_0 = \overline{\dim}_Q(P)$ a j_0 exists with $n_j^{2/\alpha}\Delta_{n_j}(\|.\|^2, P) \leqslant 1$ for $j \geqslant j_0$. But then, for $j \geqslant j_0$ (c_1, c_2 are constants),

$$\begin{aligned} 1 \geqslant n_j^{2/\alpha}\Delta_{n_j}(\|.\|^2, P) &\geqslant n_j^{2/\alpha} \int_{\tilde{E}} \min_{p\in O} \|x - p\|^2 dP(x) \\ &\geqslant n_j^{2/\alpha}\delta_j^2 N_K(E_i, \delta_j)\delta_j^{\alpha_2}/2 \geqslant c_1 N_K(E_i, \delta_j)^{1+2/\alpha}\delta_j^{2+\alpha_2}. \end{aligned}$$

Thus

$$N_K(E_i, \delta_j) \leqslant c_2 \delta_j^{-(2+\alpha_2)\alpha/(2+\alpha)}.$$

The sequence (δ_j) being arbitrary, implies

$$\begin{aligned} \overline{\dim}_B(E_i) &\leqslant (2 + \alpha_2)\alpha/(2 + \alpha) \\ &\leqslant \frac{(2 + \alpha_2)\gamma}{2 + \gamma} = \frac{(2 + \alpha_2)2(\alpha_1 - \beta)/(2 + \beta + \alpha_2 - \alpha_1)}{2 + 2(\alpha_1 - \beta)/(2 + \beta + \alpha_2 - \alpha_1)} \\ &= \frac{(2 + \alpha_2)(\alpha_1 - \beta)}{2 + \alpha_2} = \alpha_1 - \beta. \end{aligned}$$

Remark 1. Theorem 1 provides upper and lower bounds on the two quantization dimensions. These bounds, which are dimensions of a set E, depend on P only as far as $P(E) = 1$ holds. However, where for instance $\overline{\dim}_Q(P)$ lies in $[\dim^*_{\mathscr{P}}(P), \overline{\dim}^*_B(P)]$, depends on P and how it distributes its mass on E. Let for instance, E be Borel with $P(E) = 1$ and $\dim_{\mathscr{P}}(E) < \overline{\dim}_B(E)$. Let $\alpha \in]\dim_{\mathscr{P}}(E), \overline{\dim}_B(E)[$. Then $E = \bigcup_{i=1}^{\infty} E_i$, where $\overline{\dim}_B(E_i) < \alpha$. There are constants $c_i = c_{i,\alpha}$, such that $N(E_i, r) \leqslant c_i r^{-\alpha}$. This sequence of constants is unbounded, since otherwise E would be of upper box-counting dimension at most α. The upper quantization dimension of P depends on the weight of sets E_i with large c_i. In case their weight is small, $\overline{\dim}_Q(P)$ will be close to $\dim^*_{\mathscr{P}}(P)$. Otherwise, it might be near $\overline{\dim}^*_B(P)$ (see Example 1).

We provide assumptions, which guarantee that $\overline{\dim}_Q(P) = \dim^*_{\mathscr{P}}(P)$ or $\underline{\dim}_Q(P) = \dim^*_{\mathscr{H}}(P)$ for distributions P on $[0, 1]^d$. We define

$$\eta_{G,\alpha}(x) = \inf\{G(B(x, r))/r^{\alpha} \quad | \, 0 < r < d^{\frac{1}{2}}\}, \tag{14}$$

where G is a probability measure and $\alpha \in \mathbb{R}_+$.

The lower density of a set is a quantity closely related to the packing dimension of the set. $\eta_{P,\alpha}(x)$ depends also on large neighborhoods of x. Note that $\eta_{P,\alpha} > 0$ P-a.e. if $\alpha > \dim^*_{\mathscr{P}}(P)$. In the case $s = \dim^*_{\mathscr{P}}(P)$, however, $\eta_{G,s}(x) > 0$ implies that $\underline{D}^s(G, .)$ is positive in x. This is the reason why we replace $\eta_{P,s}$ by $\eta_{G,s}$ in exact results. Additionally, among random quantizers, those corresponding to P are not optimal.

Assumption (A3) controls the deviation from uniformity in the Anti-Frostman Lemma (see [**1**]). Compared to the assumptions (A1) and (A2), which are in some sense natural, considered from the viewpoint of random quantizers, assumption (A3) has no additional and obvious interpretation.

https://doi.org/10.1017/S0305004101005357 Published online by Cambridge University Press

ASSUMPTIONS.

(A1) *A Borel set E with $P(E) = 1$ and $\dim_{\mathscr{P}}(E) = \dim^*_{\mathscr{P}}(P)$ exists, such that for all $\alpha > \dim^*_{\mathscr{P}}(P)$*

$$\int_E \eta_{P,\alpha}(x)^{-2/\alpha} dP(x) < \infty.$$

(A2) *A Borel set E with $P(E) = 1$, $\dim_{\mathscr{P}}(E) = \dim^*_{\mathscr{P}}(P)$ and a distribution G exist, such that for $s = \dim^*_{\mathscr{P}}(P)$,*

$$\int_E \eta_{G,s}(x)^{-2/s} dP(x) < \infty.$$

(A3) *There exists a Borel set E with $P(E) = 1$ and $\dim_{\mathscr{H}}(E) = \dim^*_{\mathscr{H}}(P) =: s$, for all $\epsilon > 0$, $s < \alpha' < \alpha''$, there are positive constants $c(\epsilon)$ and $c(\alpha', \alpha'')$, such that for all $\delta > 0$ a measurable partition $(B_i)_{i=1}^{\infty}$ of E exists, satisfying*

(A3·1)

$$r_i := \operatorname{diam}(B_i) < \delta \quad \text{and } P(B_i) \geqslant r_i^{\alpha'} \quad \text{for all } i,$$

(A3·2)

$$\sum_{i=1}^{\infty} P(B_i)^2 \geqslant c(\alpha', \alpha'') \left(\sum_{i=1}^{\infty} P(B_i)^{1+2/\alpha'} \right)^{\alpha''/2},$$

(A3·3)

$$\sup_{n \in \mathbb{N}} \sum_{i=1}^{\infty} P(B_i) e^{-nP(B_i)} e^{n^{1-\epsilon} \sum_{i=1}^{\infty} P(B_i)^2} \leqslant c(\epsilon).$$

THEOREM 2. *Let P be a probability measure on $[0,1]^d$.*

(1) (A1) *implies* $\overline{\dim}_Q(P) = \dim^*_{\mathscr{P}}(P)$.

(2) (A2) *implies*

$$\limsup_{n \to \infty} n^{2/s} \Delta_n(\|.\|^2, P) < \infty, \tag{15}$$

where $s = \dim^*_{\mathscr{P}}(P)$.

(3) (A3) *implies* $\underline{\dim}_Q(P) = \dim^*_{\mathscr{H}}(P)$.

The subsequent proof shows that the statements of the theorem are met also by random quantizers. A random quantizer with distribution G is a sequence $\tilde{O}_n$ of independent G-distributed random vectors, i.e. $\tilde{O}_n = \{p_1, \ldots, p_n\}$ with $p_i \sim G$. $\mathbb{E}^G(\Delta_n(\|.\|^2, \tilde{O}_n, P))$, the quantization error of the random quantizer, is an upper bound of $\Delta_n(\|.\|^2, P)$.

Proof of Theorem 2. Let E be a Borel set with $P(E) = 1$ and $\dim_{\mathscr{P}}(E) = \dim^*_{\mathscr{P}}(P)$. Further, let $\alpha > \dim^*_{\mathscr{P}}(P)$. $p_1, \ldots, p_n$ are independent and distributed with the law P. If $x \in E$, then

$$\begin{aligned} P(\min_{i \leqslant n} \|p_i - x\|^2 > t) &= P(p_1 \notin B(x, t^{1/2}))^n \\ &= (1 - P(B(x, t^{1/2}))^n \leqslant (1 - \eta_{P,\alpha}(x) t^{\alpha/2})_+^n \\ &\leqslant e^{-n \eta_{P,\alpha}(x) t^{\alpha/2}}. \end{aligned}$$

https://doi.org/10.1017/S0305004101005357 Published online by Cambridge University Press

The quantization error of the random quantizer $\tilde{O}_n = (p_1, \ldots, p_n)$ is then

$$\mathbb{E}^P(\Delta_n(P, \tilde{O}_n)) = \int_0^\infty \int P(\min_{i \leqslant n} \|x - p_i\|^2 > t) dP(x) dt$$
$$\leqslant \int_0^\infty \int e^{-n\eta_{P,\alpha}(x) t^{\alpha/2}} dP(x) dt = \int \frac{\Gamma(1 + 2/\alpha)}{n^{2/\alpha} \eta_{P,\alpha}(x)^{2/\alpha}} dP(x).$$

This proves the first statement. In the proof of the second α is replaced by $s = \dim^*_{\mathscr{P}}(P)$ and G is chosen as the distribution of the random quantizer.

To prove statement three, let $\alpha > s = \dim_{\mathscr{H}}(E)$, where $P(E) = 1$. We choose α', α'' and ϵ with $s < \alpha' < \alpha'' < \alpha$, $\alpha'' < (1 - \epsilon)\alpha$ and $\epsilon' := 1 - \epsilon - \alpha''/\alpha > 0$. Let (δ_k) be a positive nullsequence. For every k let (B_i^k) be a Borel partition of E into sets of diameter at most δ_k, which satisfy the assumption (A3) with constants $c(\epsilon)$ and $c(\alpha', \alpha'')$. Let n_k be the integer part of $(\sum_{i=1}^\infty P(B_i^k)^{1+2/\alpha'})^{-\alpha/2}$. (n_k) is unbounded. A sample of P-distributed random vectors of size n_k serves as prototypes $p_1, \ldots, p_{n_k}$. Let $x \in B_i^k$. If B_i^k contains a prototype, then $\min_j \|x - p_j\|^2 \leqslant r_i^2 \leqslant P(B_i^k)^{2/\alpha'}$. Otherwise, we replace $\min_j \|x - p_j\|^2$ by d. Then

$$\begin{aligned}
\Delta_{n_k}(\|.\|^2, P) &\leqslant \mathbb{E}^P(\Delta_{n_k}(\|.\|^2, \{p_1, \ldots, p_{n_k}\}, P)) \\
&\leqslant \sum_{i=1}^\infty P(B_i^k)(P(B_i^k)^{2/\alpha'} + d(1 - P(B_i^k))^{n_k}) \\
&\leqslant (n_k - 1)^{-2/\alpha} + d \sum_{i=1}^\infty P(B_i^k) e^{-n_k P(B_i^k)} \\
&\leqslant (n_k - 1)^{-2/\alpha} + dc(\epsilon) e^{-n_k^{1-\epsilon} \sum_{i=1}^\infty P(B_i^k)^2} \\
&\leqslant (n_k - 1)^{-2/\alpha} + dc(\epsilon) e^{-n_k^{1-\epsilon} c(\alpha', \alpha'') (\sum_{i=1}^\infty P(B_i^k)^{1+2/\alpha'})^{\alpha''/2}} \\
&\leqslant (n_k - 1)^{-2/\alpha} + dc(\epsilon) e^{-n_k^{1-\epsilon} c(\alpha', \alpha'') n_k^{-\alpha''/\alpha}} \\
&= (n_k - 1)^{-2/\alpha} + dc(\epsilon) e^{-n_k^{\epsilon'} c(\alpha', \alpha'')} = O(n_k^{-2/\alpha}).
\end{aligned}$$

Theorem 3 indicates that sets, which are small in Hausdorff measure, are responsible if $\underline{\dim}_Q(P) < \overline{\dim}_Q(P)$. Restricting P to arbitrarily large sets leads to the equality of Hausdorff and quantization dimension. We define for $\epsilon > 0$ and $\dim^*_{\mathscr{H}}(P) = s$, $\overline{\dim}^\epsilon_Q$ by

$$\overline{\dim}^\epsilon_Q = \inf \{\overline{\dim}_Q(P_{\restriction E}) \mid \mathscr{H}^s(E^c) < \epsilon\}. \tag{16}$$

Theorem 3. *Let $\dim^*_{\mathscr{H}}(P) = s$, E be a bounded Borel set with $P(E) = 1$, $\dim_{\mathscr{H}}(E) = s$ and $dP/d\mathscr{H}^s_{\restriction E} \neq 0$. If $\mathscr{H}^s(\{x \in E \mid \underline{D}^s(E, x) = 0\}) = 0$, then $\overline{\dim}^\epsilon_Q = s$ for all $\epsilon > 0$.*

Proof. According to [**6**], for all $\epsilon > 0$ a set E_ϵ exists, such that $\mathscr{H}^s(E \setminus E_\epsilon) < \epsilon$ and $\overline{\dim}_B(E_\epsilon) = s$. Theorem 1(2) implies $\overline{\dim}_Q(P_{\restriction E_\epsilon}) = s$.

The subsequent example demonstrates that the inequalities in Theorem 1(1) and Theorem 1(2) cannot be improved.

Example 1. Let for $m \in \mathbb{N}$ $U_m \subseteq \mathbb{R}^2$ denote the set $\{0, 1/m, \ldots, 1 - 1/m\} \times [0, 1]$ and P_m the uniform distribution on U_m. Let $P = \sum_{m=1}^\infty q_m P_m$ be a mixture of such distributions. We show that weights (q_m) can be found, such that $\underline{\dim}_Q(P) = 2$ and weights such that $\overline{\dim}_Q(P) = 1$. Let $E = \bigcup_{q_m > 0} U_m$. Note that $\bar{E} = [0, 1]^2$ if an

https://doi.org/10.1017/S0305004101005357 Published online by Cambridge University Press

infinite number of weights q_m is positive, so that $\dim^*_{\mathscr{P}}(P) = \dim^*_{\mathscr{H}}(P) = 1$ and $\underline{\dim}^*_B(P) = \overline{\dim}^*_B(P) = 2$ and the lower and the upper quantization dimensions are necessarily in $[1, 2]$. We have for $n = m^2$,

$$\Delta_n(\|.\|^2, P_m) = \frac{1}{12n}. \tag{17}$$

A distribution P, which is a mixture of the distributions P_m, with $\underline{\dim}^*_B(P) = 2$ exists, if for all $\epsilon > 0$ mixtures P^ϵ exist with $\underline{\dim}^*_B(P^\epsilon) > 2 - \epsilon$ ($P = \sum w_k P^{1/k}$ satisfies $\underline{\dim}^*_B(P) = 2$). Let $\epsilon > 0$, $s < 2 - \epsilon$ and $a \in \mathbb{N}$ with $a > s/(2-s)$. Denote by A the set $A = \{(k+1)^a \mid k \in \mathbb{N}\}$ and define q_m by

$$q_m = \begin{cases} c_1/k^2 & \text{if } m \in A, m = (k+1)^a, \\ 0 & \text{otherwise,} \end{cases}$$

where $c_1 = 6/\pi^2$. Let $m_n = ([n^{1/2a}] + 1)^a \in A$. Thus

$$\begin{aligned} n^{2/s}\Delta_n(\|.\|^2, P) &\geqslant n^{2/s} q_{m_n} \Delta_{m_n^2}(\|.\|^2, P_{m_n}) \\ &= \frac{c_1}{12} \frac{n^{2/s}}{[n^{1/2a}]^2([n^{1/2a}]+1)^{2a}} \geqslant c_2 n^{2/s - 1 - 1/a}, \end{aligned}$$

so that $\liminf_{n\to\infty} n^{2/s}\Delta_n(\|.\|^2, P) = \infty$, implying $\underline{\dim}_Q(P) \geqslant s > 2 - \epsilon$.

We show that for $q_m = (1/2)^m$, $\overline{\dim}_Q(P) = 1$ holds. Constants c_3 and c_4 exist with $\Delta_n(\|.\|^2, P_m) \leqslant c_3/n$ and $\Delta_n(\|.\|^2, P_m) \leqslant c_4 m^2/n^2$ for all $n, m \in \mathbb{N}$. Let $i_n = [\log_2 n]$ and let $O_1, \ldots, O_{i_n}, O_{i_n+1}$ be sets of prototypes for $P_1, \ldots, P_{i_n}$ and $\sum_{m=i_n+1}^\infty (\frac{1}{2})^{m-i_n} P_m$ of sizes $[n/(2i_n)], \ldots, [n/(2i_n)]$ and $n - i_n[n/(2i_n)]$, respectively. Define $O := O_1 \cup \cdots \cup O_{i_n+1}$. Thus ($c_5, c_6, c_7$ are constants),

$$\begin{aligned} \Delta_n(\|.\|^2, P) &\leqslant \Delta_n(\|.\|^2, O, P) \\ &\leqslant \sum_{m=1}^{i_n} (\tfrac{1}{2})^m \Delta_{[n/(2i_n)]}(\|.\|^2, O_m, P_m) \\ &\quad + (\tfrac{1}{2})^{i_n+1} \Delta_{n - i_n[n/(2i_n)]} \left(\|.\|^2, O_{i_n+1}, \sum_{m=i_n+1}^\infty (\tfrac{1}{2})^{m-i_n} P_m \right) \\ &= \sum_{m=1}^{i_n} (\tfrac{1}{2})^m \Delta_{[n/(2i_n)]}(\|.\|^2, P_m) \\ &\quad + (\tfrac{1}{2})^{i_n+1} \Delta_{n - i_n[n/(2i_n)]} \left(\|.\|^2, \sum_{m=i_n+1}^\infty (\tfrac{1}{2})^{m-i_n} P_m \right) \\ &\leqslant \sum_{m=1}^{i_n} (\tfrac{1}{2})^m c_4 m^2/[n/(2i_n)]^2 + (\tfrac{1}{2})^{i_n+1} c_3/(n - i_n[n/(2i_n)]) \\ &\leqslant c_5 (i_n/n)^2 + c_6/n^2 \leqslant c_7 (\log n/n)^2. \end{aligned}$$

3. *Appendix: Results from geometric measure theory*

References [**1**–**3**, **8**] provide accounts of the concepts and results presented in this appendix.

1. *Hausdorff and packing measure.* Let $E \subseteq \mathbb{R}^d$, $\delta > 0$. A δ-cover of E is a finite or denumerably infinite family of sets $\{U_i\}$ such that $E \subseteq \bigcup_i U_i$ and for all i,

https://doi.org/10.1017/S0305004101005357 Published online by Cambridge University Press

$\operatorname{diam}(U_i) \leqslant \delta$. $\operatorname{diam}(A) := \sup\{\|x - y\| \mid x, y \in A\}$ denotes the diameter of A. For $s > 0$ let

$$\mathscr{H}^s_\delta(E) = \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam}(U_i)^s \mid \{U_i\} \text{ is a } \delta\text{-cover of } E\right\} \tag{18}$$

and

$$\mathscr{H}^s(E) = \lim_{\delta \to 0+} \mathscr{H}^s_\delta(E). \tag{19}$$

$\mathscr{H}^s(E)$ exists in $[0, \infty]$, as $\mathscr{H}^s_\delta(E)$ is increasing in δ. $\mathscr{H}^s(E)$ is the Hausdorff measure of E. $\mathscr{H}^s$ is a Borel-regular outer measure on $\mathbb{R}^d$. Borel sets are measurable. For $s = d$ $\mathscr{H}^d = c_d^{-1}\mathscr{L}^d$, where $\mathscr{L}^d$ is Lebesgue measure and $c_d = \pi^{d/2}2^{-d}/\Gamma(1 + d/2)$ is the volume of the d-dimensional ball of diameter 1.

For every set $E \subseteq \mathbb{R}^d$ a real s, the Hausdorff dimension $\dim_{\mathscr{H}}(E)$ of E, exists, such that

$$\mathscr{H}^\alpha(E) = \begin{cases} \infty & \text{for } \alpha < s, \\ 0 & \text{for } \alpha > s. \end{cases}$$

An s-set is a set E such that $0 < \mathscr{H}^s(E) < \infty$, where $s = \dim_{\mathscr{H}}(E)$.

Packing measure and packing dimension are defined analogously. A finite or denumerably infinite family of disjoint balls $\{B_i\}$ with centers in E is called a δ-packing of E, if for all i, $\operatorname{diam}(B_i) \leqslant \delta$ holds. Define

$$\mathscr{P}^s_\delta(E) = \sup\left\{\sum_{i=1}^{\infty} \operatorname{diam}(B_i)^s \mid \{B_i\} \text{ is a } \delta\text{-packing of } E\right\} \tag{20}$$

and

$$\mathscr{P}^s_0(E) = \lim_{\delta \to 0+} \mathscr{P}^s_\delta(E). \tag{21}$$

$\mathscr{P}^s_0$ itself is not an outer measure but allows the definition of the packing measure

$$\mathscr{P}^s(E) = \inf\left\{\sum_{i=1}^{\infty} \mathscr{P}^s_0(E_i) \mid E \subseteq \bigcup_i E_i\right\}. \tag{22}$$

The packing dimension of a set E is the discontinuity of the mapping $\alpha \mapsto \mathscr{P}^\alpha(E)$.

2. *Box-counting dimension.* Define the covering number $N(E, r)$ of a bounded set $E \subseteq \mathbb{R}^d$ as the minimal cardinality of a r-cover of E. The lower and the upper box-counting dimension of E are given by

$$\underline{\dim}_B(E) = \liminf_{r \to 0+} \frac{\log N(E, r)}{\log(1/r)}, \tag{23}$$

$$\overline{\dim}_B(E) = \limsup_{r \to 0+} \frac{\log N(E, r)}{\log(1/r)}. \tag{24}$$

A set E with $\overline{\dim}_B(E) = s$ is a set of exact upper box-counting dimension, if $N(E, r) = O(r^{-s})$. Restricting the r-covers to certain classes of sets leads to the same definition of box-counting dimensions. An attractive class are covers by so-called r-meshed cubes,

$$[i_1 2^{-k}, (i_1 + 1)2^{-k}[\times \cdots \times [i_d 2^{-k}, (i_d + 1)2^{-k}[, \tag{25}$$

https://doi.org/10.1017/S0305004101005357 Published online by Cambridge University Press

with $2^{-k} \leqslant r$. (r-meshed cubes on the right-hand boundary of $[0,1]^d$ are modified to cover $[0,1]^d$.) We denote the covering number of E by r-meshed-cubes by $N_K(E,r)$.

LEMMA A1.
(1) $\dim_{\mathscr{H}}(E) \leqslant \dim_{\mathscr{P}}(E) \leqslant \overline{\dim}_B(E)$ *and* $\dim_{\mathscr{H}}(E) \leqslant \underline{\dim}_B(E) \leqslant \overline{\dim}_B(E)$.
(2) $\dim(\bigcup_{i=1}^k E_i) = \max_i \dim(E_i)$ *for* $\dim = \dim_{\mathscr{H}}, \dim_{\mathscr{P}}, \dim_B$.
(3) $\underline{\dim}(\bigcup_{i=1}^\infty E_i) = \max_i \dim(E_i)$ *for* $\dim = \dim_{\mathscr{H}}, \dim_{\mathscr{P}}$.
(4) $\overline{\dim}_B(E) = \overline{\dim}_B(\bar{E})$ *and* $\underline{\dim}_B(E) = \underline{\dim}_B(\bar{E})$.
(5) *If for all* $\alpha > s$ *sets* E_i *exist, such that* $E = \bigcup_{i=1}^\infty E_i$ *and* $\overline{\dim}_B(E_i) < \alpha$, *then* $\dim_{\mathscr{P}}(E) \leqslant s$.
(6) *Let* $\dim_{\mathscr{P}}(E) = s$ *and* $\alpha > s$. E_i *exist such that* $E = \bigcup_{i=1}^\infty E_i$ *and* $\overline{\dim}_B(E_i) < \alpha$.

3. *Local dimension.* Let P be a finite Borel measure on $\mathbb{R}^d$. The lower and upper local dimension of P is defined by

$$\underline{\dim}_{\mathrm{loc}}(P,x) = \liminf_{r\to 0} \frac{\log P(B(x,r))}{\log r}, \tag{26}$$

$$\overline{\dim}_{\mathrm{loc}}(P,x) = \limsup_{r\to 0} \frac{\log P(B(x,r))}{\log r}. \tag{27}$$

LEMMA A2. *Let* $\alpha > 0$. *Then*
(1) $\dim_{\mathscr{H}}(\{x \mid \underline{\dim}_{\mathrm{loc}}(P,x) \leqslant \alpha\}) \leqslant \alpha$.
(2) $A \subseteq \{x \mid \underline{\dim}_{\mathrm{loc}}(P,x) \geqslant \alpha\}$ *and* $P(A) > 0$ *implies* $\dim_{\mathscr{H}}(A) \geqslant \alpha$.
(3) $\dim_{\mathscr{P}}(\{x \mid \overline{\dim}_{\mathrm{loc}}(P,x) \leqslant \alpha\}) \leqslant \alpha$.
(4) $A \subseteq \{x \mid \overline{\dim}_{\mathrm{loc}}(P,x) \geqslant \alpha\}$ *and* $P(A) > 0$ *implies* $\dim_{\mathscr{P}}(A) \geqslant \alpha$.

4. *Density.* The lower and the upper density of a Borel measure P in $x \in \mathbb{R}^d$ are defined by

$$\underline{D}^s(P,x) = \liminf_{r\to 0} \frac{P(B(x,r))}{(2r)^s}, \tag{28}$$

$$\overline{D}^s(P,x) = \limsup_{r\to 0} \frac{P(B(x,r))}{(2r)^s}. \tag{29}$$

$\overline{D}^s(E,x)$ and $\underline{D}^s(E,x)$ denote the densities of $\mathscr{H}^s_{|E}$, i.e. $\overline{D}^s(\mathscr{H}^s_{|E},x)$ and $\underline{D}^s(\mathscr{H}^s_{|E},x)$ respectively. The upper convex density (for $P = \mathscr{H}^s_{|E}$) is

$$\overline{D}^s_c(E,x) = \lim_{r\to 0} \sup \left\{ \frac{\mathscr{H}^s(E \cap C)}{(\mathrm{diam}(C))^s} \mid C \text{ convex}, x \in C, \mathrm{diam}(U) \leqslant r \right\}. \tag{30}$$

LEMMA A3. *Let* E *denote a Borel set.*
(1) $\overline{D}^s_c(E,x) = 1$ $\mathscr{H}^s_{|E}$-a.e.
(2) $P(E) \leqslant \mathscr{H}^s(E) \sup\{\overline{D}^s(P,x) \mid x \in E\}$.
(3) $P(E) \geqslant \mathscr{H}^s(E) \inf\{\overline{D}^s(P,x) \mid x \in E\}$.
(4) $P(E) \leqslant \mathscr{P}^s(E) \sup\{\underline{D}^s(P,x) \mid x \in E\}$.
(5) $P(E) \geqslant \mathscr{P}^s(E) \inf\{\underline{D}^s(P,x) \mid x \in E\}$.

REFERENCES

[1] C. D. CUTLER. The density theorem and Hausdorff inequality for packing measure in general metric spaces. *Illinois Journal of Mathematics* **39** (1995), 676–694.
[2] K. J. FALCONER. *The geometry of fractal sets* (Cambridge University Press, 1995).

https://doi.org/10.1017/S0305004101005357 Published online by Cambridge University Press

[**3**] K. J. Falconer. *Techniques in fractal geometry* (John Wiley, 1997).
[**4**] S. Graf and H. Luschgy. The quantization of the Cantor distribution. *Math. Nachrichten* **183** (1997), 113–133.
[**5**] S. Graf and H. Luschgy. The quantization dimension of self-similar sets. MIP-Berichte 9609, Fakultät für Mathematik und Informatik, Universität Passau (1996).
[**6**] S. Ikeda. On a fractal set with a gap between its Hausdorff dimension and box dimension. *Hiroshima Math. J.* **25** (1995), 433–439.
[**7**] K. Pötzelberger and H. Strasser. *Clustering and Quantization by MSP-partitions*, to appear in *Statistics and Decisions* **19** (2001).
[**8**] C. Tricot. Two definitions of fractal dimensions. *Math. Proc. Camb. Phil. Soc.* **91** (1982), 57–74.
[**9**] P. L. Zador. Development and Evaluation of Procedures for Quantizing Multivariate Distributions. Ph.D. Thesis, Stanford University, 1964.
[**10**] P. L. Zador. Asymptotic quantization error of continuous signals and the quantization dimension. *IEEE Trans. Inform. Theory* **28** (1982), 139–149.

https://doi.org/10.1017/S0305004101005357 Published online by Cambridge University Press