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# The Cauchy integral and analytic continuation*

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One of the most important concepts in the theory of approximation by analytic functions is that of analytic continuation. In a typical problem, for example, there is generally a region $\Omega$, a Banach space $B$ of functions analytic in $\Omega$ and a subfamily $\mathscr{F} \subset B$, each member of which is analytic in some larger open set, and one might be asked to decide whether or not $\mathscr{F}$ is dense in $B$. It often happens, however, that either $\mathscr{F}$ is dense or the only functions which can be so approximated have a natural analytic continuation across $\partial\Omega$. A similar phenomenon is also known to occur even for approximation on sets without interior. In this article we shall describe a method for proving such theorems which can be applied in a variety of settings and, in particular, to: (1) the Bernštein problem for weighted polynomial approximation on the real line; (2) the completeness problem for weighted polynomial approximation on bounded simply connected regions; (3) the Shapiro overconvergence problem for sequences of rational functions with sparse poles; (4) the Akutowicz–Carleson minimum problem for interpolating functions. Although we shall present no new results, the method of proof, which is based on an argument of the author [**6**], seems sufficiently versatile to warrant exposition.

## 1. *The Bernštein approximation problem*

Let $w(x) \geqslant 0$ be a bounded measurable function defined on the real line $\mathbb{R}$ with the property that

$$\lim_{|x|\to\infty} |x|^n w(x) = 0, \quad n = 0, 1, 2, \ldots,$$

and let $C_w = \{F: \lim_{|x|\to\infty} F(x)\, w(x) = 0, F \text{ continuous}\}$. Then, clearly, the polynomials are contained in $C_w$ and we can, therefore, pose the question: Given an arbitrary $F \in C_w$, does there exist a sequence of polynomials $Q_j, j = 1, 2, 3, \ldots$, such that

$$\sup_x |F(x) - Q_j(x)|\, w(x) \to 0 \quad \text{as} \quad j \to \infty?$$

This is the celebrated *Bernštein problem* as it was originally set in 1924 (cf. [**3**]). By now, of course, it has been largely solved due to the efforts of Bernštein himself, Ahiezer, Pollard, Mergeljan and others (cf. [**1**], [**9**], [**15**]).

The fact most pertinent to the present discussion, however, is the following result due to Mergeljan ([**15**], p. 74), but first noticed in a special case by Bernštein [**4**]:

THEOREM 1. *Either the polynomials are dense in $C_w$ or the only functions in $C_w$ which admit approximation by polynomials are restrictions of entire functions to the real axis $\mathbb{R}$.*

Note that the Weierstrass approximation theorem is an immediate consequence of

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this fact. For if $[a, b]$ is any closed, bounded interval, then for $\lambda$ real and sufficiently large $(x-\lambda)^{-1} = -\sum_{n=0}^{\infty} x^n/\lambda^{n+1}$, the series converging uniformly on $[a, b]$, and $(x-\lambda)^{-1}$ is obviously not entire.

In addition to the argument that we shall now give, another simple proof of Theorem 1 can be found in the article of Ferrier [9]. On the other hand, his reasoning is special to the Bernštein problem and does not extend to the other questions considered here.

*Proof of theorem* 1. Suppose that the polynomials are not dense in $C_w$. Then, it follows in the usual manner that there exists a nonzero measure $w d\sigma$ on $\mathbb{R}$ such that

$$(a) \qquad \int_{-\infty}^{\infty} t^n w(t)\, d\sigma(t) = 0, \quad n = 0, 1, 2, \ldots;$$

$$(b) \qquad \int_{-\infty}^{\infty} |d\sigma(t)| < \infty.$$

We may also assume without loss of generality that $\sigma$ is real.

Now let

$$f(z) = \int_{-\infty}^{\infty} \frac{w(t)\, d\sigma(t)}{t-z},$$

so that $f(z)$ is defined and analytic in both the upper and lower half planes and, because $\sigma$ is real, $f(\bar{z}) = \overline{f(z)}$. Thus, $f$ cannot vanish identically in either half plane, since $\sigma \neq 0$. Furthermore, it can easily be shown that

$$\int_{-\infty}^{\infty} \frac{|f(x+iy)|^{\frac{1}{2}}}{1+x^2} dx \leqslant C < \infty$$

uniformly for $0 < y < \infty$ (cf. [9], p. 313). Consequently, $f$ has nontangential limits a.e. $-dx$ on the real axis and those boundary values cannot vanish, except possibly on a set of measure zero.

From these remarks we readily conclude that, given any $N > 0$, there exists a set $X$ made up of circles $|z| = r$, $N \leqslant r \leqslant 2N$, with the following properties:

(1) $X$ has positive $dA$ measure;

(2) $|f| \geqslant \lambda > 0$ a.e. $-dA$ on $X$ for some $\lambda$,

where $dA$ denotes area or two-dimensional Lebesgue measure. If we then let $U$ be the disc $|z| < N/2$ we can infer from the Cauchy integral formula that there exists, for each $\zeta \in U$, a function $G_\zeta \in L^\infty$ such that

$$(3) \qquad Q(\zeta) = \int_X Q G_\zeta\, dA \quad \text{for every polynomial } Q;$$

(4) $\|G_\zeta\|_\infty \leqslant C$ for some constant $C$ which does not depend on $\zeta \in U$.

Because $\int_{-\infty}^{\infty} Q(t)\, w(t)\, d\sigma(t) = 0$ for all polynomials $Q$, we also have

$$(5) \qquad Q(\xi) = \frac{1}{f(\xi)} \int_{-\infty}^{\infty} Q(t) \frac{w(t)\, d\sigma(t)}{t-\xi},$$

the formula being valid at every point $\xi$ where $f(\xi)$ is defined and non-zero; hence, it is

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valid a.e. $-dA_\xi$ on $X$. Finally, substituting (5) into (3), applying Fubini's theorem, and making use of property (2), we conclude that

$$\begin{aligned}|Q(\zeta)| &\leqslant C\int_{-\infty}^{\infty} |Q(t)|\, w(t)\left(\int_X \frac{1}{|f(\xi)|}\cdot\frac{|G_\zeta(\xi)|}{|t-\xi|}\,dA_\xi\right)d\sigma(t)\\ &\leqslant C\sup_t |Q(t)|\, w(t)\end{aligned}$$

for every polynomial $Q$ and every $\zeta\in U$. Therefore, if a sequence of polynomials $Q_j$ converges to some function $F$ in the norm of $C_w$ the limit has a natural analytic continuation to the disc $|z| < N/2$. Since $N$ is arbitrary, $F$ is entire. ∎

A second and completely analogous problem to the one just considered is this: can every $F\in L^p(w\,dx)$, $p \geqslant 1$, be approximated by a sequence of polynomials $Q_j$, $j = 1, 2, \ldots$, in the sense that

$$\int_{-\infty}^{\infty} |F(x)-Q_j(x)|^p\, w(x)\,dx \to 0 \quad \text{as} \quad j\to\infty?$$

We must assume, of course, that

$$\int_{-\infty}^{\infty} |x|^n\, w(x)\,dx < \infty, \quad n = 0, 1, 2, \ldots.$$

By exactly the same method as outlined above one can prove: *either the polynomials are dense in $L^p(w\,dx)$ or the only functions which can be so approximated are restrictions of entire functions to the real line.*

For additional information on questions of this kind, we refer the reader to the survey article [**10**].

### 2. *Approximation on planar sets and a conjecture of Mergeljan*

A second problem which may at first appear to be rather unrelated to Bernštein's, but which has been developed along similar lines, arises in the following way: Let $\Omega$ be a bounded simply connected domain, let $dA$ denote two-dimensional Lebesgue measure and let $w(z) > 0$ be a bounded measurable function defined on $\Omega$. For each $p$, $1 \leqslant p < \infty$, we consider two spaces of functions:

(i) $H^p(\Omega, w\,dA)$, *the closure of the polynomials in* $L^p(w\,dA)$;

(ii) $L^p_a(\Omega, w\,dA)$, *the set of functions in* $L^p(w\,dA)$ *which are analytic in* $\Omega$.

If $w$ is bounded away from zero locally it is easy to see that $L^p_a$ is a closed subspace of $L^p$ and that $H^p \subseteq L^p_a$. It is an old problem to determine: *for which regions* $\Omega$ *and weights* $w$ *is* $H^p(\Omega, w\,dA) = L^p_a(\Omega, w\,dA)$? When this happens the polynomials are said to be *complete* in $L^p_a$.

Problems of this kind were first considered by Carleman in 1923 and about ten years later by Markuševič and Farrell. However, the first results for a general region $\Omega$ and weight $w$ were not obtained until the early 1940s, that having been accomplished by Keldyš [**12**]. In subsequent years the problem has been taken up by Mergeljan, Havin, Beurling and others. For a more detailed description and historical account the reader is referred to articles [**6**], [**7**], [**13**] and to the references cited therein.

The main difficulty in proving that $H^p(\Omega, w\,dA) = L^p_a$ stems from the fact that $\Omega$ may have a non-empty *inner boundary*; that is, there may be points in $\partial\Omega$ which belong

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to the interior of $\overline{\Omega}$ (viz., a Jordan domain with a cut or incision in the form of a simple arc from an interior point to a boundary point). Roughly speaking, $H^p(\Omega, w\,dA) = L^p_a$ if $w(z) \to 0$ sufficiently rapidly at every point of the inner boundary. But this is not the only factor that has to be considered, and in order to avoid certain technical difficulties we shall make the additional assumption that $w$ depends only on Green's function. By this we mean that $w(z) = W(g(z))$, where $g(z)$ is the Green function for $\Omega$ and $W(t) \to 0$ as $t \to 0$. This, in effect, makes the completeness problem 'conformally invariant' and, moreover, every significant result to date makes use of this or some equivalent fact.

Based on the evidence of known examples and influenced perhaps by his work on the Bernštein problem, Mergeljan [**14**] conjectured in 1955 that:

THEOREM 2. *Either $H^p(\Omega, w\,dA) = L^p_a$ or every $f \in H^p$ admits an analytic continuation across a fixed piece of $\partial\Omega$.*

Although the conjecture was originally stated only in the case $w(z) \equiv 1$, it is unlikely that Mergeljan was unaware of the general theorem. In any case, it was finally settled in the affirmative by the author [**6**] in 1979, using an argument similar to that described here in Section 1.

In much the same way we can also obtain a parallel result for completeness in $L^p(w\,dA)$ in which the assumptions on $w$ are less restrictive.

THEOREM 3. *Let $w\,dA$ be a measure of compact support in the complex plane $\mathbb{C}$. If $w \in L^{1+\epsilon}(dA)$ for some $\epsilon > 0$, then either $H^p(w\,dA) = L^p$ or there exists a point $x_0$ in the support of $w$ and an open set $U$ containing $x_0$ such that every $f \in H^p(w\,dA)$ admits an analytic continuation to $U$.*

For $p = 2$, Theorem 3 is of particular significance due to its connection with the invariant subspace problem for subnormal operators. In that context it has long been conjectured that the corresponding result is true for an arbitrary compactly supported measure $d\mu$. At this time, however, it is not even known if the assertion is valid for absolutely continuous measures; that is, for measures of the form $w\,dA$ with $w \in L^1(dA)$. On the other hand, Hruščev [**11**] was able to obtain a slight improvement of Theorem 3 in which the condition $w \in L^{1+\epsilon}(dA)$ is replaced by the assumption that

$$\int w(\log^+ w)^p\,dA < \infty$$

(cf. also [**6**], p. 413).

In conclusion, we shall now sketch briefly the proof of Theorem 3, but only to the extent that is necessary to give an indication as to how the argument in Section 1 can be carried over to the planar setting. A more detailed account can be found in [**6**].

*Proof of theorem* 3. Suppose that $H^p(w\,dA) \neq L^p$. Then there exists a non-zero function $g \in L^q(w\,dA)$, $q = p/(p-1)$, with the property that $\int Qg\,w\,dA = 0$ for every polynomial $Q$. Moreover, the Cauchy integral

$$f(z) = \int \frac{gw(\zeta)}{\zeta - z}\,dA_\zeta$$

must fail to vanish on some set of positive measure. If we could find a point $x_0$ and a set $X$ consisting of circles $|z - x_0| = r$ such that

(1) $X$ has positive $dA$ measure;

(2) $|f| \geqslant \lambda > 0$ a.e. $-dA$ on $X$ for some $\lambda$,

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we could then argue exactly as in the proof of Theorem 1 to obtain the desired analytic continuation.

The existence of such a set, however, is a simple consequence of the fact that $gw \in L^{1+\delta}(dA)$ and that $f$ is, therefore, 'almost continuous' almost everywhere; that is, for almost every $x_0 \in \mathbb{C}$ there is a set $E$ (depending on $x_0$) which is 'thin' at $x_0$ and for which

$$\lim_{\substack{x \to x_0 \\ x \notin E}} f(x) = f(x_0).$$

Here it is sufficient to know that if, for any $\rho > 0$, almost every circle $|z - x_0| = r, r \leqslant \rho$, meets a certain set then that set is not thin at $x_0$. We have then only to choose a point $x_0$ where $f$ is almost continuous and where $f(x_0) \neq 0$. Setting $E_\lambda = \{z: |f(z)| < \lambda\}$, it follows that, for small values of $\lambda$, $E_\lambda$ is thin at $x_0$ and , consequently, there are enough circles lying in the complement $\mathbb{C} \backslash E_\lambda$ to form the required set $X$. ∎

### 3. *The Shapiro overconvergence problem*

Let $D = \{z: |z| < 1\}$ be the open unit disc, let $D_\infty = \{z: |z| > 1\}$ be the complementary region and let $B$ be a Banach space of functions defined and analytic in $D$ with the property that $(z-a)^{-1} \in B$ for all $a \in D_\infty$. With certain mild restrictions on $B$, Shapiro [16] has shown that if a sequence of functions $F_n$, $n = 1, 2, \ldots$, converges in the norm of $B$ and if the poles of each $F_n$ are confined to a 'sparse' subset $E \subset D_\infty$ then the sequence overconverges in the sense that it also converges uniformly on compact subsets of $D_\infty \backslash E$. In general, however, the sequence $F_n$, $n = 1, 2, \ldots$, will not converge in any region intersecting $\partial D$ and the limit functions in $D$ and $D_\infty$ need not be analytic continuations of one another. Thus, we are led to ask: *What additional assumptions on $B$ and $E$ will guarantee overconvergence in a region meeting $\partial D$?* This is the *Shapiro overconvergence problem*. Similar questions have also been studied by Tumarkin [18].

In this article we shall consider only the case where $B$ is one of the spaces $H^p(d\theta)$, $1 \leqslant p < \infty$, and, moreover, our results are implicit in those of Shapiro [16]. Our argument, however, is more elementary in that it does not appeal to the log log-theorem of Beurling (cf. [8], p. 431) on the existence of a greatest subharmonic minorant. Finally, given any set $E \subset D_\infty$ we shall denote by $\mathscr{F}_E$ the set of all finite linear combinations $\sum_k c_k/(z-a_k)$ corresponding to $a_k \in E$. If $\mathscr{F}_E$ is not dense in $B$ then $E$ is, in a manner of speaking, sparse.

THEOREM 4. *Let $E$ be any subset of $D_\infty$ having no accumulation points on some arc $\alpha$ of $\partial D$ and suppose that $\mathscr{F}_E$ is not dense in $H^p(d\theta)$. If a sequence $F_n$, $n = 1, 2, \ldots$, belonging to $\mathscr{F}_E$ converges in the norm of $H^p(d\theta)$ then the limit function $F$ admits an analytic continuation across $\alpha$ to $D_\infty \backslash \bar{E}$.*

*Proof.* Because $\mathscr{F}_E$ is not dense in $H^p(d\theta)$ we can find a function $k \in L^q(d\theta)$, $q = p/(p-1)$, so that

$$\int_0^{2\pi} Qk\, d\theta = 0 \quad \text{for every} \quad Q \in \mathscr{F}_E; \tag{1}$$

$$\hat{k}(z) = \int_0^{2\pi} \frac{k(\theta)}{e^{i\theta} - z}\, d\theta \not\equiv 0 \quad \text{in} \quad D_\infty. \tag{2}$$

It may happen, however, that $\hat{k} \not\equiv 0$ in $D$, in which case we do the following: choose $g \in L^q(d\theta)$ so that

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(3) $\hat{g} \equiv 0$ in $D_\infty$ and hence $g$ is orthogonal to $\mathscr{F}_E$;

(4) $\hat{g} \not\equiv 0$ in $D$.

In particular, any $g \in H^q(d\theta)$ with $g(0) = 0$ will do. Setting $h = k + g$, we obtain a function which is orthogonal to $\mathscr{F}_E$ and whose Cauchy transform $\hat{h} = k \hat{+} g$ has non-zero non-tangential limits a.e. $-d\theta$ on $\partial D$ – from both inside and outside $D$.

Given any point $x_0 \in \alpha$ we can, as in Section 1, find a collection of circles $X$, each centred at $x_0$, and having the properties:

(5) $X$ does not surround any points of $\bar{E}$;

(6) $|\hat{h}| \geqslant \lambda > 0$ a.e. $-dA$ on $X$.

Further, $[Q(z) - Q(\xi)](z - \xi)^{-1} \in \mathscr{F}_E$ whenever $Q \in \mathscr{F}_E$ and so the representation

$$Q(\xi) = \frac{1}{\hat{h}(\xi)} \int_0^{2\pi} Q(\theta) \frac{h(\theta)}{e^{i\theta} - \xi} d\theta,$$

corresponding to formula (5) of Section 1, is valid for almost every $\xi \in X$. We can, therefore, repeat the entire argument in the proof of Theorem 1 to conclude that if $F_n \to F$ and all $F_n \in \mathscr{F}_E$ then $F$ has an analytic extension to a neighbourhood of $x_0$. The reader can easily verify that the continuation extends to all of $D_\infty \backslash \bar{E}$. ∎

## 4. *The Akutowicz–Carleson minimum problem*

Once again $D$ will denote the open unit disc and $B$ will be a Banach space of functions defined and analytic in $D$. We shall further assume that $\{z_n, n = 1, 2, \ldots\}, z_n \in D$, and $\{w_n, n = 1, 2, \ldots\}$ are two complex sequences such that the interpolation problem

$$f(z_n) = w_n, \quad n = 1, 2, \ldots, \tag{4·1}$$

with $f \in B$ is underdetermined; that is, there are at least two distinct functions in $B$ (and so infinitely many) which satisfy (4·1). Under certain conditions on $B$ which, among other things, guarantee that the interpolation problem has a unique solution of minimal norm, Akutowicz and Carleson [2] have shown that this minimal solution admits an analytic continuation across any piece of $\partial D$ not lying in the closure of the point set $\{z_n\}$. The general question as to when such a phenomenon can be expected to occur is known as the *Akutowicz–Carleson minimum problem*.

In order to illustrate how our ideas can be applied in this setting we shall restrict our attention to one special case. We shall assume that $B = \mathscr{D}$ is the Dirichlet space which consists of all functions $f$ analytic in $D$ such that $f(0) = 0$ and $\int |f'|^2 dA < \infty$. An inner product is then given by the expression

$$\langle f, g \rangle = \int_D f' \bar{g}' \, dA$$

for $f, g \in \mathscr{D}$, and we have a reproducing kernel

$$K(z, \zeta) = -\frac{1}{\pi} \log(1 - \bar{\zeta} z).$$

Here it is important to note that if $f$ is any solution to the interpolation problem (4·1)

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and if $g$ is orthogonal to the functions $K(z, z_n)$, $n = 1, 2, \ldots$, then $f \pm \epsilon g$ also solves (4·1). Consequently, if $f_*$ is the minimal solution a standard variational argument shows that

$$\langle f_*, g\rangle = \int f_*' \bar{g}'\, dA = 0,$$

from which it follows that $f_*$ lies in the closed linear span of $\{K(z, z_n), n = 1, 2, \ldots\}$. It is this fact which allows us to turn the Akutowicz–Carlson problem into an approximation question of the kind studied in Section 3. It was used explicitly by Shapiro [**16**] and in a somewhat more disguised form by Akutowicz and Carleson [**2**].

THEOREM 5. *If the interpolation problem* (4·1) *is underdetermined in the Dirichlet space* $\mathscr{D}$ *then the solution* $f_*$ *of minimal norm admits an analytic continuation across any boundary arc* $\alpha$ *which is disjoint from the closure of the point set* $\{z_n, n = 1, 2, \ldots\}$.

Continuation of the minimal solution $f_*$, is by no means an exclusive property of the Dirichlet space. We have chosen this example because the technique developed in the preceding sections can be applied here with particular ease and the resulting proof of the extension property differs substantially from that of Akutowicz and Carleson ([**2**], chapter 1).

*Proof of theorem* 5. According to our remarks the extremal function $f_*$ belongs to the space $\mathscr{F}$ spanned by the finite linear combinations $\Sigma c_n \log(1 - \bar{z}_n z)$. Hence, the derivative $f_*'$ is in the Bergman subspace $\mathscr{F}' \subseteq L_a^2(D)$ spanned by the sums $\Sigma c_n/(r - a_n)$, $a_n = 1/\bar{z}_n$, $n = 1, 2, \ldots$. If $\alpha \subseteq \partial D$ is an arc at a positive distance from $\{z_n\}$ it can be shown that every function in $\mathscr{F}'$, and in particular $f_*'$, admits an analytic continuation across $\alpha$.

To see how this is done, let $g \in \mathscr{D}$ with $g \perp \mathscr{F}$ and $g \not\equiv 0$. Such a function exists because the interpolation problem is underdetermined and we can arrange that $g(z_n) = 0$, $n = 1, 2, \ldots$. Each finite sum $f' = \Sigma c_n/(z - a_n)$ can then be represented in the form

$$f'(\lambda) = \frac{1}{\mu(\lambda)} \int_D f'(z) \frac{\bar{g}'(z)}{z - \lambda}\, dA \tag{4·2}$$

for $\lambda \in D_\infty \backslash \{a_n\}$ with

$$\mu(\lambda) = \int \frac{\bar{g}'(z)}{z - \lambda}\, dA.$$

Here the function $\mu$ is actually defined almost everywhere in the entire plane, and by virtue of Green's theorem

$$\mu(\lambda) = \int_D \frac{\bar{g}'(z)}{z - \lambda}\, dA = \frac{-i}{2} \int_{\partial D} \frac{\bar{g}(z)}{z - \lambda}\, dz = -\pi \bar{g}(1/\bar{\lambda})$$

whenever $\lambda \in D_\infty$. This means that $\mu(\lambda)$ has non-zero nontangential limits a.e. on $\partial D$ under approach from $D_\infty$, since $g$ has the corresponding property inside $D$. It follows that $\mu$ is 'almost continuous' and non-zero at almost every point of $\partial D$. About any $x_0 \in \alpha$ we can, therefore, find a collection of circles $X$ which do not surround any of the points $a_n$, $n = 1, 2, \ldots$, and on which $|\mu| \geqslant C > 0$. Consequently, both $f_*'$ and $f_*$ extend analytically to a neighbourhood of $x_0$ and thus to a neighbourhood of $\alpha$. ∎

The preceding argument can be adapted to other Hilbert spaces for which the reproducing kernel is known. It can be used, for example, to obtain an analogous result for the Hardy space $H^2(d\theta)$. On the other hand, it does not readily extend to the

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Bergman space $L_a^2(D)$ where there is no Cauchy integral formula corresponding to (4·2) and its counterparts in Sections 1–3.

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